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Research Article

The Equivalence of Convergence Results between Ishikawa and Mann Iterations with Errors for Uniformly Continuous Generalized Φ-Pseudocontractive Mappings in Normed Linear Spaces

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We prove the equivalence of the convergence of the Mann and Ishikawa iterations with errors for uniformly continuous generalized Φ-pseudocontractive mappings in normed linear spaces. Our results extend and improve the corresponding results of Xu, 1998, Kim et al., 2009, Ofoedu, 2006, Chidume and Zegeye, 2004, Chidume, 2001, Chang et al. 2002, Liu, 1995, Hirano and huang, 2003, C. E. Chidume and C. O. Chidume, 2005, and huang, 2007.

1. Introduction

Let E be a real normed linear space, E^* its dual space, and $J : E \to 2^{E^*}$ the normalized duality mapping defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\| = \|f\|^2 \right\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. The single-valued normalized duality mapping is denoted by j.

Definition 1.1. A mapping $T : E \to E$ is said to be

(1) strongly accretive if, for all $x, y \in E$, there exist a constant $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq k\|x-y\|^2, \tag{1.2}$$

(2) ϕ-strongly accretive if there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq \phi(\|x-y\|)\|x-y\|, \quad \forall x, y \in E, \tag{1.3}$$

(3) generalized Φ-accretive if, for all $x, y \in E$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq \Phi(\|x-y\|). \tag{1.4}$$

Definition 1.2. Let $N(T) = \{x \in E : Tx = 0\} \neq \emptyset$. The mapping T is called strongly quasi-accretive if, for all $x \in E$, $q \in N(T)$, there exist a constant $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that $\langle Tx - Tq, j(x-q) \rangle \geq k\|x-q\|^2$; T is called ϕ-strongly quasi-accretive if, for all $x \in E$, $q \in N(T)$, there exists a function ϕ such that $\langle Tx - Tq, j(x-q) \rangle \geq \phi(\|x-q\|)\|x-q\|$, where ϕ is as in Definition 1.1. Finally, T is called generalized Φ-quasi-accretive if, for each $x \in E$, $q \in N(T)$, there exist $j(x-q) \in J(x-q)$ and a strictly increasing function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that $\langle Tx - Tq, j(x-q) \rangle \geq \Phi(\|x-q\|)$.

Closely related to the class of accretive-type mappings are those of pseudocontractive types.

Definition 1.3. A mapping T with domain $D(T)$ and range $R(T)$ is said to be

(1) strongly pseudocontractive if there exist a constant $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that, for each $x, y \in D(T)$,

$$\langle Tx - Ty, j(x-y) \rangle \leq k\|x-y\|^2, \tag{1.5}$$

(2) ϕ-strongly pseudocontractive if there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|, \quad \forall x, y \in D(T), \tag{1.6}$$

(3) generalized Φ-pseudocontractive if, for all $x, y \in D(T)$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \Phi(\|x-y\|). \tag{1.7}$$

Definition 1.4. Let $F(T) = \{x \in E : Tx = x\} \neq \emptyset$. The mapping T is called generalized Φ-hemi-pseudocontractive if, for all $x \in D(T)$, $q \in F(T)$, there exist $j(x-q) \in J(x-q)$ and a strictly increasing function $\Phi : [0,+\infty) \to [0,+\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Tq, j(x-q)\rangle \leq \|x-q\|^2 - \Phi(\|x-q\|). \tag{1.8}$$

Obviously, a mapping T is strongly pseudocontractive, ϕ-strongly pseudocontractive, generalized Φ-pseudocontractive, and generalized Φ-hemicontractive if and only if $(I-T)$ is strongly accretive, ϕ-strongly accretive, generalized Φ-accretive, and generalized Φ-quasi-accretive, respectively.

It is shown in [1] that the class of strongly pseudocontractive mappings is a proper subclass of ϕ-strongly pseudocontractive mappings. Furthermore, an example in [2] shows that the class of ϕ-strongly pseudocontractive mappings with the nonempty fixed point set is a proper subclass of Φ-hemicontractive mappings. Hence, the class of generalized Φ-hemicontractive mappings is the most general among those defined above.

Definition 1.5. The mapping $T : E \to E$ is called Lipschitz if exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x-y\|, \quad \forall x, y \in E. \tag{1.9}$$

It is clear that if T is Lipschitz then it must be uniformly continuous. Otherwise, it is not true.

The following iteration schemes were introduced by Xu [3] in 1998. Let K be a nonempty convex subset of E. For any given $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{gathered} x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n \delta_n, \\ y_n = \alpha'_n x_n + \beta'_n T x_n + \gamma'_n \eta_n, \quad \forall n \geq 0, \end{gathered} \tag{1.10}$$

is called the Ishikawa iteration sequence with errors, where $\{\delta_n\}_{n=0}^{\infty}, \{\eta_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in K and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{\alpha'_n\}_{n=0}^{\infty}, \{\beta'_n\}_{n=0}^{\infty}$, and $\{\gamma'_n\}_{n=0}^{\infty}$ are six real sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for all $n \geq 0$ and satisfy certain conditions.

If $\beta'_n = \gamma'_n = 0$, for all $n \geq 0$, then, from (1.10), we get the Mann iteration sequence with errors $\{u_n\}_{n=0}^{\infty}$ defined by

$$\forall u_0 \in K, \quad u_{n+1} = \alpha_n u_n + \beta_n T u_n + \gamma_n \varepsilon_n, \quad \forall n \geq 0, \tag{1.11}$$

where $\{\varepsilon_n\}_{n=0}^{\infty}$ is an arbitrary bounded sequence in K.

Numerous convergence results have been proved through iterative methods of approximating fixed points of Lipschitz pseudocontractive- (accretive-) type nonlinear mappings [3–10]. Most of these results have been extended to uniformly continuous mappings by some authors. Recently, C. E. Chidume and C. O. Chidume in [11] gave the most general result for uniformly continuous generalized Φ-hemicontractive mappings in normed linear. Their results are as follows.

Theorem 1.6 (see [11, Theorem 2.3]). *Let E be a real normed linear space, K nonempty subset of E, and $T : E \to K$ a uniformly continuous generalized Φ-hemicontractive mapping, that is, there* *exist* $x^* \in F(T)$ *and a strictly increasing function* $\Phi : [0,+\infty) \to [0,+\infty)$, $\Phi(0) = 0$, *such that, for all* $x \in K$, *there exists* $j(x - x^*) \in J(x - x^*)$ *such that*

$$\langle Tx - x^*, j(x - x^*)\rangle \leq \|x - x^*\|^2 - \Phi(\|x - x^*\|). \tag{1.12}$$

(a) *If* $y^* \in K$ *is a fixed point of* T, *then* $y^* = x^*$ *and so* T *has at most one fixed point in* K.

(b) *Suppose there exists* $x_0 \in K$ *such that the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad \forall n \geq 0, \tag{1.13}$$

is contained in K, *where* $\{a_n\}, \{b_n\}$, *and* $\{c_n\}$ *are real sequences satisfying the following conditions:*

(i) $a_n + b_n + c_n = 1$,

(ii) $\sum_{n=0}^{\infty}(b_n + c_n) = \infty$,

(iii) $\sum_{n=0}^{\infty}(b_n + c_n)^2 < \infty$,

(iv) $\sum_{n=0}^{\infty} c_n < \infty$.

Then, $\{x_n\}$ *converges strongly to* x^*. *In particular, if* y^* *is a fixed point of* T *in* K, *then* $\{x_n\}$ *converges strongly to* y^*.

Unfortunately, the control conditions (iii) $\sum_{n=0}^{\infty}(b_n + c_n)^2 < \infty$ and (iv) $\sum_{n=0}^{\infty} c_n < \infty$ cannot assure that the result in [11] holds. In the proof course of p552, let $\epsilon > 0$ be any given, we can choose (also in view of conditions (iii) and (iv)) an integer $N_1 > 0$ such that for all $n > N_1$ the following inequality $M_1 c_n < (\Phi(\epsilon)/4)\alpha_n$ holds, where $\alpha_n = b_n + c_n$. The inequality above $M_1 c_n < (\Phi(\epsilon)/4)\alpha_n$ implies that $c_n = o(\alpha_n)$. But conditions (iii) and (iv) of [11] can not assure that $c_n = o(\alpha_n)$. On the one hand, let $c_n = 1/n^2$, $n = 1, 2, 3, \ldots$; $\alpha_1 = 0$, $\alpha_2 = 1/2$, $\alpha_3 = 0$, $\alpha_4 = 1/4$, $\alpha_5 = 0$, $\alpha_6 = 1/6, \ldots$; then $\sum_{n=0}^{\infty} c_n < \infty$, but $c_n \neq o(\alpha_n)$. On the other hand, set $c_n = 1/n$, $\alpha_n = 2/\sqrt{n}$; then $c_n = o(\alpha_n)$, but $\sum_{n=0}^{\infty} c_n = \infty$.

The purpose of this paper is that we obtain the convergence result of the Mann iteration with errors, and we also prove the equivalence of convergence between the Ishikawa iteration with errors defined by (1.10) and the Mann iteration with errors defined by (1.11). We also show that the Ishikawa iteration with errors defined by (1.10) converges to the unique fixed point of T. Our results extend and improve the corresponding results of [3–12]. For this, in the sequel, we will need the following lemmas.

Lemma 1.7 (see [13]). *Let* E *be a real normed space. Then, for all* $x, y \in E$, *the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y). \tag{1.14}$$

Lemma 1.8 (see [14]). *Let* $\Phi : [0,+\infty) \to [0,+\infty)$ *be a strictly increasing continuous function with* $\Phi(0) = 0$ *and* $\{\theta_n\}, \{\sigma_n\}$, *and* $\{\lambda_n\}$ *nonnegative three sequences that satisfy the following inequality:*

$$\theta_{n+1}^2 \leq \theta_n^2 - 2\lambda_n \Phi(\theta_{n+1}) + \sigma_n, \quad n \geq N, \tag{1.15}$$

where $\lambda_n \in (0,1)$, $\lim_{n\to\infty}\lambda_n = 0$, *and* $\sum_{n=0}^{\infty}\lambda_n = \infty$, $\sigma_n = o(\lambda_n)$. *Then* $\theta_n \to 0$ *as* $n \to \infty$.

2. Main Results

Theorem 2.1. *Let K be a nonempty closed convex subset of a real normed linear space E. Suppose that $T : K \to K$ is a uniformly continuous generalized Φ-hemicontractive mapping with $F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in K defined iteratively from some $u_0 \in K$ by (1.11), where $\{\varepsilon_n\}$ is an arbitrary bounded sequence in K and $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $\lim_{n\to\infty}\beta_n = 0$,

(iv) $\gamma_n = o(\beta_n)$.

Then, the iteration sequence $\{u_n\}$ converges strongly to the unique fixed point of T.

Proof. Let $q \in F(T)$. The uniqueness of the fixed point of T comes from Definition 1.4.

First, we prove that there exists $u_0 \in K$ with $u_0 \neq Tu_0$ such that $t_0 = \|u_0 - Tu_0\| \cdot \|u_0 - q\| \in R(\Phi)$. In fact, if $u_0 = Tu_0$, then we are done. Otherwise, there exists the smallest positive integer $n_0 \in N$ such that $u_{n_0} \neq Tu_{n_0}$. We denote $u_{n_0} = u_0$, and then we obtain that $t_0 = \|u_0 - Tu_0\| \cdot \|u_0 - q\| \in R(\Phi)$. Indeed, if $R(\Phi) = [0, +\infty)$, then $t_0 \in R(\Phi)$. If $R(\Phi) = [0, A)$ with $0 < A < +\infty$, then, for $q \in K$, there exists a sequence $\{w_n\} \subset K$ such that $w_n \to q$ as $n \to \infty$ with $w_n \neq q$, and we also obtain that the sequence $\{w_n - Tw_n\}$ is bounded. So there exists $n_0 \in N$ such that $\|w_n - Tw_n\| \cdot \|w_n - q\| \in R(\Phi)$ for $n \geq n_0$, and then we redefine $u_0 = w_{n_0}$. Let $\omega_0 = \Phi^{-1}(\|(u_0 - q) - (Tu_0 - q)\| \cdot \|u_0 - q\|) > 0$.

Next for $n \geq 0$ we will prove $\|u_n - q\| \leq \omega_0$ by induction. Clearly, $\|u_0 - q\| \leq \omega_0$ holds. Suppose that $\|u_n - q\| \leq \omega_0$, for some n; then we want to prove $\|u_{n+1} - q\| \leq \omega_0$. If it is not the case, then $\|u_{n+1} - q\| > \omega_0$. Since T is a uniformly continuous mapping, setting $\epsilon_0 = \Phi(\omega_0)/12\omega_0$, there exists $\delta > 0$ such that $\|Tx - Ty\| < \Phi(\omega_0)/12\omega_0$ whenever $\|x - y\| < \delta$ and T is a bounded operator. Set $M = \sup\{\|Tx\| : \|x - q\| \leq \omega_0\} + \sup_n \|\varepsilon_n\|$. Since $\lim_{n\to\infty}\beta_n = 0$, $\gamma_n = o(\beta_n)$, without loss of generality, let

$$\beta_n, \frac{\gamma_n}{\beta_n} < \min\left\{\frac{\omega_0}{4(M + \|q\|)}, \frac{\delta}{2M + 2\omega_0 + 2\|q\|}, \frac{\Phi(\omega_0)}{6\omega_0}, \frac{\Phi(\omega_0)}{6(\omega_0 + M + \|q\|)^2}\right\}, \quad n \geq 0. \tag{2.1}$$

From (1.10), we have

$$\begin{aligned}
\|u_{n+1} - q\| &= \|\alpha_n(u_n - q) + \beta_n Tu_n + \gamma_n\varepsilon - (\beta_n + \gamma_n)q\| \\
&\leq \|u_n - q\| + (\beta_n\|Tu_n\| + \gamma_n\|\varepsilon_n\| + (\beta_n + \gamma_n)\|q\|) \\
&\leq \omega_0 + \beta_n(\|Tu_n\| + \|\varepsilon_n\| + 2\|q\|) \\
&\leq \omega_0 + \beta_n(2M + 2\|q\|) \leq \frac{3}{2}\omega_0, \\
\|u_{n+1} - u_n\| &= \|\beta_n Tu_n + \gamma_n\varepsilon_n - (\beta_n + \gamma_n)u_n\| \\
&\leq \beta_n\|Tu_n\| + \gamma_n\|\varepsilon_n\| + (\beta_n + \gamma_n)\|u_n - q\| + (\beta_n + \gamma_n)\|q\| \\
&\leq \beta_n(\|Tu_n\| + \|\varepsilon_n\| + 2\|u_n - q\| + 2\|q\|) \\
&\leq \beta_n(2M + 2\omega_0 + 2\|q\|).
\end{aligned} \tag{2.2}$$

Applying Lemma 1.7, the recursion (1.11), and the inequalities above, we obtain

$$\begin{aligned}
\|u_{n+1}-q\|^2 &= \|(u_n-q)+\beta_n Tu_n+\gamma_n\varepsilon_n-(\beta_n+\gamma_n)u_n\|^2 \\
&\le \|u_n-q\|^2+2\langle \beta_n Tu_n+\gamma_n\varepsilon_n-(\beta_n+\gamma_n)u_n, j(u_{n+1}-q)\rangle \\
&\le \|u_n-q\|^2+2\beta_n\langle Tu_{n+1}-Tq, j(u_{n+1}-q)\rangle \\
&\quad +2\beta_n\langle (Tu_n-Tu_{n+1})+(u_{n+1}-u_n)-(u_{n+1}-q), j(u_{n+1}-q)\rangle \\
&\quad +2\gamma_n(\|u_n-q\|+\|q\|+\|\varepsilon_n\|)\|u_{n+1}-q\| \\
&\le \|u_n-q\|^2-2\beta_n\Phi(\|u_{n+1}-q\|)+2\beta_n\|Tu_n-Tu_{n+1}\|\cdot\|u_{n+1}-q\| \\
&\quad +2\beta_n\|u_n-u_{n+1}\|\cdot\|u_{n+1}-q\|+2\gamma_n(\omega_0+\|q\|+M)\|u_{n+1}-q\| \\
&\le \omega_0^2-2\beta_n\Phi(\omega_0)+2\beta_n\frac{\Phi(\omega_0)}{12\omega_0}\frac{3\omega_0}{2}+2\beta_n\frac{\Phi(\omega_0)}{6\omega_0}\frac{3\omega_0}{2}+2\beta_n\frac{\Phi(\omega_0)}{6\omega_0}\frac{3\omega_0}{2} \\
&< \omega_0^2,
\end{aligned} \tag{2.3}$$

which is a contraction with the assumption $\|u_{n+1}-q\|>\omega_0$. Then, $\|u_{n+1}-q\|\le\omega_0$, that is, the sequence $\{u_n\}$ is bounded. It leads to

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0, \qquad \lim_{n\to\infty}\|Tu_{n+1}-Tu_n\|=0. \tag{2.4}$$

Again using Lemma 1.7, we have

$$\begin{aligned}
\|u_{n+1}-q\|^2 &= \|(u_n-q)+\beta_n Tu_n+\gamma_n\varepsilon_n-(\beta_n+\gamma_n)u_n\|^2 \\
&\le \|u_n-q\|^2+2\langle \beta_n Tu_n+\gamma_n\varepsilon_n-(\beta_n+\gamma_n)u_n, j(u_{n+1}-q)\rangle \\
&\le \|u_n-q\|^2+2\beta_n\langle Tu_{n+1}-Tq, j(x_{u+1}-q)\rangle \\
&\quad +2\beta_n\langle (Tu_n-Tu_{n+1})+(u_{n+1}-u_n)-(u_{n+1}-q), j(u_{n+1}-q)\rangle \\
&\quad +2\gamma_n(\|u_n-q\|+\|q\|+\|\varepsilon_n\|)\|u_{n+1}-q\| \\
&\le \|u_n-q\|^2-2\beta_n\Phi(\|u_{n+1}-q\|)+2\beta_n\|Tu_n-Tu_{n+1}\|\cdot\|u_{n+1}-q\| \\
&\quad +2\beta_n\|u_n-u_{n+1}\|\cdot\|u_{n+1}-q\|+2\gamma_n(\omega_0+\|q\|+M)\|u_{n+1}-q\| \\
&\le \|u_n-q\|^2-2\beta_n\Phi(\|u_{n+1}-q\|)+A_n,
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
A_n &= 2\beta_n\|Tu_n-Tu_{n+1}\|\cdot\|u_{n+1}-q\|+2\beta_n\|u_n-u_{n+1}\|\cdot\|u_{n+1}-q\| \\
&\quad +2\gamma_n(\omega_0+\|q\|+M)\|u_{n+1}-q\| \\
&= o(\beta_n).
\end{aligned} \tag{2.6}$$

Therefore, (2.5) becomes

$$\|u_{n+1}-q\|^2\le\|u_n-q\|^2-2\beta_n\Phi(\|u_{n+1}-q\|)+o(\beta_n). \tag{2.7}$$

From Lemma 1.8, we obtain $\lim_{n\to\infty}\|u_n-q\|=0$. □

Theorem 2.2. *Let K be a nonempty closed convex subset of a real normed linear space E and $T : K \to K$ a uniformly continuous generalized Φ-pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{u_n\}, \{x_n\}$ be two sequences in K defined iteratively from some $u_0 \neq x_0 \in K$ by* (1.11) *and* (1.10)*, where $\{\delta_n\}, \{\eta_n\}, \{\varepsilon_n\}$ are three arbitrary bounded sequences in K and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$, and $\{\gamma'_n\}$ are six sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$,

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\beta'_n = \lim_{n\to\infty}\gamma'_n = 0$,

(iv) $\gamma_n = o(\beta_n)$.

Then, the following two assertions are equivalent:

(1) *Iteration* (1.11) *converges strongly to the unique fixed point of T,*

(2) *Iteration* (1.10) *converges strongly to the unique fixed point of T.*

Proof. Let $q \in F(T)$. The uniqueness of q comes from Definition 1.4. If the Ishikawa iteration with errors converges to q, then setting $\beta'_n = \gamma'_n = 0$, for all $n \geq 0$ in (1.10), we can get the convergence of the Mann iteration with errors. Conversely, we only prove that (1)⇒(2), that is, if the Mann iteration with errors converges to q, we want to prove the convergence of the Ishikawa iteration with errors.

First, we will prove that there exists $x_0 \in K$ with $x_0 \neq Tx_0$ such that $t_0 = \|x_0 - Tx_0\| \cdot \|x_0 - q\| \in R(\Phi)$.

In fact, if $x_0 = Tx_0$, then we are done. Otherwise, there exists the smallest positive integer $N_0 \in N$ such that $x_{N_0} \neq Tx_{N_0}$. We denote $x_{N_0} = x_0$, and then we obtain that $t_0 = \|x_0 - Tx_0\| \cdot \|x_0 - q\| \in R(\Phi)$. Indeed if $R(\Phi) = [0, +\infty)$, then $t_0 \in R(\Phi)$. If $R(\Phi) = [0, A)$ with $0 < A < +\infty$, then for $q \in K$ there exists a sequence $\{w_n\} \subset K$ such that $w_n \to q$ as $n \to \infty$ with $w_n \neq q$, and we also obtain that the sequence $\{w_n - Tw_n\}$ is bounded. So there exists $n_0 \in N$ such that $\|w_n - Tw_n\| \cdot \|w_n - q\| \in R(\Phi)$, $n \geq n_0$, and then we redefine $x_0 = w_{n_0}$. Let $\mu_0 = \Phi^{-1}(\|(x_0 - q) - (Tx_0 - q)\| \cdot \|x_0 - q\|) > 0$.

Second, we will prove that the sequence $\{x_n - q\}$ is a bounded sequence.

Set

$$B_1 = \{\|x - q\| \leq \mu_0 : x \in K\}, \qquad B_2 = \{\|x - q\| \leq 2\mu_0 : x \in K\},$$
$$M_1 = \max\left\{\sup_{x\in B_2}\|Tx\|; \sup_{n\in N}\|\varepsilon_n\|; \sup_{n\in N}\|\delta_n\|; \sup_{n\in N}\|\eta_n\|; \sup_{n\in N}\|u_n\|; \sup_{n\in N}\|Tu_n\|\right\}. \tag{2.8}$$

Since T is uniformly continuous, for $\epsilon = \Phi(\mu_0)/5\mu_0$, there exists $\delta > 0$ such that $\|Tx - Ty\| < \epsilon$ whenever $\|x - y\| < \delta$. Now let $k = \min\{1, \mu_0/4(M_1 + \mu_0 + \|q\|), \delta/2(4M_1 + \mu_0 + \|q\|), \Phi(\mu_0)/20\mu_0(M_1 + \mu_0 + \|q\|)\}$. By the control conditions (iii) and (iv), without loss of generality, set $|\alpha_n - \alpha'_n|, \beta_n, \gamma_n, \beta'_n, \gamma'_n < k$, for all $n \geq 0$.

Observe that if $x_n \in B_1$, we obtain $y_n \in B_2$. Indeed

$$\begin{aligned}\|y_n - q\| &= \|(1 - \beta'_n - \gamma'_n)x_n + \beta'_n Tx_n + \gamma'_n\eta_n - q\| \\ &\leq \|x_n - q\| + \beta'_n\|Tx_n\| + (\beta'_n + \gamma'_n)\|x_n - q\| + \gamma'_n\|\eta_n\| + (\beta'_n + \gamma'_n)\|q\| \\ &\leq \mu_0 + k(2M_1 + 2\mu_0 + 2\|q\|) \\ &\leq 2\mu_0.\end{aligned} \tag{2.9}$$

Next, by induction, we prove $x_n \in B_1$. Clearly, from (2.1), we obtain $\|x_0 - q\| \le \mu_0$, that is, $x_0 \in B_1$. Set $\|x_n - q\| \le \mu_0$, for some n; then we will prove that $\|x_{n+1} - q\| \le \mu_0$. If it is not the case, we assume that $\|x_{n+1} - q\| > \mu_0$. From (1.10), we obtain the following inequalities:

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n x_n + \beta_n T y_n + \gamma_n \delta_n - q\| \\
&\le \|(1-\beta_n-\gamma_n)(x_n - q) + \beta_n T y_n + \gamma_n \delta_n + (\beta_n+\gamma_n) q\| \\
&\le \|x_n - q\| + (\beta_n+\gamma_n)\|x_n - q\| + \beta_n \|T y_n\| + \gamma_n \|\delta_n\| + (\beta_n + \gamma_n)\|q\| \\
&\le \mu_0 + k(2\mu_0 + 2M_1 + 2\|q\|) \le \frac{5}{4}\mu_0, \\
\|x_{n+1} - x_n\| &= \|\beta_n T y_n + \gamma_n \delta_n - (\beta_n+\gamma_n) x_n\| \\
&\le (\beta_n+\gamma_n)\|x_n - q\| + \beta_n \|T y_n\| + \gamma_n \|\delta_n\| + (\beta_n + \gamma_n)\|q\| \\
&\le k(2\mu_0 + 2M_1 + 2\|q\|) \le \frac{\Phi(\mu_0)}{10\mu_0}, \\
\|x_{n+1} - y_n\| &= \|(\alpha_n - \alpha'_n) x_n + \beta_n T y_n - \beta'_n T x_n + \gamma_n \delta_n - \gamma'_n \eta_n\| \\
&\le |\alpha_n - \alpha'_n| \cdot \|x_n - q\| + \beta_n \|T y_n\| + \beta'_n \|T x_n\| + \gamma_n \|\delta_n\| + \gamma'_n \|\eta_n\| + |\alpha_n - \alpha'_n| \cdot \|q\| \\
&\le k(\mu_0 + 4M_1 + \|q\|) \le \delta.
\end{aligned} \tag{2.10}$$

Since T is a uniformly continuous mapping, then

$$\|T x_{n+1} - T y_n\| \le \epsilon = \frac{\Phi(\mu_0)}{5\mu_0}. \tag{2.11}$$

Since $\gamma_n = o(\beta_n)$, we let

$$\gamma_n < \beta_n \cdot \frac{1}{8} \cdot \Phi(\mu_0) \cdot \left(\frac{4}{5\mu_0}\right)^2. \tag{2.12}$$

From (2.9)–(2.12), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1-\beta_n-\gamma_n) x_n + \beta_n T y_n + \gamma_n \delta_n - q\|^2 \\
&= \|(x_n - q) + \beta_n T y_n + \gamma_n \delta_n - (\beta_n+\gamma_n) x_n\|^2 \\
&\le \|x_n - q\|^2 + 2\langle \beta_n T y_n + \gamma_n \delta_n - (\beta_n + \gamma_n) x_n, j(x_{n+1} - q)\rangle \\
&\le \|x_n - q\|^2 + 2\beta_n \langle T x_{n+1} - Tq, j(x_{n+1} - q)\rangle + 2\beta_n \langle (T y_n - T x_{n+1}), j(x_{n+1} - q)\rangle \\
&\quad + 2(\beta_n+\gamma_n)\langle (x_{n+1} - x_n), j(x_{n+1} - q)\rangle + 2\gamma_n \|\delta_n\| \cdot \|x_{n+1} - q\| \\
&\quad - 2(\beta_n+\gamma_n)\langle (x_{n+1} - q), j(x_{n+1} - q)\rangle \\
&\le \|x_n - q\|^2 - 2\beta_n \Phi(\|x_{n+1} - q\|) + 2\beta_n \|T x_{n+1} - T y_n\| \cdot \|x_{n+1} - q\| \\
&\quad + 2(\beta_n+\gamma_n)\|x_{n+1} - x_n\| \cdot \|x_{n+1} - q\| + 2\gamma_n \|\delta_n\| \|x_{n+1} - q\| + 2\gamma_n \|x_{n+1} - q\|^2
\end{aligned}$$

$$
\begin{aligned}
&< \mu_0^2 - 2\beta_n\Phi(\mu_0) + 2\beta_n\frac{\Phi(\mu_0)}{5\mu_0}\cdot\frac{5\mu_0}{4} + 4\beta_n\frac{\Phi(\mu_0)}{10\mu_0}\cdot\frac{5\mu_0}{4} + 2\beta_n\frac{\Phi(\mu_0)}{5\mu_0}\cdot\frac{5\mu_0}{4} \\
&\quad + 2\beta_n\cdot\frac{1}{8}\cdot\Phi(\mu_0)\cdot\left(\frac{4}{5\mu_0}\right)^2\cdot\left(\frac{5\mu_0}{4}\right)^2 \\
&< \mu_0^2,
\end{aligned}
\tag{2.13}
$$

which is a contradiction with the assumption $\|x_{n+1} - q\| > \mu_0$. Hence, $x_n \in B_1$, that is, the sequence $\{x_n\}$ is bounded. From (2.9), the sequence $\{y_n\}$ is also bounded. So we obtain

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|u_{n+1} - u_n\| = \lim_{n\to\infty}\left\|x_{n+1} - y_n\right\| = 0. \tag{2.14}
$$

Since T is a uniformly continuous mapping, then

$$
\lim_{n\to\infty}\|Tu_{n+1} - Tu_n\| = \lim_{n\to\infty}\left\|Tx_{n+1} - Ty_n\right\| = 0. \tag{2.15}
$$

Using the recursion formula (1.10), we compute as follows:

$$
\begin{aligned}
\|x_{n+1} - u_{n+1}\|^2 &\le \|x_n - u_n\|^2 - 2\beta_n\Phi(\|x_{n+1} - u_{n+1}\|) + 2\beta_n\left\|Tx_{n+1} - Ty_n\right\|\cdot\|x_{n+1} - u_{n+1}\| \\
&\quad + 2\beta_n\|Tu_n - Tu_{n+1}\|\cdot\|x_{n+1} - u_{n+1}\| + 2(\beta_n + \gamma_n)\|u_n - u_{n+1}\|\cdot\|x_{n+1} - u_{n+1}\| \\
&\quad + 2(\beta_n + \gamma_n)\|x_{n+1} - x_n\|\cdot\|x_{n+1} - u_{n+1}\| + 2\gamma_n(\|\delta_n\| + \|\varepsilon_n\|)\|x_{n+1} - u_{n+1}\| \\
&= \|x_n - u_n\|^2 - 2\beta_n\Phi(\|x_{n+1} - u_{n+1}\|) + A_n,
\end{aligned}
\tag{2.16}
$$

where

$$
\begin{aligned}
A_n &= 2\beta_n\left\|Tx_{n+1} - Ty_n\right\|\cdot\|x_{n+1} - u_{n+1}\| + 2\beta_n\|Tu_n - Tu_{n+1}\|\cdot\|x_{n+1} - u_{n+1}\| \\
&\quad + 2(\beta_n + \gamma_n)\|u_n - u_{n+1}\|\cdot\|x_{n+1} - u_{n+1}\| + 2(\beta_n + \gamma_n)\|x_{n+1} - x_n\|\cdot\|x_{n+1} - u_{n+1}\| \\
&\quad + 2\gamma_n(\|\delta_n\| + \|\varepsilon_n\|)\|x_{n+1} - u_{n+1}\|.
\end{aligned}
\tag{2.17}
$$

Since the sequence $\{x_n - u_n\}$ is bounded, from (2.15), (2.16), and the control condition (iv), we get $A_n = o(\beta_n)$. Then, in (2.18), set $\theta_n = \|x_n - u_n\|$, $\lambda_n = \beta_n$, and $\sigma_n = A_n$. From Lemma 1.8, we obtain $\lim_{n\to\infty}\|x_{n+1} - u_{n+1}\| = 0$. Since $\lim_{n\to\infty}\|u_n - q\| = 0$, using inequality $0 \le \|x_n - q\| \le \|x_n - u_n\| + \|u_n - q\|$, then $\lim_{n\to\infty}\|x_n - q\| = 0$, and we complete the proof of Theorem 2.1. □

From Theorems 2.1 and 2.2, we obtain the following corollary.

Corollary 2.3. *Let K be a nonempty convex subset of a real normed linear space E. $T : K \to K$ is a uniformly continuous generalized Φ-hemicontractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in K defined iteratively from some $x_0 \in K$ by (1.10), where $\{\delta_n\}, \{\eta_n\}$ are arbitrary* *bounded sequences in K and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$, and $\{\gamma'_n\}$ are sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$,

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\beta'_n = \lim_{n\to\infty}\gamma'_n = 0$,

(iv) $\gamma_n = o(\beta_n)$.

Then, the iteration sequence $\{x_n\}$ defined by (1.10) converges strongly to the unique fixed point of T.

Corollary 2.4. *Let $T : E \to E$ be a uniformly continuous generalized Φ-accretive mapping with $N(T) \neq \emptyset$. For some $u_0, x_0 \in E$, the iteration sequences $\{u_n\}, \{x_n\}$ in E are defined as follows:*

$$u_{n+1} = \alpha_n u_n + \beta_n S u_n + \gamma_n \varepsilon_n, \quad \forall n \geq 0, \tag{2.18}$$

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n S y_n + \gamma_n \delta_n, \\ y_n &= \alpha'_n x_n + \beta'_n S x_n + \gamma'_n \eta_n, \quad \forall n \geq 0, \end{aligned} \tag{2.19}$$

where $Sx := x - Tx$ for all $x \in E$. $\{\delta_n\}, \{\eta_n\}$, and $\{\varepsilon_n\}$ are arbitrary bounded sequences in K and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$, and $\{\gamma'_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$,

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\beta'_n = \lim_{n\to\infty}\gamma'_n = 0$,

(iv) $\gamma_n = o(\beta_n)$.

Then, the following two assertions are equivalent:

(1) *$\{x_n\}$ converges strongly to the unique fixed point of S,*

(2) *$\{u_n\}$ converges strongly to the unique fixed point of S.*

Proof. Let $S = I - T$, and observe that T is a uniformly continuous generalized Φ-accretive mapping if and only if S is a uniformly continuous generalized Φ-pseudocontractive mapping. The result follows from Theorem 2.2. □

Remark 2.5. Our results improve and extend the corresponding results of [11] in the following sense.

(i) We point out the gaps of C. E. Chidume and C. O. Chidume [11] in their proof and provide a counterexample.

(ii) In the proof method, Theorem 2.1 differs from Theorem 2.3 in [11].

(iii) The control conditions $\sum_{n=0}^{\infty}(b_n + c_n)^2 < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$ in Theorem 2.3 in [11] are replaced by the condition $b_n = o(c_n)$. Under the new condition, we obtain the convergence theorem of the Mann iteration sequence with errors.

(iv) We also obtain the equivalence of convergence results between the Ishikawa and Mann iterations with errors.

(v) The Mann iteration sequence with errors is extended to the Ishikawa iteration sequence with errors.

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