

Research Article

Approximation of Mixed-Type Functional Equations in Menger PN-Spaces

M. Eshaghi Gordji,¹ H. Khodaei,¹ Y. W. Lee,² and G. H. Kim³

¹ Department of Mathematics, Semnan University, Semnan 35195-363, Iran

² Department of Computer Hacking, Information Security, Daejeon University,
Youngwoondong Donggu Daejeon 300-716, Republic of Korea

³ Department of Mathematics, Kangnam University, Yongin, Gyeonggi 446-702, Republic of Korea

Correspondence should be addressed to Y. W. Lee, ywlee@dju.kr

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Let X and Y be vector spaces. We show that a function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies $\Delta f(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in X$, if and only if there exist functions $C : X \times X \times X \rightarrow Y$, $B : X \times X \rightarrow Y$ and $A : X \rightarrow Y$ such that $f(x) = C(x, x, x) + B(x, x) + A(x)$ for all $x \in X$, where the function C is symmetric for each fixed one variable and is additive for fixed two variables, B is symmetric bi-additive, A is additive and $\Delta f(x_1, \dots, x_n) = \sum_{k=2}^n (\sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \dots \sum_{i_{n-k+1}=i_{n-k}+1}^n) f(\sum_{i=1, i \neq i_1, \dots, i_{n-k+1}}^n x_i - \sum_{r=1}^{n-k+1} x_{i_r}) + f(\sum_{i=1}^n x_i) - 2^{n-2} \sum_{i=2}^n (f(x_1 + x_i) + f(x_1 - x_i)) + 2^{n-1} (n-2) f(x_1)$ ($n \in \mathbb{N}$, $n \geq 3$) for all $x_1, \dots, x_n \in X$. Furthermore, we solve the stability problem for a given function f satisfying $\Delta f(x_1, \dots, x_n) = 0$, in the Menger probabilistic normed spaces.

1. Introduction and Preliminaries

Menger [1] introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has developed in many directions [2]. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. A probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis. Probabilistic normed spaces were introduced by Sherstnev in 1962 [3] by means of a definition that was closely modelled on the theory of (classical) normed spaces and used to study the problem of best approximation in statistics.

In the sequel, we will adopt the usual terminology, notation and conventions of the theory of probabilistic normed spaces as in [2, 4–6, 6, 7, 7–18].

Throughout this paper, let Δ^+ be the space of distribution functions, that is,

$$\begin{aligned} \Delta^+ := \{F : \mathbb{R} \cup \{-\infty, \infty\} \rightarrow [0, 1] : F \text{ is left-continuous,} \\ \text{nondecreasing on } \mathbb{R}, F(0) = 0, F(+\infty) = 1\}, \end{aligned} \quad (1.1)$$

and the subset $D^+ \subseteq \Delta^+$ is the set:

$$D^+ = \{F \in \Delta^+ : \Gamma F(+\infty) = 1\}, \quad (1.2)$$

where, $\Gamma f(x)$ denotes the left limit of the function f at the point x . The space Δ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases} \quad (1.3)$$

Definition 1.1 (see [2]). A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous triangular norm (briefly, a continuous t -norm) if T satisfies the following conditions:

- (1) T is commutative and associative;
- (2) T is continuous;
- (3) $T(a, 1) = a$ for all $a \in [0, 1]$;
- (4) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Two typical examples of continuous t -norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$. Recall (see [19, 20]) that if T is a t -norm and $\{x_n\}$ is a given sequence of numbers in $[0, 1]$, $T_{i=1}^n x_i$ is defined recurrently by

$$T_{i=1}^n x_i = \begin{cases} x_1, & \text{if } n = 1, \\ T(T_{i=1}^{n-1} x_i, x_n), & \text{if } n \geq 2. \end{cases} \quad (1.4)$$

$T_{i=n+1}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$.

Definition 1.2. A Menger probabilistic normed space (briefly, Menger PN-space) is a triple (X, Λ, T) where, X is a vector space, T is a continuous t -norm, and Λ is a mapping from X into D^+ such that the following conditions hold:

- (PN₁) $\Lambda_x(0) = 0$ for all $x \in X$;
- (PN₂) $\Lambda_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
- (PN₃) $\Lambda_{\alpha x}(t) = \Lambda_x(t/|\alpha|)$ for all $x \in X$, $\alpha \neq 0$ and all $t > 0$;
- (PN₄) $\Lambda_{x+y}(t+s) \geq T(\Lambda_x(t), \Lambda_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

Clearly, every Menger PN-space is probabilistic metric space having a metrizable uniformity on X if $\sup_{a < 1} T(a, a) = 1$.

Definition 1.3. Let (X, Λ, T) be a Menger PN-space.

- (i) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\Lambda_{x_n-x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.
- (ii) A sequence $\{x_n\}$ in X is called *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\Lambda_{x_n-x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.
- (iii) A Menger PN-space (X, Λ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X .

Theorem 1.4. *If (X, Λ, T) is a Menger PN-space and $\{x_n\}$ is a sequence such that $x_n \rightarrow x$, then $\lim_{n \rightarrow \infty} \Lambda_{x_n}(t) = \Lambda_x(t)$.*

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem concerning group homomorphisms was raised by Ulam [21] in 1940 and affirmatively solved by Hyers [22]. The result of Hyers was generalized by Aoki [23] for approximate additive function and by Rassias [24] for approximate linear functions by allowing the difference Cauchy equation $\|f(x + y) - f(x) - f(y)\|$ to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. Taking into consideration a lot of influence of Ulam, Hyers, and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias is called the Hyers-Ulam-Rassias stability. In 1994, a generalization of Rassias' theorem was obtained by Găvruta [25], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. The functional equation,

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2), \tag{1.5}$$

is related to symmetric biadditive function and is called a quadratic functional equation and every solution of the quadratic equation (1.5) is said to be a quadratic function. For more details about the results concerning such problems, the reader is referred to [26–28].

The functional equation,

$$f(2x_1 + x_2) + f(2x_1 - x_2) = 2f(x_1 + x_2) + 2f(x_1 - x_2) + 12f(x_1), \tag{1.6}$$

is called the cubic functional equation, since the function $f(x) = cx^3$ is its solution. Every solution of the cubic functional equation is said to be a cubic mapping. The stability results for the cubic functional equation were proved by Jun and Kim [29].

Eshaghi Gordji and Khodaei [30] have established the general solution and investigated the Hyers-Ulam-Rassias stability for a mixed type of cubic, quadratic, and additive functional equations, with $f(0) = 0$,

$$f(x_1 + kx_2) + f(x_1 - kx_2) = k^2 f(x_1 + x_2) + k^2 f(x_1 - x_2) + 2(1 - k^2) f(x_1) \tag{1.7}$$

in quasi-Banach spaces, where k is nonzero integer numbers with $k \neq \pm 1$. It is easy to see that the function $f(x) = ax + bx^2 + cx^3$ is a solution of the functional equation (1.7), see [31, 32].

The stability of different functional equations in probabilistic normed spaces, RN-spaces, and fuzzy normed spaces has been studied in [6, 7, 33–37].

Now, we introduce the new mixed type of cubic, quadratic, and additive functional equation in n -variables as follows:

$$\begin{aligned} & \sum_{k=2}^n \left(\sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f \left(\sum_{i=1, i \neq i_1, \dots, i_{n-k+1}}^n x_i - \sum_{r=1}^{n-k+1} x_{i_r} \right) + f \left(\sum_{i=1}^n x_i \right) \\ & = 2^{n-2} \sum_{i=2}^n (f(x_1 + x_i) + f(x_1 - x_i)) - 2^{n-1} (n-2) f(x_1), \end{aligned} \quad (1.8)$$

where $n \geq 3$ and $f(0) = 0$. As a special case, if $n = 3$ in (1.8), then (1.8) reduces to

$$\begin{aligned} & \sum_{i_1=2}^2 \sum_{i_2=i_1+1}^3 f \left(\sum_{i=1, i \neq i_1, i_2}^3 x_i - \sum_{r=1}^2 x_{i_r} \right) + \sum_{i_1=2}^3 f \left(\sum_{i=1, i \neq i_1}^3 x_i - x_{i_1} \right) + f \left(\sum_{i=1}^3 x_i \right) \\ & = 2 \sum_{i=2}^3 (f(x_1 + x_i) + f(x_1 - x_i)) - 2^2 f(x_1), \end{aligned} \quad (1.9)$$

that is,

$$\begin{aligned} & f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 + x_2 + x_3) \\ & = 2(f(x_1 + x_2) + f(x_1 - x_2) + f(x_1 + x_3) + f(x_1 - x_3)) - 4f(x_1). \end{aligned} \quad (1.10)$$

The main purpose of this paper is to prove the stability for (1.8), in Menger probabilistic normed spaces.

2. Results in Menger Probabilistic Normed Spaces

We start our work with a general solution for the mixed functional equation (1.8) and then investigate the stability of this equation in Menger PN-space.

Theorem 2.1. *Let X and Y be vector spaces. A function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies (1.8) for all $x_1, \dots, x_n \in X$ if and only if there exist functions $C : X \times X \times X \rightarrow Y$, $B : X \times X \rightarrow Y$ and $A : X \rightarrow Y$ such that $f(x) = C(x, x, x) + B(x, x) + A(x)$ for all $x \in X$, where the function C is symmetric for each fixed one variable and is additive for fixed two variables, B is symmetric biadditive and A is additive.*

Proof. If there exists a function C that is symmetric for each fixed one variable and is additive for fixed two variables, B is biadditive, and A is additive, then by a simple computation one can show that the functions $x \mapsto C(x, x, x)$, $x \mapsto B(x, x)$, and $x \mapsto A(x)$ satisfy the functional equation (1.8). Therefore, the function f satisfies (1.8).

Conversely, let f with $f(0) = 0$ satisfies (1.8). Hence, according to (1.8), we get

$$\begin{aligned}
 & \sum_{i_1=2}^2 \sum_{i_2=i_1+1}^3 \cdots \sum_{i_{n-1}=i_{n-2}+1}^n f\left(\sum_{i=1, i \neq i_1, \dots, i_{n-1}}^n x_i - \sum_{r=1}^{n-1} x_{i_r}\right) \\
 & + \sum_{i_1=2}^3 \sum_{i_2=i_1+1}^4 \cdots \sum_{i_{n-2}=i_{n-3}+1}^n f\left(\sum_{i=1, i \neq i_1, \dots, i_{n-2}}^n x_i - \sum_{r=1}^{n-2} x_{i_r}\right) \\
 & + \cdots + \sum_{i_1=2}^n f\left(\sum_{i=1, i \neq i_1}^n x_i - x_{i_1}\right) + f\left(\sum_{i=1}^n x_i\right) \\
 & = 2^{n-2} \sum_{i=2}^n (f(x_1 + x_i) + f(x_1 - x_i)) - 2^{n-1}(n-2)f(x_1)
 \end{aligned} \tag{2.1}$$

for all $x_1, \dots, x_n \in X$. Setting $x_i = 0$ ($i = 4, \dots, n$) in (2.1), we have

$$\begin{aligned}
 & f(x_1 - x_2 - x_3) + ((n-3)f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3)) \\
 & + \left(\binom{n-3}{2} f(x_1 - x_2 - x_3) + \binom{n-3}{n-4} f(x_1 - x_2 + x_3) + \binom{n-3}{n-4} f(x_1 + x_2 - x_3) \right. \\
 & \quad \left. + \binom{n-3}{n-3} f(x_1 + x_2 + x_3) \right) \\
 & + \left(\binom{n-3}{3} f(x_1 - x_2 - x_3) + \binom{n-3}{n-5} f(x_1 - x_2 + x_3) + \binom{n-3}{n-5} f(x_1 + x_2 - x_3) \right. \\
 & \quad \left. + \binom{n-3}{n-4} f(x_1 + x_2 + x_3) \right) + \cdots \\
 & + \left(\binom{n-3}{n-3} f(x_1 - x_2 - x_3) + \binom{n-3}{1} f(x_1 - x_2 + x_3) + \binom{n-3}{1} f(x_1 + x_2 - x_3) \right. \\
 & \quad \left. + \binom{n-3}{2} f(x_1 + x_2 + x_3) \right) \\
 & + (f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + (n-3)f(x_1 + x_2 + x_3)) + f(x_1 + x_2 + x_3) \\
 & = 2^{n-2} \sum_{i=2}^3 (f(x_1 + x_i) + f(x_1 - x_i)) - 2^{n-1} f(x_1),
 \end{aligned} \tag{2.2}$$

that is,

$$\begin{aligned} & \left(1 + \sum_{\ell=1}^{n-3} \binom{n-3}{\ell} \right) (f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 + x_2 + x_3)) \\ &= 2^{n-2} (f(x_1 + x_2) + f(x_1 - x_2) + f(x_1 + x_3) + f(x_1 - x_3)) - 2^{n-1} f(x_1) \end{aligned} \quad (2.3)$$

for all $x_1, x_2, x_3 \in X$. On the other hand, we have the relation:

$$1 + \sum_{\ell=1}^{n-3} \binom{n-3}{\ell} = \sum_{i=0}^{n-3} \binom{n-3}{i} = 2^{n-3} \quad (2.4)$$

for all $n \geq 3$. Hence, we obtain from (2.3) and (2.4) that

$$\begin{aligned} & f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 + x_2 + x_3) \\ &= 2(f(x_1 + x_2) + f(x_1 - x_2) + f(x_1 + x_3) + f(x_1 - x_3)) - 4f(x_1) \end{aligned} \quad (2.5)$$

for all $x_1, x_2, x_3 \in X$. Replacing x_3 by x_2 in (2.5), one gets

$$f(x_1 + 2x_2) + f(x_1 - 2x_2) = 4f(x_1 + x_2) + 4f(x_1 - x_2) - 6f(x_1) \quad (2.6)$$

for all $x_1, x_2 \in X$. Therefore, f satisfies (1.7) for $k = 2$. By Theorem 2.3 of [30], there exist an additive function $A : X \rightarrow Y$, symmetric biadditive function $B : X \times X \rightarrow Y$, and $C : X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x) + B(x, x) + A(x)$ for all $x \in X$, where the function C is symmetric for each fixed one variable and is additive for fixed two variables. \square

From now on, let X be a real linear space and let (Y, Λ, T) be a complete Menger PN-space. For convenience, we use the following abbreviation for a given function $f : X \rightarrow Y$:

$$\begin{aligned} \Delta f(x_1, \dots, x_n) &= \sum_{k=2}^n \left(\sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f \left(\sum_{i=1, i \neq i_1, \dots, i_{n-k+1}}^n x_i - \sum_{r=1}^{n-k+1} x_{i_r} \right) \\ &+ f \left(\sum_{i=1}^n x_i \right) - 2^{n-2} \sum_{i=2}^n (f(x_1 + x_i) + f(x_1 - x_i)) + 2^{n-1} (n-2) f(x_1) \end{aligned} \quad (2.7)$$

for all $x_1, \dots, x_n \in X$.

Theorem 2.2. Let $\xi : X^n \rightarrow D^+$ ($n \in \mathbb{N}$, $n \geq 3$ and $\xi(x_1, \dots, x_n)$ is denoted by ξ_{x_1, \dots, x_n}) be a function such that

$$\lim_{m \rightarrow \infty} \xi_{2^m x_1, \dots, 2^m x_n} (2^{2^m} t) = 1 \quad (2.8)$$

for all $x_1, \dots, x_n \in X, t > 0$ and

$$\lim_{m \rightarrow \infty} T_{\ell=1}^{\infty} \left(\xi_{0, \dots, 0, 2^{m+\ell-1}x, 2^{m+\ell-1}x} \left(2^{2m+\ell+n-2}t \right) \right) = 1 \quad (2.9)$$

for all $x \in X$ and $t > 0$. Suppose that an even function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality:

$$\Lambda_{\Delta f(x_1, \dots, x_n)}(t) \geq \xi_{x_1, \dots, x_n}(t) \quad (2.10)$$

for all $x_1, \dots, x_n \in X$ and $t > 0$. Then, there exists a unique quadratic function $Q : X \rightarrow Y$ such that

$$\Lambda_{f(x)-Q(x)}(t) \geq T_{\ell=1}^{\infty} \left(\xi_{0, \dots, 0, 2^{\ell-1}x, 2^{\ell-1}x} \left(2^{\ell+n-2}t \right) \right) \quad (2.11)$$

for all $x \in X$ and $t > 0$.

Proof. Setting $x_i = 0$ ($i = 1, \dots, n - 2$) and $x_{n-1} = x_n = x$ in (2.10) and then use $f(0) = 0$, we obtain that

$$\Lambda_{\left(1 + \sum_{\ell=1}^{n-3} \binom{n-3}{\ell}\right) (f(2x)+f(-2x)) - 2^{n-1} (f(x)+f(-x))}(t) \geq \xi_{0, \dots, 0, x, x}(t) \quad (2.12)$$

for all $x \in X$ and $t > 0$. By using evenness of f and the relation $1 + \sum_{\ell=1}^{n-3} \binom{n-3}{\ell} = \sum_{i=0}^{n-3} \binom{n-3}{\ell} = 2^{n-3}$, we get

$$\Lambda_{2^{n-2}f(2x) - 2^n f(x)}(t) \geq \xi_{0, \dots, 0, x, x}(t) \quad (2.13)$$

for all $x \in X$ and $t > 0$. So,

$$\Lambda_{f(2x)/2^2 - f(x)}(t) \geq \xi_{0, \dots, 0, x, x}(2^n t) \geq \xi_{0, \dots, 0, x, x}(2^{n-1} t) \quad (2.14)$$

for all $x \in X$ and $t > 0$, which implies that

$$\Lambda_{f(2^{\ell+1}x)/2^{2(\ell+1)} - f(2^{\ell}x)/2^{2\ell}}(t) \geq \xi_{0, \dots, 0, 2^{\ell}x, 2^{\ell}x}(2^{n+2\ell} t) \quad (2.15)$$

for all $x \in X$, $t > 0$ and $\ell \in \mathbb{N}$. It follows from (2.15) and (PN₄) that

$$\begin{aligned}
& \Lambda_{f(2^2x)/2^4-f(x)}(t) \\
& \geq T\left(\Lambda_{f(2^2x)/2^4-f(2x)/2^2}\left(\frac{t}{2}\right), \Lambda_{f(2x)/2^2-f(x)}\left(\frac{t}{2}\right)\right) \geq T\left(\xi_{0,\dots,0,2x,2x}(2^{n+1}t), \xi_{0,\dots,0,x,x}(2^{n-1}t)\right) \\
& \geq T\left(\xi_{0,\dots,0,2x,2x}(2^n t), \xi_{0,\dots,0,x,x}(2^{n-1}t)\right), \\
& \Lambda_{f(2^3x)/2^6-f(x)}(t) \\
& \geq T\left(\Lambda_{f(2^3x)/2^6-f(2x)/2^2}\left(\frac{t}{2}\right), \Lambda_{f(2x)/2^2-f(x)}\left(\frac{t}{2}\right)\right) \\
& \geq T\left(T\left(\Lambda_{f(2^3x)/2^6-f(2^2x)/2^4}\left(\frac{t}{4}\right), \Lambda_{f(2^2x)/2^4-f(2x)/2^2}\left(\frac{t}{4}\right)\right), \Lambda_{f(2x)/2^2-f(x)}\left(\frac{t}{2}\right)\right) \\
& \geq T\left(T\left(\xi_{0,\dots,0,2^2x,2^2x}(2^{n+2}t), \xi_{0,\dots,0,2x,2x}(2^n t)\right), \xi_{0,\dots,0,x,x}(2^{n-1}t)\right) \\
& \geq T\left(T\left(\xi_{0,\dots,0,2^2x,2^2x}(2^{n+1}t), \xi_{0,\dots,0,2x,2x}(2^n t)\right), \xi_{0,\dots,0,x,x}(2^{n-1}t)\right) \\
& = T\left(\xi_{0,\dots,0,x,x}(2^{n-1}t), T\left(\xi_{0,\dots,0,2x,2x}(2^n t), \xi_{0,\dots,0,2^2x,2^2x}(2^{n+1}t)\right)\right) \\
& = T\left(T\left(\xi_{0,\dots,0,x,x}(2^{n-1}t), \xi_{0,\dots,0,2x,2x}(2^n t)\right), \xi_{0,\dots,0,2^2x,2^2x}(2^{n+1}t)\right)
\end{aligned} \tag{2.16}$$

for all $x \in X$ and $t > 0$. Thus,

$$\Lambda_{f(2^m x)/2^{2m}-f(x)}(t) \geq T_{\ell=1}^m \left(\xi_{0,\dots,0,2^{\ell-1}x,2^{\ell-1}x}(2^{\ell+n-2}t) \right) \tag{2.17}$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{f(2^m x)/2^{2m}\}$, we replace x with $2^{m'}x$ in (2.17) to find that

$$\Lambda_{f(2^{m+m'}x)/2^{2(m+m')}-f(2^{m'}x)/2^{2m'}}(t) \geq T_{\ell=1}^m \left(\xi_{0,\dots,0,2^{m'+\ell-1}x,2^{m'+\ell-1}x}(2^{2m'+\ell+n-2}t) \right) \tag{2.18}$$

for all $x \in x$ and $t > 0$. Since the right-hand side of the inequality tends to 1 as m' and m tend to infinity, the sequence $\{f(2^m x)/2^{2m}\}$ is a Cauchy sequence. Therefore, one can define the function $Q : X \rightarrow Y$ by $Q(x) := \lim_{m \rightarrow \infty} (1/2^{2m})f(2^m x)$ for all $x \in X$. Now, if we replace x_1, \dots, x_n with $2^m x_1, \dots, 2^m x_n$ in (2.10), respectively, it follows that

$$\Lambda_{\Delta f(2^m x_1, \dots, 2^m x_n)/2^{2m}}(t) \geq \xi_{2^m x_1, \dots, 2^m x_n}(2^{2m}t) \tag{2.19}$$

for all $x_1, \dots, x_n \in x$ and $t > 0$. By letting $m \rightarrow \infty$ in (2.19), we find that $\Lambda_{\Delta Q(x_1, \dots, x_n)}(t) = 1$ for all $t > 0$, which implies that $\Delta Q(x_1, \dots, x_n) = 0$ thus Q satisfies (1.8). Hence, by Theorem 2.1 (see [30, Lemma 2.1]), the function $Q : X \rightarrow Y$ is quadratic.

To prove (2.11), take the limit as $m \rightarrow \infty$ in (2.17).

Finally, to prove the uniqueness of the quadratic function Q subject to (2.11), let us assume that there exists a quadratic function Q' which satisfies (2.11). Since $Q(2^m x) = 2^{2m}Q(x)$ and $Q'(2^m x) = 2^{2m}Q'(x)$ for all $x \in X$ and $m \in \mathbb{N}$, from (2.11), it follows that

$$\begin{aligned} \Lambda_{Q(x)-Q'(x)}(t) &= \Lambda_{Q(2^m x)-Q'(2^m x)}(2^{2m}t) \\ &\geq T\left(\Lambda_{Q(2^m x)-f(2^m x)}(2^{2m-1}t), \Lambda_{f(2^m x)-Q'(2^m x)}(2^{2m-1}t)\right) \\ &\geq T\left(T_{\ell=1}^\infty\left(\xi_{0,\dots,0,2^{m+\ell-1}x,2^{m+\ell-1}x}(2^{2m+\ell+n-2}t)\right), T_{\ell=1}^\infty\left(\xi_{0,\dots,0,2^{m+\ell-1}x,2^{m+\ell-1}x}(2^{2m+\ell+n-2}t)\right)\right) \end{aligned} \quad (2.20)$$

for all $x \in X$ and $t > 0$. By letting $m \rightarrow \infty$ in (2.20), we find that $Q = Q'$. \square

Theorem 2.3. Let $\xi : X^n \rightarrow D^+$ be a function such that

$$\lim_{m \rightarrow \infty} T\left(\xi_{2^m x_1, \dots, 2^m x_n}(2^m t), \xi_{2^m x_1, \dots, 2^m x_n}(2^{m-4} t)\right) = 1 \quad (2.21)$$

for all $x_1, \dots, x_n \in X, t > 0$ and

$$\lim_{m \rightarrow \infty} T_{\ell=1}^\infty\left(T\left(\xi_{2^{m+\ell}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}(2^{m+n-5}t), \xi_{2^{m+\ell-1}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}(2^{m+n-7}t)\right)\right) = 1 \quad (2.22)$$

for all $x \in X$ and $t > 0$. Suppose that an odd function $f : X \rightarrow Y$ satisfies (2.10) for all $x_1, \dots, x_n \in X$ and $t > 0$. Then, there exists a unique additive function $A : X \rightarrow Y$ such that

$$\Lambda_{f(2x)-8f(x)-A(x)}(t) \geq T_{\ell=1}^\infty\left(T\left(\xi_{2^\ell x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}(2^{n-4}t), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}(2^{n-6}t)\right)\right) \quad (2.23)$$

for all $x \in X$ and $t > 0$.

Proof. Setting $x_1 = x_2 = x_3 = x$ and $x_i = 0$ ($i = 4, \dots, n$) in (2.10), we obtain

$$\Lambda_{\sum_{\ell=0}^{n-3} \binom{n-3}{\ell} (f(3x)+2f(x)+f(-x))-2^{n-1}f(2x)+2^{n-1}f(x)}(t) \geq \xi_{x,x,x,0,\dots,0}(t) \quad (2.24)$$

for all $x \in X$ and $t > 0$. By using oddness of f and the relation $\sum_{i=0}^{n-3} \binom{n-3}{i} = 2^{n-3}$, we lead to

$$\Lambda_{f(3x)-4f(2x)+5f(x)}(t) \geq \xi_{x,x,x,0,\dots,0}(2^{n-3}t) \quad (2.25)$$

for all $x \in X$ and $t > 0$. Putting $x_1 = 2x, x_2 = x_3 = x$ and $x_i = 0$ ($i = 4, \dots, n$) in (2.10), we have

$$\Lambda_{\sum_{\ell=0}^{n-3} \binom{n-3}{\ell} (f(4x)+2f(2x))-2^{n-2}(2f(3x)+2f(x))+2^{n-1}f(2x)}(t) \geq \xi_{2x,x,x,0,\dots,0}(t) \quad (2.26)$$

for all $x \in X$ and $t > 0$. So

$$\Lambda_{f(4x)-4f(3x)+6f(2x)-4f(x)}(t) \geq \xi_{2x,x,x,0,\dots,0}(2^{n-3}t) \quad (2.27)$$

for all $x \in X$ and $t > 0$. It follows from (2.25), (2.27), and (PN₄) that

$$\Lambda_{f(4x)-10f(2x)+16f(x)}(t) \geq T\left(\xi_{2x,x,x,0,\dots,0}(2^{n-4}t), \xi_{x,x,x,0,\dots,0}(2^{n-6}t)\right) \quad (2.28)$$

for all $x \in X$ and $t > 0$. Let $g : X \rightarrow Y$ be a function defined by $g(x) := f(2x) - 8f(x)$ for all $x \in X$. From (2.28), we conclude that

$$\begin{aligned} \Lambda_{g(2x)/2-g(x)}(t) &\geq T\left(\xi_{2x,x,x,0,\dots,0}(2^{n-3}t), \xi_{x,x,x,0,\dots,0}(2^{n-5}t)\right) \\ &\geq T\left(\xi_{2x,x,x,0,\dots,0}(2^{n-4}t), \xi_{x,x,x,0,\dots,0}(2^{n-6}t)\right) \end{aligned} \quad (2.29)$$

for all $x \in X$ and $t > 0$, which implies that

$$\Lambda_{g(2^{\ell+1}x)/2^{\ell+1}-g(2^\ell x)/2^\ell}(t) \geq T\left(\xi_{2^{\ell+1}x,2^\ell x,2^\ell x,0,\dots,0}(2^{\ell+n-3}t), \xi_{2^\ell x,2^\ell x,2^\ell x,0,\dots,0}(2^{\ell+n-5}t)\right) \quad (2.30)$$

for all $x \in X$, $t > 0$ and $\ell \in \mathbb{N}$. It follows from (2.30) and (PN₄) that

$$\begin{aligned} \Lambda_{g(2^2x)/2^2-g(x)}(t) &\geq T\left(\Lambda_{g(2^2x)/2^2-g(2x)/2}\left(\frac{t}{2}\right), \Lambda_{g(2x)/2-g(x)}\left(\frac{t}{2}\right)\right) \\ &\geq T\left(T\left(\xi_{2^2x,2x,2x,0,\dots,0}(2^{n-3}t), \xi_{2x,2x,2x,0,\dots,0}(2^{n-5}t)\right), \right. \\ &\quad \left. T\left(\xi_{2x,x,x,0,\dots,0}(2^{n-4}t), \xi_{x,x,x,0,\dots,0}(2^{n-6}t)\right)\right) \\ &\geq T\left(T\left(\xi_{2^2x,2x,2x,0,\dots,0}(2^{n-4}t), \xi_{2x,2x,2x,0,\dots,0}(2^{n-6}t)\right), \right. \\ &\quad \left. T\left(\xi_{2x,x,x,0,\dots,0}(2^{n-4}t), \xi_{x,x,x,0,\dots,0}(2^{n-6}t)\right)\right), \\ \Lambda_{g(2^3x)/2^3-g(x)}(t) &\geq T\left(\Lambda_{g(2^3x)/2^3-g(2x)/2}\left(\frac{t}{2}\right), \Lambda_{g(2x)/2-g(x)}\left(\frac{t}{2}\right)\right) \\ &\geq T\left(T\left(\Lambda_{g(2^3x)/2^3-g(2^2x)/2^2}\left(\frac{t}{4}\right), \Lambda_{g(2^2x)/2^2-g(2x)/2}\left(\frac{t}{4}\right)\right), \Lambda_{g(2x)/2-g(x)}\left(\frac{t}{2}\right)\right) \\ &\geq T\left(T\left(T\left(\xi_{2^3x,2^2x,2^2x,0,\dots,0}(2^{n-3}t), \xi_{2^2x,2^2x,2^2x,0,\dots,0}(2^{n-5}t)\right), \right. \right. \end{aligned}$$

$$\begin{aligned}
 & T\left(\xi_{2^2x,2x,2x,0,\dots,0}\left(2^{n-4}t\right), \xi_{2x,2x,2x,0,\dots,0}\left(2^{n-6}t\right)\right), \\
 & T\left(\xi_{2x,x,x,0,\dots,0}\left(2^{n-4}t\right), \xi_{x,x,x,0,\dots,0}\left(2^{n-6}t\right)\right) \\
 \geq & T\left(T\left(T\left(\xi_{2^3x,2^2x,2^2x,0,\dots,0}\left(2^{n-4}t\right), \xi_{2^2x,2^2x,2^2x,0,\dots,0}\left(2^{n-6}t\right)\right)\right), \right. \\
 & \left. T\left(\xi_{2^2x,2x,2x,0,\dots,0}\left(2^{n-4}t\right), \xi_{2x,2x,2x,0,\dots,0}\left(2^{n-6}t\right)\right)\right), \\
 & T\left(\xi_{2x,x,x,0,\dots,0}\left(2^{n-4}t\right), \xi_{x,x,x,0,\dots,0}\left(2^{n-6}t\right)\right) \\
 = & T\left(T\left(T\left(\xi_{2x,x,x,0,\dots,0}\left(2^{n-4}t\right), \xi_{x,x,x,0,\dots,0}\left(2^{n-6}t\right)\right)\right), \right. \\
 & \left. T\left(\xi_{2^2x,2x,2x,0,\dots,0}\left(2^{n-4}t\right), \xi_{2x,2x,2x,0,\dots,0}\left(2^{n-6}t\right)\right)\right), \\
 & T\left(\xi_{2^3x,2^2x,2^2x,0,\dots,0}\left(2^{n-4}t\right), \xi_{2^2x,2^2x,2^2x,0,\dots,0}\left(2^{n-6}t\right)\right)
 \end{aligned} \tag{2.31}$$

for all $x \in X$ and $t > 0$. Thus,

$$\Lambda_{g(2^m x)/2^m - g(x)}(t) \geq T_{\ell=1}^m \left(T\left(\xi_{2^\ell x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}\left(2^{n-4}t\right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}\left(2^{n-6}t\right)\right) \right) \tag{2.32}$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{g(2^m x)/2^m\}$, we replace x with $2^{m'} x$ in (2.32) to find that

$$\begin{aligned}
 & \Lambda_{g(2^{m+m'} x)/2^{m+m'} - g(2^{m'} x)/2^{m'}}(t) \\
 & \geq T_{\ell=1}^m \left(T\left(\xi_{2^{m'+\ell}x, 2^{m'+\ell-1}x, 2^{m'+\ell-1}x, 0, \dots, 0}\left(2^{m'+n-4}t\right), \xi_{2^{m'+\ell-1}x, 2^{m'+\ell-1}x, 2^{m'+\ell-1}x, 0, \dots, 0}\left(2^{m'+n-6}t\right)\right) \right)
 \end{aligned} \tag{2.33}$$

for all $x \in X$ and $t > 0$. Since the right-hand side of the inequality tends to 1 as m' and m tend to infinity, the sequence $\{g(2^m x)/2^m\}$ is a Cauchy sequence. Therefore, one can define the function $A : X \rightarrow Y$ by $A(x) := \lim_{m \rightarrow \infty} (1/2^m)g(2^m x)$ for all $x \in X$. Now, if we replace x_1, \dots, x_n with $2^m x_1, \dots, 2^m x_n$ in (2.10), respectively, it follows that

$$\begin{aligned}
 \Lambda_{\Delta g(2^m x_1, \dots, 2^m x_n)/2^m}(t) &= \Lambda_{\Delta f(2^{m+1}x_1, \dots, 2^{m+1}x_n)/2^{m+1} - 8(\Delta f(2^m x_1, \dots, 2^m x_n)/2^m)}(t) \\
 &\geq T\left(\Lambda_{\Delta f(2^{m+1}x_1, \dots, 2^{m+1}x_n)/2^{m+1}}\left(\frac{t}{2}\right), \Lambda_{\Delta f(2^m x_1, \dots, 2^m x_n)/2^{m-3}}\left(\frac{t}{2}\right)\right) \\
 &\geq T\left(\xi_{2^{m+1}x_1, \dots, 2^{m+1}x_n}\left(2^{m-1}t\right), \xi_{2^m x_1, \dots, 2^m x_n}\left(2^{m-4}t\right)\right)
 \end{aligned} \tag{2.34}$$

for all $x_1, \dots, x_n \in X$ and $t > 0$. By letting $m \rightarrow \infty$ in (2.34), we find that $\Lambda_{\Delta A(x_1, \dots, x_n)}(t) = 1$ for all $t > 0$, which implies $\Delta A(x_1, \dots, x_n) = 0$, thus A satisfies (1.8). Hence, by Theorem 2.1 (see [30, Lemma 2.2]) the function $A : X \rightarrow Y$ is additive.

To prove (2.23), take the limit as $m \rightarrow \infty$ in (2.32).

Finally, to prove the uniqueness of the additive function A subject to (2.23), let us assume that there exists a additive function A' which satisfies (2.23). Since $A(2^m x) = 2^m A(x)$ and $A'(2^m x) = 2^m A'(x)$ for all $x \in X$ and $m \in \mathbb{N}$, from (2.23), it follows that

$$\begin{aligned} & \Lambda_{A(x)-A'(x)}(t) \\ &= \Lambda_{A(2^m x)-A'(2^m x)}(2^m t) \geq T\left(\Lambda_{A(2^m x)-g(2^m x)}(2^{m-1}t), \Lambda_{g(2^m x)-A'(2^m x)}(2^{m-1}t)\right) \\ &\geq T\left(T_{\ell=1}^{\infty}\left(T\left(\xi_{2^{m+\ell}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(2^{m+n-5}t\right), \xi_{2^{m+\ell-1}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(2^{m+n-7}t\right)\right)\right), \right. \\ &\quad \left. T_{\ell=1}^{\infty}\left(T\left(\xi_{2^{m+\ell}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(2^{m+n-5}t\right), \xi_{2^{m+\ell-1}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(2^{m+n-7}t\right)\right)\right)\right) \end{aligned} \quad (2.35)$$

for all $x \in X$ and $t > 0$. By letting $m \rightarrow \infty$ in (2.35), we find that $A = A'$. \square

Theorem 2.4. Let $\xi : X^n \rightarrow D^+$ be a function such that

$$\lim_{m \rightarrow \infty} T\left(\xi_{2^m x_1, \dots, 2^m x_n}\left(2^{3m}t\right), \xi_{2^m x_1, \dots, 2^m x_n}\left(2^{3m-2}t\right)\right) = 1 \quad (2.36)$$

for all $x_1, \dots, x_n \in X$, $t > 0$ and

$$\lim_{m \rightarrow \infty} T_{\ell=1}^{\infty}\left(T\left(\xi_{2^{m+\ell}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(2^{3m+2\ell+n-5}t\right), \xi_{2^{m+\ell-1}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(2^{3m+2\ell+n-7}t\right)\right)\right) = 1 \quad (2.37)$$

for all $x \in X$ and $t > 0$. Suppose that an odd function $f : X \rightarrow Y$ satisfies (2.10) for all $x_1, \dots, x_n \in X$ and $t > 0$. Then, there exists a unique cubic function $C : X \rightarrow Y$ such that

$$\Lambda_{f(2x)-2f(x)-C(x)}(t) \geq T_{\ell=1}^{\infty}\left(T\left(\xi_{2^\ell x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}\left(2^{2\ell+n-4}t\right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}\left(2^{2\ell+n-6}t\right)\right)\right) \quad (2.38)$$

for all $x \in X$ and $t > 0$.

Proof. Similar to proof of Theorem 2.3, we obtain (2.28) for all $x \in X$ and $t > 0$. Let $h : X \rightarrow Y$ be a function defined by $h(x) := f(2x) - 2f(x)$ for all $x \in X$. Therefore, (2.28) implies that

$$\begin{aligned} \Lambda_{h(2x)/2^3-h(x)}(t) &\geq T\left(\xi_{2x, x, x, 0, \dots, 0}\left(2^{n-1}t\right), \xi_{x, x, x, 0, \dots, 0}\left(2^{n-3}t\right)\right) \\ &\geq T\left(\xi_{2x, x, x, 0, \dots, 0}\left(2^{n-2}t\right), \xi_{x, x, x, 0, \dots, 0}\left(2^{n-4}t\right)\right) \end{aligned} \quad (2.39)$$

for all $x \in X$ and $t > 0$, which implies that

$$\Lambda_{h(2^{\ell+1}x)/2^{3(\ell+1)}-h(2^\ell x)/2^{3\ell}}(t) \geq T\left(\xi_{2^{\ell+1}x, 2^\ell x, 2^\ell x, 0, \dots, 0}\left(2^{3\ell+n-1}t\right), \xi_{2^\ell x, 2^\ell x, 2^\ell x, 0, \dots, 0}\left(2^{3\ell+n-3}t\right)\right) \quad (2.40)$$

for all $x \in X$, $t > 0$, and $\ell \in \mathbb{N}$. It follows from (2.40) and (PN₄) that

$$\Lambda_{h(2^m x)/2^{3m}-h(x)}(t) \geq T_{\ell=1}^m \left(T \left(\xi_{2^\ell x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(2^{2\ell+n-4}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(2^{2\ell+n-6}t \right) \right) \right) \quad (2.41)$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{h(2^m x)/2^{3m}\}$, we replace x with $2^{m'} x$ in (2.41) to find that

$$\begin{aligned} &\Lambda_{h(2^{m+m'} x)/2^{3(m+m')} - h(2^{m'} x)/2^{3m'}}(t) \\ &\geq T_{\ell=1}^m \left(T \left(\xi_{2^{m'+\ell}x, 2^{m'+\ell-1}x, 2^{m'+\ell-1}x, 0, \dots, 0} \left(2^{3m'+2\ell+n-4}t \right), \xi_{2^{m'+\ell-1}x, 2^{m'+\ell-1}x, 2^{3m'+\ell-1}x, 0, \dots, 0} \left(2^{3m'+2\ell+n-6}t \right) \right) \right) \end{aligned} \quad (2.42)$$

for all $x \in x$ and $t > 0$. Since the right-hand side of the inequality tends to 1 as m' and m tend to infinity, the sequence $\{h(2^m x)/2^{3m}\}$ is a Cauchy sequence. Therefore, one can define the function $C : X \rightarrow Y$ by $C(x) := \lim_{m \rightarrow \infty} (1/2^{3m})h(2^m x)$ for all $x \in X$. Now, if we replace x_1, \dots, x_n with $2^m x_1, \dots, 2^m x_n$ in (2.10), respectively, it follows that

$$\Lambda_{\Delta h(2^m x_1, \dots, 2^m x_n)/2^{3m}}(t) \geq T \left(\xi_{2^{m+1}x_1, \dots, 2^{m+1}x_n} \left(2^{3m-1}t \right), \xi_{2^m x_1, \dots, 2^m x_n} \left(2^{3m-2}t \right) \right) \quad (2.43)$$

for all $x_1, \dots, x_n \in x$ and $t > 0$. By letting $m \rightarrow \infty$ in (2.43), we find that $\Lambda_{\Delta C(x_1, \dots, x_n)}(t) = 1$ for all $t > 0$, which implies $\Delta C(x_1, \dots, x_n) = 0$, thus C satisfies (1.8). Hence, by Theorem 2.1 (see [30, Lemma 2.2]), the function $C : X \rightarrow Y$ is cubic. The rest of the proof is similar to the proof of Theorem 2.3. \square

Theorem 2.5. *Let $\xi : X^n \rightarrow D^+$ be a function satisfies (2.21) for all $x_1, \dots, x_n \in X$, $t > 0$ and (2.22) for all $x \in X$ and $t > 0$. Suppose that an odd function $f : X \rightarrow Y$ satisfies (2.10) for all $x_1, \dots, x_n \in X$ and $t > 0$. Then, there exists a unique additive function $A : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ such that*

$$\begin{aligned} &\Lambda_{f(x)-A(x)-C(x)}(t) \\ &\geq T \left(T_{\ell=1}^\infty \left(T \left(\xi_{2^\ell x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{n-4}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{n-6}t \right) \right) \right), \right. \\ &\quad \left. T_{\ell=1}^\infty \left(T \left(\xi_{2^\ell x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{2\ell+n-4}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{2\ell+n-6}t \right) \right) \right) \right) \end{aligned} \quad (2.44)$$

for all $x \in X$ and $t > 0$.

Proof. By Theorems 2.3 and 2.4, there exist an additive function $A_0 : X \rightarrow Y$ and a cubic function $C_0 : X \rightarrow Y$ such that

$$\begin{aligned} &\Lambda_{f(2x)-8f(x)-A_0(x)}(t) \geq T_{\ell=1}^\infty \left(T \left(\xi_{2^\ell x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(2^{n-4}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(2^{n-6}t \right) \right) \right), \\ &\Lambda_{f(2x)-2f(x)-C_0(x)}(t) \geq T_{\ell=1}^\infty \left(T \left(\xi_{2^\ell x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(2^{2\ell+n-4}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(2^{2\ell+n-6}t \right) \right) \right) \end{aligned} \quad (2.45)$$

for all $x \in X$ and $t > 0$. It follows from (2.45) that

$$\Lambda_{f(x)+(1/6)A_0(x)-(1/6)C_0(x)}(t) \geq T(\Lambda_{f(2x)-8f(x)-A_0(x)}(3t), \Lambda_{f(2x)-2f(x)-C_0(x)}(3t)) \quad (2.46)$$

for all $x \in X$ and $t > 0$. Thus, we obtain (2.44) by letting $A(x) = -(1/6)A_0(x)$ and $C(x) = (1/6)C_0(x)$ for all $x \in X$.

To prove the uniqueness property of A and C , let $A', C' : X \rightarrow Y$ be another additive and cubic functions satisfying (2.44). Let $\bar{A} = A - A'$ and $\bar{C} = C - C'$. So,

$$\begin{aligned} & \Lambda_{\bar{A}(x)+\bar{C}(x)}(t) \\ & \geq T\left(\Lambda_{f(x)-A(x)-C(x)}\left(\frac{t}{2}\right), \Lambda_{f(x)-A'(x)-C'(x)}\left(\frac{t}{2}\right)\right) \\ & \geq T\left(T\left(T_{\ell=1}^{\infty}\left(T\left(\xi_{2^{\ell}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{n-5}t\right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{n-7}t\right)\right)\right), \right. \\ & \quad \left. T_{\ell=1}^{\infty}\left(T\left(\xi_{2^{\ell}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{2\ell+n-5}t\right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{2\ell+n-7}t\right)\right)\right)\right), \\ & T\left(T_{\ell=1}^{\infty}\left(T\left(\xi_{2^{\ell}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{n-5}t\right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{n-7}t\right)\right)\right), \right. \\ & \quad \left. T_{\ell=1}^{\infty}\left(T\left(\xi_{2^{\ell}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{2\ell+n-5}t\right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{2\ell+n-7}t\right)\right)\right)\right) \end{aligned} \quad (2.47)$$

for all $x \in X$ and $t > 0$, then (2.47) implies that

$$\begin{aligned} & \Lambda_{\bar{A}(2^m x)/2^{3m}+\bar{C}(2^m x)/2^{3m}}(t) \\ & \geq T\left(T\left(T_{\ell=1}^{\infty}\left(T\left(\xi_{2^{m+\ell}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{3m+n-5}t\right), \xi_{2^{m+\ell-1}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{3m+n-7}t\right)\right)\right), \right. \\ & \quad \left. T_{\ell=1}^{\infty}\left(T\left(\xi_{2^{m+\ell}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{3m+2\ell+n-5}t\right), \right. \right. \\ & \quad \left. \left. \xi_{2^{m+\ell-1}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{3m+2\ell+n-7}t\right)\right)\right)\right), \\ & T\left(T_{\ell=1}^{\infty}\left(T\left(\xi_{2^{m+\ell}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{3m+n-5}t\right), \xi_{2^{m+\ell-1}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{3m+n-7}t\right)\right)\right), \right. \\ & \quad \left. T_{\ell=1}^{\infty}\left(T\left(\xi_{2^{m+\ell}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{3m+2\ell+n-5}t\right), \right. \right. \\ & \quad \left. \left. \xi_{2^{m+\ell-1}x, 2^{m+\ell-1}x, 2^{m+\ell-1}x, 0, \dots, 0}\left(3 \cdot 2^{3m+2\ell+n-7}t\right)\right)\right)\right) \end{aligned} \quad (2.48)$$

for all $x \in X$ and $t > 0$. Since the right-hand side of the inequality tends to 1 as m tends to infinity, thus $\lim_{m \rightarrow \infty} \bar{A}(2^m x)/2^{3m} + \bar{C}(2^m x)/2^{3m} = 0$ for all $x \in X$, which implies that $\bar{C} = 0$. So, from (2.47), we lead to $\bar{A} = 0$. \square

Now, we are ready to prove the main theorem concerning the stability results for (1.8), in Menger PN-space.

Theorem 2.6. Let $\xi : X^n \rightarrow D^+$ be a function satisfies (2.8) and (2.21) for all $x_1, \dots, x_n \in X, t > 0$ and satisfies (2.9) and (2.22) for all $x \in X$ and $t > 0$. Suppose that a function $f : X \rightarrow Y$ satisfies (2.10) for all $x_1, \dots, x_n \in X$ and $t > 0$. Furthermore, assume that $f(0) = 0$ in (2.10) for the case f is even. Then, there exists a unique additive function $A : X \rightarrow Y$, a unique quadratic function $Q : X \rightarrow Y$, and a unique cubic function $C : X \rightarrow Y$ satisfying

$$\begin{aligned} & \Lambda_{f(x)-A(x)-Q(x)-C(x)}(t) \\ & \geq T \left[T \left(T_{\ell=1}^{\infty} \left(\xi_{0, \dots, 0, 2^{\ell-1}x, 2^{\ell-1}x} \left(2^{\ell+n-3}t \right) \right), T_{\ell=1}^{\infty} \left(\xi_{0, \dots, 0, 2^{\ell-1}x, 2^{\ell-1}x} \left(2^{\ell+n-3}t \right) \right) \right), \right. \\ & \quad T \left(T \left(T_{\ell=1}^{\infty} \left(T \left(\xi_{2^{\ell}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{n-5}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{n-7}t \right) \right) \right), \right. \\ & \quad \left. T_{\ell=1}^{\infty} \left(T \left(\xi_{2^{\ell}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{2\ell+n-5}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{2\ell+n-7}t \right) \right) \right) \right) \right), \\ & \quad T \left(T_{\ell=1}^{\infty} \left(T \left(\xi_{2^{\ell}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{n-5}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{n-7}t \right) \right) \right), \right. \\ & \quad \left. T_{\ell=1}^{\infty} \left(T \left(\xi_{2^{\ell}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{2\ell+n-5}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{2\ell+n-7}t \right) \right) \right) \right) \right] \end{aligned} \tag{2.49}$$

for all $x \in X$ and $t > 0$.

Proof. Let $f_e(x) = (1/2)(f(x) + f(-x))$ for all $x \in X$. Then, $f_e(0) = 0, f_e(-x) = f_e(x)$ and

$$\begin{aligned} \Lambda_{\Delta f_e(x_1, \dots, x_n)}(t) &= \Lambda_{(\Delta f(x_1, \dots, x_n) + \Delta f(-x_1, \dots, -x_n))/2}(t) \geq T(\Lambda_{\Delta f(x_1, \dots, x_n)}(t), \Lambda_{\Delta f(-x_1, \dots, -x_n)}(t)) \\ &\geq T(\xi_{x_1, \dots, x_n}(t), \xi_{-x_1, \dots, -x_n}(t)) = T(\xi_{x_1, \dots, x_n}(t), \xi_{x_1, \dots, x_n}(t)) \end{aligned} \tag{2.50}$$

for all $x_1, \dots, x_n \in X$ and $t > 0$. Hence, in view of Theorem 2.2, there exist a unique quadratic function $Q : X \rightarrow Y$ such that

$$\Lambda_{f_e(x)-Q(x)}(t) \geq T \left(T_{\ell=1}^{\infty} \left(\xi_{0, \dots, 0, 2^{\ell-1}x, 2^{\ell-1}x} \left(2^{\ell+n-2}t \right) \right), T_{\ell=1}^{\infty} \left(\xi_{0, \dots, 0, 2^{\ell-1}x, 2^{\ell-1}x} \left(2^{\ell+n-2}t \right) \right) \right) \tag{2.51}$$

for all $x \in X$ and $t > 0$. On the other hand, let $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in X$. Then, $f_o(0) = 0, f_o(-x) = -f_o(x)$ and by using the above method, from Theorem 2.5, there exist a unique additive function $A : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ such that

$$\begin{aligned} & \Lambda_{f_o(x)-A(x)-C(x)}(t) \\ & \geq T \left(T \left(T_{\ell=1}^{\infty} \left(T \left(\xi_{2^{\ell}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{n-4}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{n-6}t \right) \right) \right), \right. \\ & \quad \left. T_{\ell=1}^{\infty} \left(T \left(\xi_{2^{\ell}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{2\ell+n-4}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{2\ell+n-6}t \right) \right) \right) \right) \right), \tag{2.52} \\ & \quad T \left(T_{\ell=1}^{\infty} \left(T \left(\xi_{2^{\ell}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{n-4}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{n-6}t \right) \right) \right), \right. \\ & \quad \left. T_{\ell=1}^{\infty} \left(T \left(\xi_{2^{\ell}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{2\ell+n-4}t \right), \xi_{2^{\ell-1}x, 2^{\ell-1}x, 2^{\ell-1}x, 0, \dots, 0} \left(3 \cdot 2^{2\ell+n-6}t \right) \right) \right) \right) \right) \end{aligned}$$

for all $x \in X$ and $t > 0$. Hence, (2.49) follows from (2.51) and (2.52). □

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