

*Research Article*

# A New Positive Definite Expanded Mixed Finite Element Method for Parabolic Integrodifferential Equations

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A new positive definite expanded mixed finite element method is proposed for parabolic partial integrodifferential equations. Compared to expanded mixed scheme, the new expanded mixed element system is symmetric positive definite and both the gradient equation and the flux equation are separated from its scalar unknown equation. The existence and uniqueness for semidiscrete scheme are proved and error estimates are derived for both semidiscrete and fully discrete schemes. Finally, some numerical results are provided to confirm our theoretical analysis.

## 1. Introduction

In this paper, we consider the following initial-boundary value problem of parabolic partial integrodifferential equations:

$$\begin{aligned} u_t - \nabla \cdot \left( a(x, t) \nabla u + b(x, t) \int_0^t \nabla u ds \right) &= f(x, t), \quad (x, t) \in \Omega \times J, \\ u(x, t) &= 0, \quad (x, t) \in \partial\Omega \times \bar{J}, \\ u(x, 0) &= u_0(x), \quad x \in \Omega, \end{aligned} \tag{1.1}$$

where  $\Omega$  is a bounded convex polygonal domain in  $\mathbb{R}^d$ , ( $d = 1, 2, 3$ ) with a smooth boundary  $\partial\Omega$ ,  $J = (0, T]$  is the time interval with  $0 < T < \infty$ . The coefficients  $a = a(x, t)$ ,  $b = b(x, t)$  are

two functions, which satisfy the property that there exist some positive constants  $a_{\min}$ ,  $a_{\max}$ ,  $b_{\min}$ , and  $b_{\max}$  such that  $0 < a_{\min} \leq a(x, t) \leq a_{\max}$  and  $0 < b_{\min} \leq b(x, t) \leq b_{\max}$ .

Parabolic integrodifferential equations are a class of very important evolution equations which describe many physical phenomena such as heat conduction in material with memory, compression of viscoelastic media, and nuclear reactor dynamics. In recent years, a lot of researchers have studied the numerical methods for parabolic integrodifferential equations, such as finite element methods [1–5], mixed finite element methods [6–9], and finite volume element method [10] and so forth.

In 1994, Chen [11, 12] proposed a new mixed method, which is called a expanded mixed finite element method and proved some mathematical theories for second-order linear elliptic equation. Compared to standard mixed element methods, the expanded mixed method is expanded in the sense that three variables are explicitly approximated, namely, the scalar unknown, its gradient, and its flux (the tensor coefficient times the gradient). In 1997, Arbogast et al. [13] derived and exploited a connection between the expanded mixed method and a certain cell-centered finite difference method. And Chen proved some mathematical theories for second-order quasilinear elliptic equation [14] and fourth-order elliptic problems [15]. With the development of the expanded mixed finite element method, the method was applied to many evolution equations. In [16], some error estimates of the expanded mixed element for a kind of parabolic equation were given. Woodward and Dawson [17] studied the expanded mixed finite element method for nonlinear parabolic equation. Wu and Chen et al. [18–22] studied the two-grid methods for expanded mixed finite-element solution of semilinear reaction-diffusion equations. Song and Yuan [23] proposed the expanded upwind-mixed multistep method for the miscible displacement problem in three dimensions. Guo and Chen [24] developed and analysed an expanded characteristic-mixed finite element method for a convection-dominated transport problem. In 2010, Chen and Wang [25] proposed an  $H^1$ -Galerkin expanded mixed method for a nonlinear parabolic equation in porous medium flow, and Liu and Li [26] studied the  $H^1$ -Galerkin expanded mixed method for pseudo-hyperbolic equation. Liu [27], studied the  $H^1$ -Galerkin expanded mixed method for RLW-Burgers equation and proved semidiscrete and fully discrete optimal error estimates. Jiang and Li [28] studied an expanded mixed semidiscrete scheme for the problem of purely longitudinal motion of a homogeneous bar. In [29, 30], the expanded mixed covolume method was studied for the linear integrodifferential equation of parabolic type and elliptic problems, respectively. In [31], a posteriori error estimator for expanded mixed hybrid methods was studied and analysed.

In 2001, Yang [32] proposed a new mixed finite element method called the splitting positive definite mixed finite element procedure to treat the pressure equation of parabolic type in a nonlinear parabolic system describing a model for compressible flow displacement in a porous medium. Compared to standard mixed methods whose numerical solutions have been quite difficult because of losing positive definite properties, the proposed one does not lead to some saddle point problems. From then on, the method was applied to the hyperbolic equations [33] and pseudo-hyperbolic equations [34].

In this paper, our purpose is to propose and analyse a new expanded mixed method based on the positive definite system [32–34] for parabolic integrodifferential equations. Compared to expanded mixed methods, the proposed mixed element system is symmetric positive definite and avoids some saddle point problems. What is more, both the gradient equation and the flux equation are separated from its scalar unknown equation. The existence and uniqueness for semidiscrete scheme are proved and error estimates are derived for both semidiscrete and fully discrete schemes.

The layout of the paper is as follows. In Section 2, the positive definite expanded mixed weak formulation and semidiscrete mixed scheme are formulated, and the proof of the existence and uniqueness of the discrete solutions is given. Error estimates are derived for both semidiscrete and fully discrete schemes for problems, respectively, in Sections 3 and 4. In Section 5, some numerical results are provided to illustrate the efficiency of our method. Finally in Section 6, we will give some concluding remarks about the positive definite expanded mixed finite element method.

Throughout this paper,  $C$  will denote a generic positive constant which does not depend on the spatial mesh parameters  $h_u, h_\sigma$  and time-discretization parameter  $\delta$  and may be different at their occurrences. Usual definitions, notations, and norms of the Sobolev spaces as in [35–37] are used. We denote the natural inner product in  $L^2(\Omega)$  or  $[L^2(\Omega)]^d$  by  $(\cdot, \cdot)$  with norm  $\|\cdot\|_{L^2(\Omega)}$  or  $\|\cdot\|_{[L^2(\Omega)]^d}$  and introduce the function space  $\mathbf{W} = H(\text{div}; \Omega) = \{\omega \in [L^2(\Omega)]^d; \nabla \cdot \omega \in L^2(\Omega)\}$ .

## 2. A New Expanded Mixed Variational Formulation

Introducing the auxiliary variables:

$$\lambda = \nabla u, \quad \sigma = a(x, t)\nabla u + b(x, t) \int_0^t \nabla u \, ds = a\lambda + b \int_0^t \lambda \, ds, \quad (2.1)$$

Then we obtain the equivalent system of parabolic partial integrodifferential equations for the problem (1.1):

$$\begin{aligned} \text{(a)} \quad & u_t - \nabla \cdot \sigma = f(x, t), \quad (x, t) \in \Omega \times J, \\ \text{(b)} \quad & \lambda - \nabla u = 0, \quad (x, t) \in \Omega \times J, \\ \text{(c)} \quad & \sigma - a\lambda - b \int_0^t \lambda \, ds = 0, \quad (x, t) \in \Omega \times J, \end{aligned} \quad (2.2)$$

with the initial values  $\lambda(x, 0) = \nabla u_0(x)$ ,  $\sigma(x, 0) = a\nabla u_0(x)$ , and  $u(x, 0) = u_0(x)$ .

Then, the following expanded mixed weak formulation of (2.2) can be given by:

$$\begin{aligned} \text{(a)} \quad & (u_t, v) - (\nabla \cdot \sigma, v) = (f(x, t), v), \quad \forall v \in L^2(\Omega), \\ \text{(b)} \quad & (\lambda, \mathbf{w}) + (u, \nabla \cdot \mathbf{w}) = 0, \quad \forall \mathbf{w} \in \mathbf{W}, \\ \text{(c)} \quad & (\sigma, \mathbf{z}) - (a\lambda, \mathbf{z}) - \left( b \int_0^t \lambda \, ds, \mathbf{z} \right) = 0, \quad \forall \mathbf{z} \in \mathbf{W}. \end{aligned} \quad (2.3)$$

From (2.3)(b) we derive

$$(\lambda_t, \mathbf{w}) + (u_t, \nabla \cdot \mathbf{w}) = 0. \quad (2.4)$$

Taking  $v = \nabla \cdot \mathbf{w}$  in (2.3)(a) for  $\mathbf{w} \in \mathbf{W}$  and then substituting it into (2.4), we derive a new equivalent expanded mixed weak formulation of the system (2.3):

$$\begin{aligned} \text{(a)} \quad & (\lambda_t, \mathbf{w}) + (\nabla \cdot \sigma, \nabla \cdot \mathbf{w}) = -(f(x, t), \nabla \cdot \mathbf{w}), \quad \forall \mathbf{w} \in \mathbf{W}, \\ \text{(b)} \quad & (\sigma, \mathbf{z}) - (a\lambda, \mathbf{z}) - \left( b \int_0^t \lambda ds, \mathbf{z} \right) = 0, \quad \forall \mathbf{z} \in \mathbf{W}, \\ \text{(c)} \quad & (u_t, v) - (\nabla \cdot \sigma, v) = (f(x, t), v), \quad \forall v \in L^2(\Omega). \end{aligned} \quad (2.5)$$

Let  $\mathcal{T}_{h_u}$  and  $\mathcal{T}_{h_\sigma}$  be two families of quasi-regular partitions of the domain  $\Omega$ , which may be the same one or not, such that the elements in the partitions have the diameters bounded by  $h_u$  and  $h_\sigma$ , respectively. Let  $X_{h_u} \subset L^2(\Omega)$  and  $\mathbf{V}_{h_\sigma} \subset \mathbf{W}$  be finite element spaces defined on the partitions  $\mathcal{T}_{h_u}$  and  $\mathcal{T}_{h_\sigma}$ .

Now the semidiscrete positive definite expanded mixed finite element method for (2.5) consists in determining  $(u_h, \lambda_h, \sigma_h) \in X_{h_u} \times \mathbf{V}_{h_\sigma} \times \mathbf{V}_{h_\sigma}$  such that

$$\begin{aligned} \text{(a)} \quad & (\lambda_{ht}, \mathbf{w}_h) + (\nabla \cdot \sigma_h, \nabla \cdot \mathbf{w}_h) = -(f(x, t), \nabla \cdot \mathbf{w}_h), \quad \forall \mathbf{w}_h \in \mathbf{V}_{h_\sigma}, \\ \text{(b)} \quad & (\sigma_h, \mathbf{z}_h) - (a\lambda_h, \mathbf{z}_h) - \left( b \int_0^t \lambda_h ds, \mathbf{z}_h \right) = 0, \quad \forall \mathbf{z}_h \in \mathbf{V}_{h_\sigma}, \\ \text{(c)} \quad & (u_{ht}, v_h) - (\nabla \cdot \sigma_h, v_h) = (f(x, t), v_h), \quad \forall v_h \in X_{h_u}, \end{aligned} \quad (2.6)$$

with given an initial approximation  $(u_h^0, \lambda_h^0, \sigma_h^0) \in X_{h_u} \times \mathbf{V}_{h_\sigma} \times \mathbf{V}_{h_\sigma}$ .

*Remark 2.1.* Compared to expanded mixed weak formulation (2.3), the new expanded mixed element system (2.6) is symmetric positive definite, that is to say the gradient function and the flux function system (2.6)(a,b) is symmetric positive definite. And both the gradient equation and the flux equation are separated from its scalar unknown equation (2.6)(c).

**Theorem 2.2.** *There exists a unique discrete solution to the system (2.6).*

*Proof.* Let  $\{\varphi_i(x)\}_{i=1}^{N_1}$  and  $\{\varphi_j(x)\}_{j=1}^{N_2}$  be bases of  $X_{h_u}$  and  $\mathbf{V}_{h_\sigma}$ , respectively. Then, we have the following expressions:

$$u_h = \sum_{i=1}^{N_1} u_i(t) \varphi_i(x), \quad \lambda_h = \sum_{j=1}^{N_2} \lambda_j(t) \varphi_j(x), \quad \sigma_h = \sum_{j=1}^{N_2} \sigma_j(t) \varphi_j(x). \quad (2.7)$$

Substituting these expressions into (2.6) and choosing  $v_h = \psi_m$ ,  $w_h = z_h = \varphi_l$ , then the problems (2.6) can be written in vector matrix form as: find  $\{U(t), \Lambda(t), \Sigma(t)\}$  such that, for all  $t \in (0, T]$

$$\begin{aligned}
 \text{(a)} \quad & A\Lambda'(t) + B\Sigma(t) = F(t), \\
 \text{(b)} \quad & A\Sigma(t) - C\Lambda(t) - H \int_0^t \Lambda(s)ds = \mathbf{0}, \\
 \text{(c)} \quad & DU'(t) - E\Sigma(t) = G(t),
 \end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
 A &= ((\varphi_j, \varphi_l))_{N_2 \times N_2}, & B &= ((\nabla \cdot \varphi_j, \nabla \cdot \varphi_l))_{N_2 \times N_2}, \\
 C &= ((a\varphi_j, \varphi_l))_{N_2 \times N_2}, & D &= ((\psi_i, \psi_m))_{N_1 \times N_1}, & H &= ((b\varphi_j, \varphi_l))_{N_2 \times N_2}, \\
 E &= ((\nabla \cdot \varphi_j, \psi_m))_{N_1 \times N_2}, & U(t) &= (u_1(t), u_2(t), \dots, u_{N_1}(t))^T, \\
 \Lambda(t) &= (\lambda_1(t), \lambda_2(t), \dots, \lambda_{N_2}(t))^T, & \Sigma(t) &= (\sigma_1(t), \sigma_2(t), \dots, \sigma_{N_2}(t))^T, \\
 F(t) &= ((-f(t), \nabla \cdot \varphi_l))_{1 \times N_2}^T, & G(t) &= ((f(t), \psi_m))_{1 \times N_1}^T.
 \end{aligned} \tag{2.9}$$

It is easy to see that both  $A$  and  $D$  are symmetric positive definite. From (2.8), the problems can be written as follows:

$$\begin{aligned}
 \text{(a)} \quad & \Lambda'(t) = -A^{-1}BA^{-1} \left( C\Lambda(t) + H \int_0^t \Lambda(s)ds \right) + A^{-1}F(t), \\
 \text{(b)} \quad & \Sigma(t) = A^{-1}C\Lambda(t) + A^{-1}H \int_0^t \Lambda(s)ds, \\
 \text{(c)} \quad & U'(t) = D^{-1}E\Sigma(t) + D^{-1}G(t).
 \end{aligned} \tag{2.10}$$

Thus, by the theory of differential equations [38, 39], (2.10) has a unique solution, and equivalently (2.6) has a unique solution.  $\square$

*Remark 2.3.* It is easy to see that the coefficient matrixes  $A, B, C$ , and  $H$  of system (2.8) are symmetric positive definite. In view of this, the new expanded mixed element system (2.6) is symmetric positive definite.

### 3. Semidiscrete Error Estimates

Let  $X_{h_u}$  and  $\mathbf{V}_{h_\sigma}$  be finite dimensional subspaces of  $L^2(\Omega)$  and  $\mathbf{W}$ , respectively, with the inverse property (see [36]) and the following approximation properties (see [40–44]): for  $0 \leq p \leq +\infty$  and  $r, r^*, k$  positive integers

$$\begin{aligned} \inf_{\mathbf{w}_h \in \mathbf{V}_{h_\sigma}} \|\mathbf{w} - \mathbf{w}_h\|_{L^p(\Omega)} &\leq Ch_\sigma^{r+1} \|\mathbf{w}\|_{\mathbf{W}^{r+1,p}(\Omega)}, \quad \forall \mathbf{w} \in H(\operatorname{div}; \Omega) \cap \left[ W^{r+1,p}(\Omega) \right]^d, \\ \inf_{\mathbf{w}_h \in \mathbf{V}_{h_\sigma}} \|\nabla \cdot (\mathbf{w} - \mathbf{w}_h)\|_{L^p(\Omega)} &\leq Ch_\sigma^{r^*} \|\nabla \cdot \mathbf{w}\|_{W^{r^*,p}(\Omega)}, \quad \forall \mathbf{w} \in H(\operatorname{div}; \Omega) \cap \left[ W^{r+1,p}(\Omega) \right]^d, \\ \inf_{v_h \in X_{h_u}} \|v - v_h\|_{L^p(\Omega)} &\leq Ch_u^{k+1} \|v\|_{W^{k+1,p}(\Omega)}, \quad \forall v \in L^2(\Omega) \cap W^{k+1,p}(\Omega), \end{aligned} \quad (3.1)$$

where  $r^* = r + 1$  for the Brezzi-Douglas-Fortin-Marini spaces [43] and the Raviart-Thomas spaces [42] and  $r^* = r$  for the Brezzi-Douglas-Marini spaces [40, 43].

For our subsequent error analysis, we introduce two operators. It is well known that, in any one of the classical mixed finite element spaces, there exists an operator  $R_h$  from  $H(\operatorname{div}; \Omega)$  onto  $\mathbf{V}_{h_\sigma}$ , see [40–44], such that, for  $1 \leq p \leq +\infty$ ,

$$\begin{aligned} (\nabla \cdot (\sigma - R_h \sigma), \phi_h) &= 0, \quad \forall \phi_h \in \nabla \cdot \mathbf{V}_{h_\sigma} = \{\phi_h = \nabla \cdot \mathbf{w}_h, \mathbf{w}_h \in \mathbf{V}_{h_\sigma}\}; \\ \|\sigma - R_h \sigma\|_{L^p(\Omega)} &\leq Ch_\sigma^{r+1} \|\sigma\|_{\mathbf{W}^{r+1,p}(\Omega)}; \\ \|\nabla \cdot (\sigma - R_h \sigma)\|_{L^p(\Omega)} &\leq Ch_\sigma^{r^*} \|\nabla \cdot \sigma\|_{W^{r^*,p}(\Omega)}. \end{aligned} \quad (3.2)$$

We also define the  $L^2$ -project operator  $P_h$  from  $L^2(\Omega)$  onto  $X_{h_u}$  such that

$$\begin{aligned} (v - P_h v, v_h) &= 0, \quad \forall v \in L^2(\Omega), v_h \in X_{h_u}; \\ \|v - P_h v\|_{L^p(\Omega)} &\leq Ch_u^{k+1} \|v\|_{H^{k+1,p}(\Omega)}, \quad \forall v \in H^{k+1}(\Omega). \end{aligned} \quad (3.3)$$

Using the definitions of the operators  $R_h$  and  $P_h$ , we can easily obtain the following lemma.

**Lemma 3.1.** *Assume that the solution of system (2.5) has the regular properties that  $u_t \in L^2(H^{k+1}(\Omega))$ ,  $\lambda_t, \lambda_{tt}, \sigma_t \in L^2(\mathbf{H}^{r+1}(\Omega))$ , then one has the following estimates:*

$$\begin{aligned} \|(\lambda - R_h \lambda)_t\|_{L^p(\Omega)} &\leq Ch_\sigma^{r+1} \|\lambda_t\|_{\mathbf{W}^{r+1,p}(\Omega)}, \\ \|(\lambda - R_h \lambda)_{tt}\|_{L^p(\Omega)} &\leq Ch_\sigma^{r+1} \|\lambda_{tt}\|_{\mathbf{W}^{r+1,p}(\Omega)}, \\ \|(\sigma - R_h \sigma)_t\|_{L^p(\Omega)} &\leq Ch_\sigma^{r+1} \|\sigma_t\|_{\mathbf{W}^{r+1,p}(\Omega)}, \\ \|(u - P_h u)_t\|_{L^p(\Omega)} &\leq Ch_u^{k+1} \|u_t\|_{H^{k+1,p}(\Omega)}. \end{aligned} \quad (3.4)$$

Let

$$\begin{aligned}
 u - u_h &= u - P_h u + P_h u - u_h = \eta + \varsigma, \\
 \lambda - \lambda_h &= \lambda - R_h \lambda + R_h \lambda - \lambda_h = \rho + \xi, \\
 \sigma - \sigma_h &= \sigma - R_h \sigma + R_h \sigma - \sigma_h = \gamma + \theta.
 \end{aligned} \tag{3.5}$$

Subtracting (2.6) from (2.5) and using projections (3.2) and (3.3), one obtains

$$\begin{aligned}
 (\xi_t, \mathbf{w}_h) + (\nabla \cdot \theta, \nabla \cdot \mathbf{w}_h) &= -(\rho_t, \mathbf{w}_h), \quad \forall \mathbf{w}_h \in \mathbf{V}_{h_\sigma}, \\
 (\theta, \mathbf{z}_h) - (a\xi, \mathbf{z}_h) - \left( b \int_0^t \xi ds, \mathbf{z}_h \right) &= -(\gamma, \mathbf{z}_h) + (a\rho, \mathbf{z}_h) + \left( b \int_0^t \rho ds, \mathbf{z}_h \right), \quad \forall \mathbf{z}_h \in \mathbf{V}_{h_\sigma}, \\
 (\varsigma_t, \mathbf{v}_h) - (\nabla \cdot (\sigma - \sigma_h), \mathbf{v}_h) &= 0, \quad \forall \mathbf{v}_h \in \mathbf{X}_{h_u}.
 \end{aligned} \tag{3.6}$$

**Theorem 3.2.** Assume that the approximate properties (3.1) hold, and the solution of system (2.5) has regular properties that  $u_t, u_{tt} \in L^2(H^{k+1}(\Omega))$ ,  $\lambda_t, \lambda_{tt}, \sigma_t, \sigma_{tt} \in L^2(\mathbf{H}^{r+1}(\Omega))$ . Then one has the error estimates

$$\begin{aligned}
 \|\lambda - \lambda_h\|_{L^\infty(L^2(\Omega))} + \|\sigma - \sigma_h\|_{L^\infty(L^2(\Omega))} &\leq Ch_\sigma^{r+1}, \\
 \|\nabla \cdot (\sigma - \sigma_h)\|_{L^\infty(L^2(\Omega))} &\leq Ch_\sigma^{r^*}, \\
 \|u - u_h\|_{L^\infty(L^2(\Omega))} &\leq C(h_\sigma^{r^*} + h_u^{k+1}).
 \end{aligned} \tag{3.7}$$

*Proof.* Choose  $\mathbf{w}_h = \theta$  in (3.6)(a) and  $\mathbf{z}_h = -\xi_t$  in (3.6)(b), and add the two equations to obtain

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} \|a^{1/2} \xi\|_{L^2(\Omega)}^2 + \|\nabla \cdot \theta\|_{L^2(\Omega)}^2 \\
 &= \frac{1}{2} (a_t \xi, \xi) + (\gamma, \xi_t) - (a\rho, \xi_t) - (\rho_t, \theta) - \left( b \int_0^t \xi ds, \xi_t \right) - \left( b \int_0^t \rho ds, \xi_t \right) \\
 &= \frac{1}{2} (a_t \xi, \xi) + \frac{d}{dt} (\gamma, \xi) - (\gamma_t, \xi) - \frac{d}{dt} (a\rho, \xi) + (a\rho_t + a_t \rho, \xi) \\
 &\quad - (\rho_t, \theta) - \frac{d}{dt} \left( b \int_0^t \xi ds, \xi \right) + \left( b_t \int_0^t \xi ds, \xi \right) + (b\xi, \xi) \\
 &\quad - \frac{d}{dt} \left( b \int_0^t \rho ds, \xi \right) + \left( b_t \int_0^t \rho ds, \xi \right) + (b\rho, \xi).
 \end{aligned} \tag{3.8}$$

Integrate with respect to time from 0 to  $t$  and apply the Cauchy-Schwarz's inequality and the Young's inequality to obtain

$$\begin{aligned} & \left\| a^{1/2} \xi \right\|_{L^2(\Omega)}^2 + 2 \int_0^t \|\nabla \cdot \theta\|_{L^2(\Omega)}^2 ds \\ & \leq \left\| a^{1/2} \xi(0) \right\|_{L^2(\Omega)}^2 + \|\rho\|_{L^2(\Omega)}^2 + \|\gamma\|_{L^2(\Omega)}^2 + \varepsilon \|\xi\|_{L^2(\Omega)}^2 \\ & \quad + C \int_0^t \left( \|\rho\|_{L^2(\Omega)}^2 + \|\rho_s\|_{L^2(\Omega)}^2 + \|\gamma_s\|_{L^2(\Omega)}^2 + \|\xi\|_{L^2(\Omega)}^2 + \|\theta\|_{L^2(\Omega)}^2 \right) ds. \end{aligned} \quad (3.9)$$

Choose  $\mathbf{z}_h = \theta$  in (3.6)(b) to get

$$\|\theta\|_{L^2(\Omega)}^2 \leq C \left( \left\| a^{1/2} \xi \right\|_{L^2(\Omega)}^2 + \|\gamma\|_{L^2(\Omega)}^2 + \|\rho\|_{L^2(\Omega)}^2 + \int_0^t \left( \|\xi\|_{L^2(\Omega)}^2 + \|\rho\|_{L^2(\Omega)}^2 \right) ds \right). \quad (3.10)$$

Combining (3.9) and (3.10), we obtain

$$\begin{aligned} & \|\xi\|_{L^2(\Omega)}^2 + \|\theta\|_{L^2(\Omega)}^2 + 2 \int_0^t \|\nabla \cdot \theta\|_{L^2(\Omega)}^2 ds \\ & \leq \left\| a^{1/2} \xi(0) \right\|_{L^2(\Omega)}^2 + \|\rho\|_{L^2(\Omega)}^2 + \|\gamma\|_{L^2(\Omega)}^2 + \varepsilon \|\xi\|_{L^2(\Omega)}^2 \\ & \quad + C \int_0^t \left( \|\rho\|_{L^2(\Omega)}^2 + \|\rho_s\|_{L^2(\Omega)}^2 + \|\gamma_s\|_{L^2(\Omega)}^2 + \|\xi\|_{L^2(\Omega)}^2 + \|\theta\|_{L^2(\Omega)}^2 \right) ds. \end{aligned} \quad (3.11)$$

Using the fact that  $\xi(0) = 0$  and Gronwall's lemma, we obtain

$$\begin{aligned} & \|\xi\|_{L^2(\Omega)}^2 + \|\theta\|_{L^2(\Omega)}^2 + 2 \int_0^t \|\nabla \cdot \theta\|_{L^2(\Omega)}^2 ds \\ & \leq \|\rho\|_{L^2(\Omega)}^2 + \|\gamma\|_{L^2(\Omega)}^2 + C \int_0^t \left( \|\rho\|_{L^2(\Omega)}^2 + \|\rho_s\|_{L^2(\Omega)}^2 + \|\gamma_s\|_{L^2(\Omega)}^2 \right) ds. \end{aligned} \quad (3.12)$$

Differentiating (3.6)(b) and taking  $\mathbf{z}_h = \xi_t$ , we obtain

$$\begin{aligned} (\theta_t, \xi_t) &= \|a \xi_t\|_{L^2(\Omega)}^2 + (a_t \xi, \xi_t) - (\gamma_t, \xi_t) + (a_t \rho + a \rho_t, \xi_t) \\ & \quad + \left( b_t \int_0^t \xi ds, \xi_t \right) + (b \xi, \xi_t) + \left( b_t \int_0^t \rho ds, \xi_t \right) + (b \rho, \xi_t). \end{aligned} \quad (3.13)$$

Choosing  $\mathbf{w}_h = \theta_t$  in (3.6)(a), we obtain

$$(\xi_t, \theta_t) + \frac{1}{2} \frac{d}{dt} \|\nabla \cdot \theta\|_{L^2(\Omega)}^2 = -\frac{d}{dt} (\rho_t, \theta) + (\rho_{tt}, \theta). \quad (3.14)$$



Add (3.13) and (3.14) to obtain

$$\begin{aligned} \|a\xi_t\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \cdot \theta\|_{L^2(\Omega)}^2 &= -\frac{d}{dt} (\rho_t, \theta) + (\rho_{tt}, \theta) - (a_t \xi, \xi_t) + (\gamma_t, \xi_t) - (a_t \rho + a \rho_t, \xi_t) \\ &\quad - \left( b_t \int_0^t \xi ds, \xi_t \right) - (b \xi, \xi_t) - \left( b_t \int_0^t \rho ds, \xi_t \right) - (b \rho, \xi_t). \end{aligned} \quad (3.15)$$

Integrate (3.15) with respect to time from 0 to  $t$  to obtain

$$\begin{aligned} \int_0^t \|\xi_s\|_{L^2(\Omega)}^2 ds + \|\nabla \cdot \theta\|_{L^2(\Omega)}^2 &\leq \|\rho\|_{L^2(\Omega)}^2 + \|\gamma\|_{L^2(\Omega)}^2 + C \left( \|\xi\|_{L^2(\Omega)}^2 + \int_0^t \|\xi\|_{L^2(\Omega)}^2 ds \right) \\ &\quad + C \int_0^t \left( \|\rho\|_{L^2(\Omega)}^2 + \|\rho_s\|_{L^2(\Omega)}^2 + \|\rho_{ss}\|_{L^2(\Omega)}^2 + \|\gamma_s\|_{L^2(\Omega)}^2 \right) ds. \end{aligned} \quad (3.16)$$

Substitute (3.12) into (3.16) to have

$$\begin{aligned} \int_0^t \|\xi_s\|_{L^2(\Omega)}^2 ds + \|\nabla \cdot \theta\|_{L^2(\Omega)}^2 \\ \leq \|\rho\|_{L^2(\Omega)}^2 + \|\gamma\|_{L^2(\Omega)}^2 + C \int_0^t \left( \|\rho\|_{L^2(\Omega)}^2 + \|\gamma\|_{L^2(\Omega)}^2 \right. \\ \left. + \|\rho_s\|_{L^2(\Omega)}^2 + \|\rho_{ss}\|_{L^2(\Omega)}^2 + \|\gamma_s\|_{L^2(\Omega)}^2 \right) ds. \end{aligned} \quad (3.17)$$

Choosing  $v_h = \zeta$  in (3.6)(b) and applying Cauchy-Schwarz's inequality, we obtain

$$\|\zeta\|_{L^2(\Omega)} \leq \|\nabla \cdot (\sigma - \sigma_h)\|_{L^2(\Omega)}. \quad (3.18)$$

Using Lemma 3.1, (3.17), and Gronwall's lemma, we get

$$\|\zeta\|_{L^\infty(L^2(\Omega))} \leq C \left( h_\sigma^{r^*} + h_u^{k+1} \right). \quad (3.19)$$

Using (3.12), (3.17), (3.19), (3.2), (3.3), and Lemma 3.1, we apply the triangle inequality to complete the proof.  $\square$

#### 4. Fully Discrete Error Estimates

In this section, we get the error estimates of fully discrete schemes. For the backward Euler procedure, let  $0 = t_0 < t_1 < t_2 < \dots < t_M = T$  be a given partition of the time interval  $[0, T]$  with step length  $\delta = T/M$ , for some positive integer  $M$ . For a smooth function  $\phi$  on  $[0, T]$ , define

$\phi^n = \phi(t_n)$  and  $\partial_t \phi^n = (\phi^n - \phi^{n-1})/\delta$ . For approximating the integrals, we use the composite left rectangle rule

$$\delta \sum_{j=0}^{n-1} \phi^j \approx \int_0^{t_n} \phi(s) ds. \quad (4.1)$$

Note that  $\phi \in C^1 [0, T]$ , the quadrature error satisfies

$$\left| \delta \sum_{j=0}^{n-1} \phi^j - \int_0^{t_n} \phi(s) ds \right| \leq C \delta \int_0^{t_n} |\phi_s(s)| ds. \quad (4.2)$$

Equation (2.5) has the following equivalent formulation:

$$\begin{aligned} \text{(a)} \quad & (\partial_t \lambda^n, \mathbf{w}) + (\nabla \cdot \sigma^n, \nabla \cdot \mathbf{w}) = (-f^n, \nabla \cdot \mathbf{w}) + (R_1^n, \mathbf{w}), \quad \forall \mathbf{w} \in \mathbf{W}, \\ \text{(b)} \quad & (\sigma^n, \mathbf{z}) - (a^n \lambda^n, \mathbf{z}) - \left( b^n \delta \sum_{j=0}^{n-1} \lambda^j, \mathbf{z} \right) = -(R_3^n, \mathbf{z}), \quad \forall \mathbf{z} \in \mathbf{W}, \\ \text{(c)} \quad & (\partial_t u^n, v) - (\nabla \cdot \sigma^n, v) = (f^n, v) + (R_2^n, v), \quad \forall v \in L^2(\Omega), \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} R_1^n &= \partial_t \lambda^n - \lambda_t = O\left(\delta \frac{\partial^2 \lambda}{\partial t^2}\right), \quad R_2^n = \partial_t u^n - u_t = O\left(\delta \frac{\partial^2 u}{\partial t^2}\right), \\ R_3^n &= \delta \sum_{j=0}^{n-1} \lambda^j - \int_0^{t_n} \lambda(s) ds = O\left(\delta \frac{\partial \lambda}{\partial t}\right). \end{aligned} \quad (4.4)$$

Now we can formulate a fully discrete scheme based on (4.3).

Fully discrete scheme: find  $(u_h^n, \lambda_h^n, \sigma_h^n) \in X_{h_u} \times \mathbf{V}_{h_\sigma} \times \mathbf{V}_{h_\sigma}$ ,  $(n = 1, 2, \dots, M-1)$  such that

$$\begin{aligned} \text{(a)} \quad & (\partial_t \lambda_h^n, \mathbf{w}_h) + (\nabla \cdot \sigma_h^n, \nabla \cdot \mathbf{w}_h) = (-f^n, \nabla \cdot \mathbf{w}_h), \quad \forall \mathbf{w}_h \in \mathbf{V}_{h_\sigma}, \\ \text{(b)} \quad & (\sigma_h^n, \mathbf{z}_h) - (a^n \lambda_h^n, \mathbf{z}_h) - \left( b^n \delta \sum_{j=0}^{n-1} \lambda_h^j, \mathbf{z}_h \right) = 0, \quad \forall \mathbf{z}_h \in \mathbf{V}_{h_\sigma}, \\ \text{(c)} \quad & (\partial_t u_h^n, v_h) - (\nabla \cdot \sigma_h^n, v_h) = (f^n, v_h), \quad \forall v_h \in X_{h_u}, \end{aligned} \quad (4.5)$$

with given an initial approximation  $(u_h^0, \lambda_h^0, \sigma_h^0) \in X_{h_u} \times \mathbf{V}_{h_\sigma} \times \mathbf{V}_{h_\sigma}$ .

For fully discrete error estimates, we now split the errors

$$\begin{aligned} u^n - u_h^n &= u^n - P_h u^n + P_h u^n - u_h^n = \eta^n + \zeta^n, \\ \lambda^n - \lambda_h^n &= \lambda^n - R_h \lambda^n + R_h \lambda^n - \lambda_h^n = \rho^n + \xi^n, \\ \sigma^n - \sigma_h^n &= \sigma^n - R_h \sigma^n + R_h \sigma^n - \sigma_h^n = \gamma^n + \theta^n. \end{aligned} \quad (4.6)$$

From (4.3) to (4.5), we then obtain

$$\begin{aligned} (a) \quad & (\partial_t \xi^n, \mathbf{w}_h) + (\nabla \cdot \theta^n, \nabla \cdot \mathbf{w}_h) = -(\partial_t \rho^n, \mathbf{w}_h) + (R_1^n, \mathbf{w}_h), \quad \forall \mathbf{w}_h \in \mathbf{V}_{h_\sigma}, \\ (b) \quad & (\theta^n, \mathbf{z}_h) - (a^n \xi^n, \mathbf{z}_h) - \left( b^n \delta \sum_{j=0}^{n-1} \xi^j, \mathbf{z}_h \right) \\ & = -(\gamma^n, \mathbf{z}_h) + (a^n \rho^n, \mathbf{z}_h) + \left( b^n \delta \sum_{j=0}^{n-1} \rho^j, \mathbf{z}_h \right) - (R_3^n, \mathbf{z}_h), \quad \forall \mathbf{z}_h \in \mathbf{V}_{h_\sigma}, \\ (c) \quad & (\partial_t \xi^n, v_h) - (\nabla \cdot (\theta^n + \gamma^n), v_h) = (R_2^n, v_h), \quad \forall v_h \in X_{h_u}. \end{aligned} \quad (4.7)$$

**Lemma 4.1.** Assume that the solution of system (2.5) has regular properties that  $u_t \in L^2(H^{k+1}(\Omega))$ ,  $\lambda_t, \sigma_t \in L^2(\mathbf{H}^{r+1}(\Omega))$ . Then one has the estimates

$$\begin{aligned} \max_{0 \leq n \leq M} \|\partial_t (\lambda - R_h \lambda)^n\|_{L^2(\Omega)} + \max_{0 \leq n \leq M} \|\partial_t (\sigma - R_h \sigma)^n\|_{L^2(\Omega)} &\leq C h_\sigma^{r+1}, \\ \max_{0 \leq n \leq M} \|\partial_t (u - P_h u)^n\|_{L^2(\Omega)} &\leq C h_u^{k+1}. \end{aligned} \quad (4.8)$$

**Theorem 4.2.** Assume that  $\partial^2 u / \partial t^2, \partial u / \partial t \in L^2(H^{k+1}(\Omega))$ ,  $\partial \lambda / \partial t, \partial^2 \lambda / \partial t^2, \partial \sigma / \partial t, \partial^2 \sigma / \partial t^2 \in L^2(\mathbf{H}^{r+1}(\Omega))$ ,  $u \in L^\infty(H^{k+1}(\Omega))$ , and  $\lambda, \sigma \in L^\infty(\mathbf{H}^{r+1}(\Omega))$ , then there exists a constant  $C$  such that

$$\begin{aligned} \max_{0 \leq n \leq M} \|(\lambda - \lambda_h)^n\|_{L^2(\Omega)} + \max_{0 \leq n \leq M} \|(\sigma - \sigma_h)^n\|_{L^2(\Omega)} &\leq C (h_\sigma^{r+1} + \delta), \\ \max_{0 \leq n \leq M} \|\nabla \cdot (\sigma - \sigma_h)^n\|_{L^2(\Omega)} &\leq C (h_\sigma^{r^*} + \delta), \\ \max_{0 \leq n \leq M} \|(u - u_h)^n\|_{L^2(\Omega)} &\leq C (h_u^{k+1} + h_\sigma^{r^*} + \delta). \end{aligned} \quad (4.9)$$

*Proof.* Set  $\mathbf{w}_h = \theta^n$  in (4.7)(a) and  $\mathbf{z}_h = -\partial_t \xi^n$  in (4.7)(b) and add the two equations to obtain

$$\begin{aligned} \|\nabla \cdot \theta^n\|_{L^2(\Omega)}^2 + (a^n \partial_t \xi^n, \xi^n) &= -(\partial_t \rho^n, \theta^n) + (R_1^n, \theta^n) + (\gamma^n, \partial_t \xi^n) - (a^n \rho^n, \partial_t \xi^n) \\ &\quad - \left( b^n \delta \sum_{j=0}^{n-1} \xi^j, \partial_t \xi^n \right) - \left( b^n \delta \sum_{j=0}^{n-1} \rho^j, \partial_t \xi^n \right) + (R_3^n, \partial_t \xi^n). \end{aligned} \quad (4.10)$$

Note that

$$\begin{aligned}
\partial_t \|a^n \xi^n\|_{L^2(\Omega)}^2 &= \frac{(a^n \xi^n, \xi^n) - (a^{n-1} \xi^{n-1}, \xi^{n-1})}{\delta} \\
&= \frac{(a^n \xi^n, \xi^n) - (a^n \xi^n, \xi^{n-1}) + (a^n \xi^n, \xi^{n-1}) - (a^{n-1} \xi^{n-1}, \xi^{n-1})}{\delta} \\
&= (a^n \partial_t \xi^n, \xi^n) + (a^n \partial_t \xi^n, \xi^{n-1}) + (\partial_t a^n \xi^{n-1}, \xi^{n-1}) \\
&= 2(a^n \partial_t \xi^n, \xi^n) - \frac{\|(a^n)^{1/2}(\xi^n - \xi^{n-1})\|_{L^2(\Omega)}^2}{\delta} + (\partial_t a^n \xi^{n-1}, \xi^{n-1}).
\end{aligned} \tag{4.11}$$

So, we get

$$(a^n \partial_t \xi^n, \xi^n) = \frac{1}{2} \partial_t \|a^n \xi^n\|_{L^2(\Omega)}^2 + \frac{\|(a^n)^{1/2}(\xi^n - \xi^{n-1})\|_{L^2(\Omega)}^2}{2\delta} - \frac{1}{2} (\partial_t a^n \xi^{n-1}, \xi^{n-1}). \tag{4.12}$$

Note that

$$\begin{aligned}
(\gamma^n, \partial_t \xi^n) &= \frac{(\xi^n, \gamma^n) - (\xi^{n-1}, \gamma^{n-1})}{\delta} - (\partial_t \gamma^n, \xi^{n-1}), \\
(a^n \rho^n, \partial_t \xi^n) &= \frac{(\xi^n, a^n \rho^n) - (\xi^{n-1}, a^{n-1} \rho^{n-1})}{\delta} - (\partial_t a^n \rho^n, \xi^{n-1}), \\
(R_3^n, \partial_t \xi^n) &= \frac{(\xi^n, R_3^n) - (\xi^{n-1}, R_3^{n-1})}{\delta} - (\partial_t R_3^n, \xi^{n-1}). \\
\left( b^n \delta \sum_{j=0}^{n-1} \xi^j, \partial_t \xi^n \right) &= \frac{(b^n \delta \sum_{j=0}^{n-1} \xi^j, \xi^n) - (b^{n-1} \delta \sum_{j=0}^{n-2} \xi^j, \xi^{n-1})}{\delta} \\
&\quad - \left( \partial_t b^n \delta \sum_{j=0}^{n-1} \xi^j, \xi^{n-1} \right) - (b^{n-1} \xi^{n-1}, \xi^{n-1}), \\
\left( b^n \delta \sum_{j=0}^{n-1} \rho^j, \partial_t \xi^n \right) &= \frac{(b^n \delta \sum_{j=0}^{n-1} \rho^j, \xi^n) - (b^{n-1} \delta \sum_{j=0}^{n-2} \rho^j, \xi^{n-1})}{\delta} \\
&\quad - \left( \partial_t b^n \delta \sum_{j=0}^{n-1} \rho^j, \xi^{n-1} \right) - (b^{n-1} \rho^{n-1}, \xi^{n-1}).
\end{aligned} \tag{4.13}$$

Substitute (4.12)-(4.13) into (4.10) to get

$$\begin{aligned}
& \|\nabla \cdot \theta^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \partial_t \|a^n \xi^n\|_{L^2(\Omega)}^2 + \frac{\|(a^n)^{1/2}(\xi^n - \xi^{n-1})\|_{L^2(\Omega)}^2}{2\delta} \\
&= -(\partial_t \rho^n, \theta^n) + (R_1^n, \theta^n) + \frac{1}{2} (\partial_t a^n \xi^{n-1}, \xi^{n-1}) + \frac{(\xi^n, \gamma^n) - (\xi^{n-1}, \gamma^{n-1})}{\delta} \\
&\quad - (\partial_t \gamma^n, \xi^{n-1}) - \frac{(\xi^n, a^n \rho^n) - (\xi^{n-1}, a^{n-1} \rho^{n-1})}{\delta} + (\partial_t a^n \rho^n, \xi^{n-1}) \\
&\quad - \frac{(b^n \delta \sum_{j=0}^{n-1} (\xi^j + \rho^j), \xi^n) - (b^{n-1} \delta \sum_{j=0}^{n-2} (\xi^j + \rho^j), \xi^{n-1})}{\delta} \\
&\quad + \left( \partial_t b^n \delta \sum_{j=0}^{n-1} (\xi^j + \rho^j), \xi^{n-1} \right) + (b^{n-1} (\xi^{n-1} + \rho^{n-1}), \xi^{n-1}) \\
&\quad + \frac{(\xi^n, R_3^n) - (\xi^{n-1}, R_3^{n-1})}{\delta} - (\partial_t R_3^n, \xi^{n-1}).
\end{aligned} \tag{4.14}$$

Summing from 1 to  $n$ , we find that

$$\begin{aligned}
& \|(a^n)^{1/2} \xi^n\|_{L^2(\Omega)}^2 + 2\delta \sum_{j=1}^n \|\nabla \cdot \theta^j\|_{L^2(\Omega)}^2 + \sum_{j=1}^n \|(a^j)^{1/2} (\xi^j - \xi^{j-1})\|_{L^2(\Omega)}^2 \\
&\leq \|(a^0)^{1/2} \xi^0\|_{L^2(\Omega)}^2 - (\xi^0, \gamma^0) + (\xi^0, a^0 \rho^0) + \varepsilon \|\xi^n\|_{L^2(\Omega)}^2 + C \|\rho^n\|_{L^2(\Omega)}^2 + C \|R_3^n\|_{L^2(\Omega)}^2 \\
&\quad + C \|\gamma^n\|_{L^2(\Omega)}^2 + C\delta \sum_{j=1}^{n-1} \|\xi^j\|_{L^2(\Omega)}^2 + C\delta \sum_{j=1}^n \|\theta^j\|_{L^2(\Omega)}^2 \\
&\quad + C\delta \sum_{j=1}^n \left[ \|\partial_t \rho^j\|_{L^2(\Omega)}^2 + \|\partial_t \gamma^j\|_{L^2(\Omega)}^2 + \|\partial_t a^j \rho^j\|_{L^2(\Omega)}^2 + \|\rho^j\|_{L^2(\Omega)}^2 + \|R_1^j\|_{L^2(\Omega)}^2 + \|\partial_t R_3^j\|_{L^2(\Omega)}^2 \right].
\end{aligned} \tag{4.15}$$

Choose  $z_h = \theta^n$  in (3.6)(b) to get

$$\|\theta^n\|_{L^2(\Omega)}^2 \leq C \left( \|(a^n)^{1/2} \xi^n\|_{L^2(\Omega)}^2 + \|\gamma^n\|_{L^2(\Omega)}^2 + \|\rho^n\|_{L^2(\Omega)}^2 \right) + C\delta \sum_{j=1}^{n-1} \left( \|\xi^j\|_{L^2(\Omega)}^2 + \|\rho^j\|_{L^2(\Omega)}^2 \right). \tag{4.16}$$

Substitute (4.16) into (4.15) and note that  $\xi^0 = 0$  to get

$$\begin{aligned}
& \|\xi^n\|_{L^2(\Omega)}^2 + \|\theta^n\|_{L^2(\Omega)}^2 + 2\delta \sum_{j=1}^n \|\nabla \cdot \theta^j\|_{L^2(\Omega)}^2 \\
& \leq C\|\rho^n\|_{L^2(\Omega)}^2 + C\|\gamma^n\|_{L^2(\Omega)}^2 + C\|R_3^n\|_{L^2(\Omega)}^2 \\
& \quad + C\delta \sum_{j=1}^{n-1} \|\xi^j\|_{L^2(\Omega)}^2 + C\delta \sum_{j=1}^n \|\theta^j\|_{L^2(\Omega)}^2 \\
& \quad + C\delta \sum_{j=1}^n \left[ \|\partial_t \rho^j\|_{L^2(\Omega)}^2 + \|\partial_t \gamma^j\|_{L^2(\Omega)}^2 + \|\partial_t a^j \rho^j\|_{L^2(\Omega)}^2 \right. \\
& \quad \left. + \|\rho^j\|_{L^2(\Omega)}^2 + \|R_1^j\|_{L^2(\Omega)}^2 + \|\partial_t R_3^j\|_{L^2(\Omega)}^2 \right].
\end{aligned} \tag{4.17}$$

Using Gronwall's lemma, we obtain

$$\begin{aligned}
& \|\xi^n\|_{L^2(\Omega)}^2 + (1 - C\delta)\|\theta^n\|_{L^2(\Omega)}^2 + 2\delta \sum_{j=1}^n \|\nabla \cdot \theta^j\|_{L^2(\Omega)}^2 \\
& \leq C\|\rho^n\|_{L^2(\Omega)}^2 + C\|\gamma^n\|_{L^2(\Omega)}^2 + C\|R_3^n\|_{L^2(\Omega)}^2 \\
& \quad + C\delta \sum_{j=1}^n \left[ \|\partial_t \rho^j\|_{L^2(\Omega)}^2 + \|\partial_t \gamma^j\|_{L^2(\Omega)}^2 \right. \\
& \quad \left. + \|\partial_t a^j \rho^j\|_{L^2(\Omega)}^2 + \|\rho^j\|_{L^2(\Omega)}^2 + \|R_1^j\|_{L^2(\Omega)}^2 + \|\partial_t R_3^j\|_{L^2(\Omega)}^2 \right].
\end{aligned} \tag{4.18}$$

Note that

$$\begin{aligned}
& \text{(a) } \delta \sum_{j=1}^n \|R_1^j\|_{L^2(\Omega)}^2 \leq C\delta^3 \sum_{j=1}^n \left\| \frac{\partial^2 \lambda}{\partial t^2} \right\|_{L^2(L^2(\Omega))}^2 \leq C\delta^2 \left\| \frac{\partial^2 \lambda}{\partial t^2} \right\|_{L^\infty(L^2(\Omega))}^2, \\
& \text{(b) } \delta \sum_{j=1}^n \|\partial_t \rho^j\|_{L^2(\Omega)}^2 \leq C\delta \sum_{j=1}^n \left\| \frac{\partial \rho^j}{\partial t} \right\|_{L^2(\Omega)}^2 \leq C \frac{T}{M} \cdot nh_\sigma^{2r+2} \leq Ch_\sigma^{2r+2}, \\
& \text{(c) } \delta \sum_{j=1}^n \|\partial_t \gamma^j\|_{L^2(\Omega)}^2 \leq C\delta \sum_{j=1}^n \left\| \frac{\partial \gamma^j}{\partial t} \right\|_{L^2(\Omega)}^2 \leq C \frac{T}{M} \cdot nh_\sigma^{2r+2} \leq Ch_\sigma^{2r+2}, \\
& \text{(d) } \delta \sum_{j=1}^n \|\partial_t R_3^j\|_{L^2(\Omega)}^2 = C\delta \sum_{j=1}^n \left\| \frac{\int_{t_{j-1}}^{t_j} (\lambda^{j-1} - \lambda) ds}{\delta} \right\|_{L^2(\Omega)}^2 \leq C\delta^2 \left\| \frac{\partial \lambda}{\partial t} \right\|_{L^\infty(L^2(\Omega))}^2.
\end{aligned} \tag{4.19}$$

Therefore, substituting the above estimates into (4.18) and choosing  $\delta_0$  in such a way that for  $0 < \delta \leq \delta_0$ ,  $(1 - C\delta) > 0$ , we obtain

$$\|\theta^n\|_{L^2(\Omega)}^2 + \|\xi^n\|_{L^2(\Omega)}^2 + \delta \sum_{j=1}^n \|\nabla \cdot \theta^j\|_{L^2(\Omega)}^2 \leq C(h_\sigma^{2r+2} + \delta^2). \quad (4.20)$$

By (4.7)(b), we obtain

$$\begin{aligned} (\partial_t \theta^n, \mathbf{z}_h) &= \left( \frac{a^n \xi^n - a^{n-1} \xi^{n-1}}{\delta}, \mathbf{z}_h \right) - (\partial_t \gamma^n, \mathbf{z}_h) + \left( \frac{a^n \rho^n - a^{n-1} \rho^{n-1}}{\delta}, \mathbf{z}_h \right) \\ &\quad + \frac{(b^n \delta \sum_{j=0}^{n-1} (\xi^j + \rho^j), \mathbf{z}_h) - (b^{n-1} \delta \sum_{j=0}^{n-2} (\xi^j + \rho^j), \mathbf{z}_h)}{\delta} - (\partial_t R_3^n, \mathbf{z}_h) \\ &= \left( \frac{a^n \xi^n - a^{n-1} \xi^{n-1}}{\delta}, \mathbf{z}_h \right) - (\partial_t \gamma^n, \mathbf{z}_h) + \left( \frac{a^n \rho^n - a^{n-1} \rho^{n-1}}{\delta}, \mathbf{z}_h \right) \\ &\quad + (b^n (\xi^{n-1} + \rho^{n-1}), \mathbf{z}_h) + \left( \partial_t b^n \delta \sum_{j=0}^{n-2} (\xi^j + \rho^j), \mathbf{z}_h \right) - (\partial_t R_3^n, \mathbf{z}_h). \end{aligned} \quad (4.21)$$

Set  $\mathbf{z}_h = \partial_t \xi^n$  in (4.21) to obtain

$$\begin{aligned} (\partial_t \theta^n, \partial_t \xi^n) &= \left( \frac{a^n \xi^n - a^{n-1} \xi^{n-1}}{\delta}, \partial_t \xi^n \right) - (\partial_t \gamma^n, \partial_t \xi^n) + \left( \frac{a^n \rho^n - a^{n-1} \rho^{n-1}}{\delta}, \partial_t \xi^n \right) \\ &\quad + (b^n (\xi^{n-1} + \rho^{n-1}), \partial_t \xi^n) + \left( \partial_t b^n \delta \sum_{j=0}^{n-2} (\xi^j + \rho^j), \partial_t \xi^n \right) - (\partial_t R_3^n, \partial_t \xi^n) \\ &= \|(a^n)^{1/2} \partial_t \xi^n\|_{L^2(\Omega)}^2 + (\partial_t a^n \xi^{n-1}, \partial_t \xi^n) + (a^n \partial_t \rho^n, \partial_t \xi^n) \\ &\quad + (\partial_t a^n \rho^{n-1}, \partial_t \xi^n) - (\partial_t \gamma^n, \partial_t \xi^n) + (b^n (\xi^{n-1} + \rho^{n-1}), \partial_t \xi^n) \\ &\quad + \left( \partial_t b^n \delta \sum_{j=0}^{n-2} (\xi^j + \rho^j), \partial_t \xi^n \right) - (\partial_t R_3^n, \partial_t \xi^n). \end{aligned} \quad (4.22)$$

Set  $\mathbf{w}_h = \partial_t \theta^n$  in (4.7)(a) to obtain

$$(\partial_t \xi^n, \partial_t \theta^n) + \frac{1}{2} \partial_t \|\nabla \cdot \theta^n\|_{L^2(\Omega)}^2 + \frac{\|\nabla \cdot (\theta^n - \theta^{n-1})\|_{L^2(\Omega)}^2}{2\delta} = -(\partial_t \rho^n, \partial_t \theta^n) + (R_1^n, \partial_t \theta^n). \quad (4.23)$$

Substitute (4.22) into (4.23) to get

$$\begin{aligned}
& \left\| (a^n)^{1/2} \partial_t \xi^n \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \partial_t \|\nabla \cdot \theta^n\|_{L^2(\Omega)}^2 + \frac{\|\nabla \cdot (\theta^n - \theta^{n-1})\|_{L^2(\Omega)}^2}{2\delta} \\
& = -(\partial_t \rho^n, \partial_t \theta^n) - (\partial_t a^n \xi^{n-1}, \partial_t \xi^n) - (a^n \partial_t \rho^n, \partial_t \xi^n) \\
& \quad - (\partial_t a^n \rho^{n-1}, \partial_t \xi^n) + (\partial_t \gamma^n, \partial_t \xi^n) - (b^n (\xi^{n-1} + \rho^{n-1}), \partial_t \xi^n) \\
& \quad + (R_1^n, \partial_t \theta^n) - \left( \partial_t b^n \delta \sum_{j=0}^{n-2} (\xi^j + \rho^j), \partial_t \xi^n \right) + (\partial_t R_3^n, \partial_t \xi^n).
\end{aligned} \tag{4.24}$$

Take  $z_h = \partial_t \theta^n$  in (4.21) to obtain

$$\begin{aligned}
\|\partial_t \theta^n\|_{L^2(\Omega)}^2 & = \left( \frac{a^n \xi^n - a^{n-1} \xi^{n-1}}{\delta}, \partial_t \theta^n \right) - (\partial_t \gamma^n, \partial_t \theta^n) \\
& \quad + \left( \frac{a^n \rho^n - a^{n-1} \rho^{n-1}}{\delta}, \partial_t \theta^n \right) + (b^n (\xi^{n-1} + \rho^{n-1}), \partial_t \theta^n) \\
& \quad + \left( \partial_t b^n \delta \sum_{j=0}^{n-2} (\xi^j + \rho^j), \partial_t \theta^n \right) - (\partial_t R_3^n, \partial_t \theta^n) \\
& = (a^n \partial_t \xi^n, \partial_t \theta^n) + (\partial_t a^n \xi^{n-1}, \partial_t \theta^n) - (\partial_t \gamma^n, \partial_t \theta^n) + (a^n \partial_t \rho^n, \partial_t \theta^n) \\
& \quad + (\partial_t a^n \rho^{n-1}, \partial_t \theta^n) + (b^n (\xi^{n-1} + \rho^{n-1}), \partial_t \theta^n) \\
& \quad + \left( \partial_t b^n \delta \sum_{j=0}^{n-2} (\xi^j + \rho^j), \partial_t \theta^n \right) - (\partial_t R_3^n, \partial_t \theta^n).
\end{aligned} \tag{4.25}$$

Add (4.24) and (4.25) to get

$$\begin{aligned}
& \left\| (a^n)^{1/2} \partial_t \xi^n \right\|_{L^2(\Omega)}^2 + \|\partial_t \theta^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \partial_t \|\nabla \cdot \theta^n\|_{L^2(\Omega)}^2 + \frac{\|\nabla \cdot (\theta^n - \theta^{n-1})\|_{L^2(\Omega)}^2}{2\delta} \\
& = -(\partial_t \rho^n, \partial_t \theta^n) - (\partial_t a^n \xi^{n-1}, \partial_t \xi^n) - (a^n \partial_t \rho^n, \partial_t \xi^n) - (\partial_t a^n \rho^{n-1}, \partial_t \xi^n) \\
& \quad + (\partial_t \gamma^n, \partial_t \xi^n) + (a^n \partial_t \xi^n, \partial_t \theta^n) + (\partial_t a^n \xi^{n-1}, \partial_t \theta^n) - (\partial_t \gamma^n, \partial_t \theta^n) \\
& \quad + (a^n \partial_t \rho^n, \partial_t \theta^n) + (\partial_t a^n \rho^{n-1}, \partial_t \theta^n) + (R_1^n, \partial_t \theta^n) - (\partial_t R_3^n, \partial_t \theta^n + \partial_t \xi^n) \\
& \quad + (b^n (\xi^{n-1} + \rho^{n-1}), \partial_t \theta^n + \partial_t \xi^n) + \left( \partial_t b^n \delta \sum_{j=0}^{n-2} (\xi^j + \rho^j), \partial_t \theta^n + \partial_t \xi^n \right).
\end{aligned} \tag{4.26}$$



Apply Cauchy-Schwarz's inequality and Young's inequality to obtain

$$\begin{aligned}
& a_{\min} \|\partial_t \xi^n\|_{L^2(\Omega)}^2 + \|\partial_t \theta^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \partial_t \|\nabla \cdot \theta^n\|_{L^2(\Omega)}^2 \\
& \leq \varepsilon \left( \|\partial_t \xi^n\|_{L^2(\Omega)}^2 + \|\partial_t \theta^n\|_{L^2(\Omega)}^2 \right) \\
& \quad + C \left[ \|\xi^{n-1}\|_{L^2(\Omega)}^2 + \|\partial_t \rho^n\|_{L^2(\Omega)}^2 + \|\partial_t \gamma^n\|_{L^2(\Omega)}^2 + \|\rho^{n-1}\|_{L^2(\Omega)}^2 + \|R_1^n\|_{L^2(\Omega)}^2 + \|\partial_t R_3^n\|_{L^2(\Omega)}^2 \right] \\
& \quad + C \delta^2 \sum_{j=0}^{n-2} \left( \|\xi^j\|_{L^2(\Omega)}^2 + \|\rho^j\|_{L^2(\Omega)}^2 \right).
\end{aligned} \tag{4.27}$$

Using (4.19) and (4.20) and summing from 1 to  $n$ , we obtain

$$\|\nabla \cdot \theta^n\|_{L^2(\Omega)}^2 + \delta \sum_{j=1}^n \left( \|\partial_t \xi^n\|_{L^2(\Omega)}^2 + \|\partial_t \theta^n\|_{L^2(\Omega)}^2 \right) \leq C \left( h_\sigma^{2r+2} + \delta^2 \right). \tag{4.28}$$

Choosing  $v_h = \zeta^n$  in (4.7)(c) and applying Cauchy-Schwarz's inequality, Young's inequality, and (4.28), we have

$$\begin{aligned}
\|\zeta^n\|_{L^2(\Omega)}^2 - \|\zeta^{n-1}\|_{L^2(\Omega)}^2 &= 2\delta (\nabla \cdot (\sigma - \sigma_h)^n, \zeta^n) + 2\delta (R_2^n, \zeta^n) \\
&\leq C\delta \left[ \|\nabla \cdot (\sigma - \sigma_h)^n\|_{L^2(\Omega)}^2 + \|R_2^n\|_{L^2(\Omega)}^2 + \|\zeta^n\|_{L^2(\Omega)}^2 \right].
\end{aligned} \tag{4.29}$$

Summing from 1 to  $n$  and using the Gronwall lemma, we obtain

$$\|\zeta^n\|_{L^2(\Omega)}^2 \leq \|\zeta^0\|_{L^2(\Omega)}^2 + C\delta \sum_{j=1}^n \left[ \|\nabla \cdot (\sigma - \sigma_h)^j\|_{L^2(\Omega)}^2 + \|R_2^j\|_{L^2(\Omega)}^2 \right]. \tag{4.30}$$

Note that

$$\delta \sum_{j=1}^n \|R_2^j\|_{L^2(\Omega)}^2 \leq C\delta^3 \sum_{j=1}^n \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(L^2(\Omega))}^2 \leq C\delta^2 \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^\infty(L^2(\Omega))}^2. \tag{4.31}$$

Substituting (4.31) into (4.30) and using (3.2), (4.20), and the triangle inequality, we get

$$\|\zeta^n\|_{L^2(\Omega)}^2 \leq C \left( h_u^{2k+2} + h_\sigma^{2r^*} + \delta^2 \right). \tag{4.32}$$

Combining (3.2), (3.3), (4.20), (4.28), (4.32), and Lemma 4.1, we apply the triangle inequality to complete the proof.  $\square$

**Table 1:** The errors and convergence order.

$(h, \delta)$	$\ u - u_h\ _{L^\infty(L^2(\Omega))}$	Order	$\ \lambda - \lambda_h\ _{L^\infty(L^2(\Omega))}$	Order
$\left(\frac{\sqrt{2}}{8}, \frac{1}{8}\right)$	$1.4527e - 002$		$1.3532e - 001$	
$\left(\frac{\sqrt{2}}{16}, \frac{1}{16}\right)$	$6.5250e - 003$	1.15	$6.9202e - 002$	0.97
$\left(\frac{\sqrt{2}}{32}, \frac{1}{32}\right)$	$3.0841e - 003$	1.08	$3.5363e - 002$	0.97
$(h, \delta)$	$\ \sigma - \sigma_h\ _{L^\infty(L^2(\Omega))}$	Order	$\ \sigma - \sigma_h\ _{L^\infty(H(\text{div}; \Omega))}$	Order
$\left(\frac{\sqrt{2}}{8}, \frac{1}{8}\right)$	$3.3512e - 001$		$3.8892e - 001$	
$\left(\frac{\sqrt{2}}{16}, \frac{1}{16}\right)$	$1.6734e - 001$	1.00	$1.7291e - 001$	1.17
$\left(\frac{\sqrt{2}}{32}, \frac{1}{32}\right)$	$8.3746e - 002$	1.00	$8.4431e - 002$	1.03

## 5. Numerical Example

In this section, we analyse some numerical results to illustrate the efficiency of the proposed method. We consider the following 2D parabolic partial integrodifferential equations with initial-boundary value condition:

$$\begin{aligned}
 u_t - \nabla \cdot \left( a(x, t) \nabla u + b(x, t) \int_0^t \nabla u \, ds \right) &= f(x, t), \quad (x, t) \in \Omega \times J, \\
 u(x, t) &= 0, \quad (x, t) \in \partial\Omega \times \bar{J}, \\
 u(x, 0) &= \sin(\pi x_1) \sin(\pi x_2), \quad x \in \Omega,
 \end{aligned} \tag{5.1}$$

where  $\Omega = [0, 1] \times [0, 1]$ ,  $J = (0, 1]$ ,  $a(x, t) = 1 + x_1^2 + 2x_2^2$ ,  $b(x, t) = 1 + 2x_1^2 + x_2^2$ ,  $x = (x_1, x_2)$ , and  $f(x, t)$  is chosen so that the exact solution for the scalar unknown function is

$$u(x, t) = e^{-t} \sin(\pi x_1) \sin(\pi x_2). \tag{5.2}$$

The corresponding exact gradient is

$$\lambda(x, t) = (\lambda_1, \lambda_2) = (\pi e^{-t} \cos(\pi x_1) \sin(\pi x_2), \pi e^{-t} \sin(\pi x_1) \cos(\pi x_2)), \tag{5.3}$$

The exact solution  $u$  at  $t = 1$

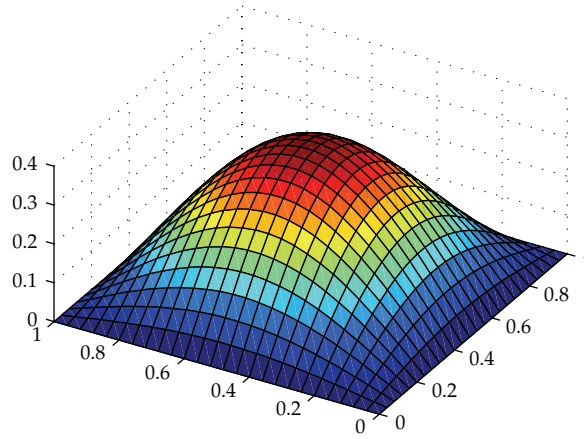


Figure 1: The exact solution  $u$ .

The numerical solution  $u_h$  at  $t = 1$

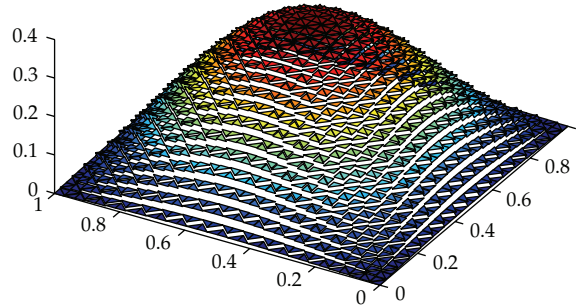
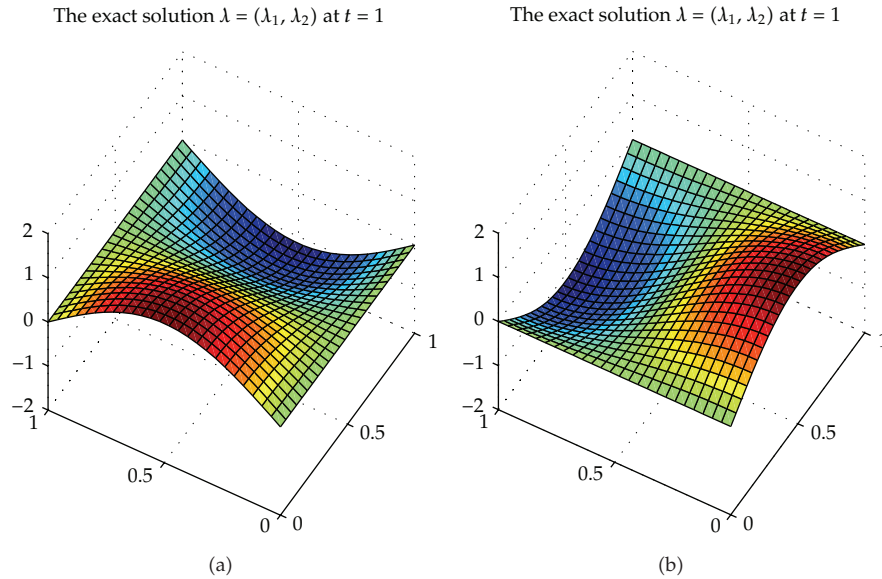


Figure 2: The numerical solution  $u_h$ .

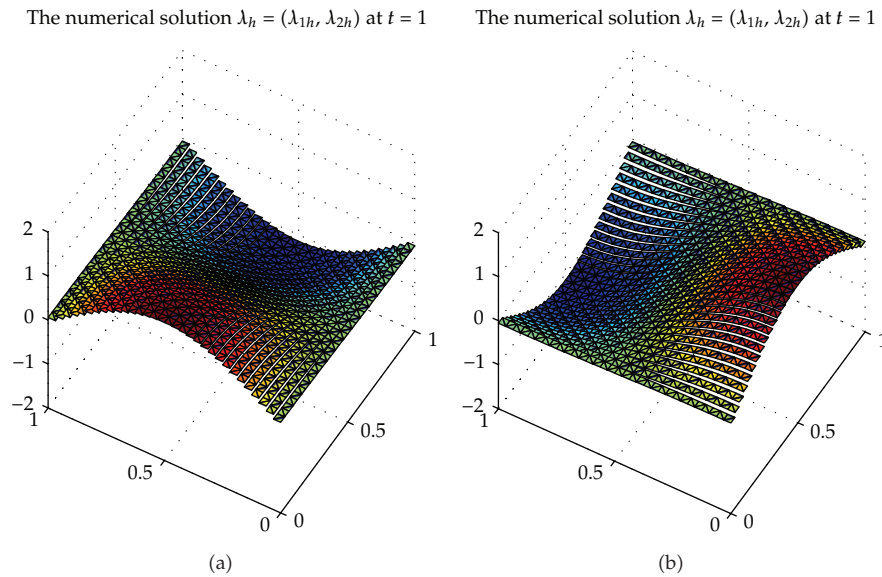
and its exact flux is

$$\begin{aligned} \sigma(x, t) = (\sigma_1, \sigma_2) = & \left( \pi \left( 1 + (2 - e^{-t})x_1^2 + (1 + e^{-t})x_2^2 \right) \cos(\pi x_1) \sin(\pi x_2), \right. \\ & \left. \pi \left( 1 + (2 - e^{-t})x_1^2 + (1 + e^{-t})x_2^2 \right) \sin(\pi x_1) \cos(\pi x_2) \right). \end{aligned} \tag{5.4}$$

Dividing the domain  $\Omega$  into the triangulations of mesh size  $h_u = h_\sigma = h$  uniformly, considering the piecewise constant space  $X_{h_u}$  with index  $k = 0$  for the scalar unknown function  $u$  and the lowest-order Raviart-Thomas triangular space  $\mathbf{V}_{h_\sigma}$  [42, 45] for the gradient  $\lambda$  and the flux  $\sigma$  and using the backward Euler procedure with uniform time step length  $\delta = 1/M$ , we obtain some convergence results for  $\|u - u_h\|_{L^\infty(L^2(\Omega))}$ ,  $\|\lambda - \lambda_h\|_{L^\infty(L^2(\Omega))}$ ,  $\|\sigma - \sigma_h\|_{L^\infty(L^2(\Omega))}$  and  $\|\sigma - \sigma_h\|_{L^\infty(H(\text{div}; \Omega))}$  with  $h = \sqrt{2}\delta = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32$  in Table 1. With time  $t = 1$ ,  $h = \sqrt{2}\delta = \sqrt{2}/32$ , the exact solution  $u$ ,  $\lambda$ ,  $\sigma$  is shown in Figures 1, 3, and 5, respectively, and the corresponding numerical solution  $u_h$ ,  $\lambda_h$ ,  $\sigma_h$  is shown in Figures 2, 4, and 6, respectively.



**Figure 3:** The exact gradient  $\lambda_1$ (a) and  $\lambda_2$ (b).



**Figure 4:** The numerical gradient  $\lambda_{1h}$  (a) and  $\lambda_{2h}$  (b).

We can see from Table 1 that the convergence rate is order 1 which confirms the theoretical results of Theorem 4.2 for the above chosen spaces  $X_{h_u}$  and  $\mathbf{V}_{h_\sigma}$ . The numerical results in Table 1 and Figures 1–6 show that new positive definite expanded mixed scheme is efficient.

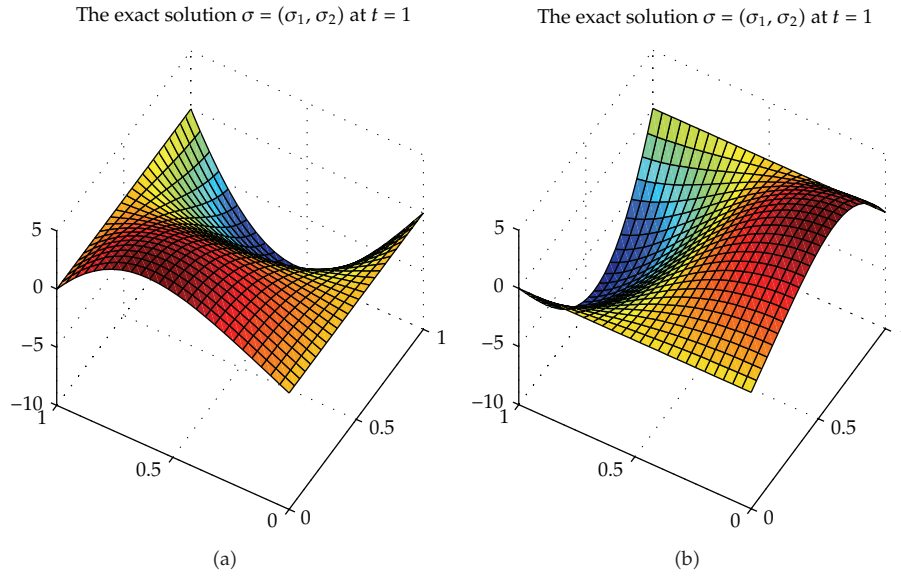


Figure 5: The exact flux  $\sigma_1$  (a) and  $\sigma_2$  (b).

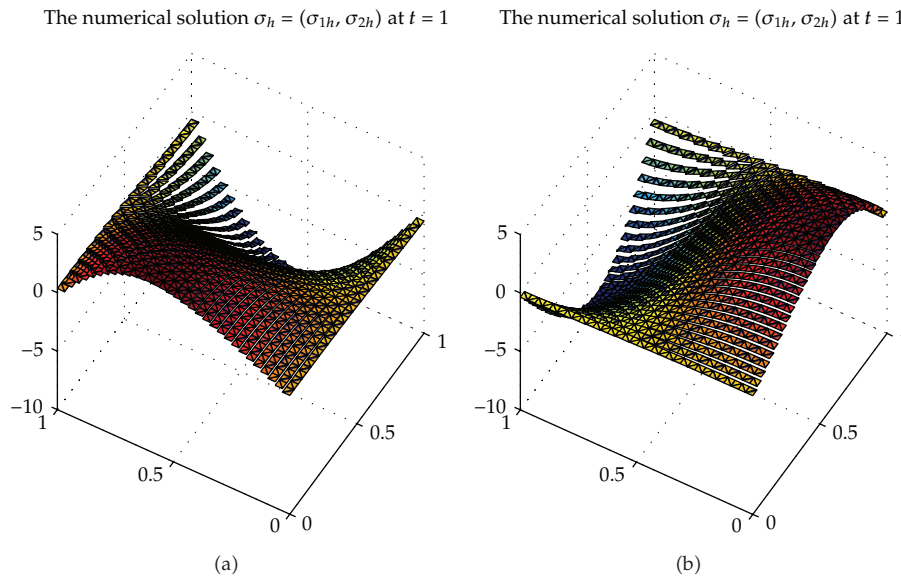


Figure 6: The numerical flux  $\sigma_{1h}$ (a) and  $\sigma_{2h}$ (b).

## 6. Concluding Remarks

In the paper, a new expanded mixed finite element method based on a positive definite system is proposed for parabolic partial integrodifferential equation. Compared to expanded mixed method and standard mixed methods, the new expanded mixed element system is symmetric positive definite and both the gradient equation and the flux equation are separated from its scalar unknown equation. The existence and uniqueness for semidiscrete

scheme are proved and error estimates are derived for both semidiscrete and fully discrete schemes. Finally, some numerical results are provided to confirm our theoretical analysis. In the near future, we will study the other evolution equations such as hyperbolic wave equation, and miscible displacement of compressible flow in porous media.

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