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Research Article

The Expression of the Drazin Inverse with Rank Constraints

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By using the matrix decomposition and the reverse order law, we provide some new expressions of the Drazin inverse for any 2×2 block matrix with rank constraints.

1. Introduction

Let A be a square complex matrix. The symbols $r(A)$ and $A^{\dagger}$ stand for the rank and the Moore-Penrose inverse of the matrix A, respectively. The Drazin inverse A^D of A is the unique matrix satisfying

$$A^{k+1}A^D = A^k, \qquad A^D A A^D = A^D, \qquad A A^D = A^D A, \tag{1.1}$$

where $k = \operatorname{ind}(A)$ is the index of A, the smallest nonnegative integer such that $r(A^{k+1}) = r(A^k)$. We write $A^{\pi} = I - AA^D$.

The Drazin inverse of a square matrix plays an important role in various fields like singular differential equations and singular difference equations, Markov chains, and iterative methods.

The problem of finding explicit representations for the Drazin inverse of a complex block matrix,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.2}$$

in terms of its blocks was posed by Campbell and Meyer [1, 2] in 1979. Many authors have considered this problem and have provided formulas for M^D under some specific conditions [3–6].

In this paper, under rank constraints, we will present some new representations of M^D which have not been discussed before.

2. Preliminary

Lemma 2.1 (see [4]). *Let P and Q be square matrices of the same order.*

If $PQ=0$, then

$$(P+Q)^D = Q^{\pi}\left[\sum_{i=0}^{k-1} Q^i\left(P^D\right)^i\right]P^D + Q^D\left[\sum_{i=0}^{k-1}\left(Q^D\right)^i P^i\right]P^{\pi}, \tag{2.1}$$

where $\max\{\operatorname{ind}(P),\ \operatorname{ind}(Q)\} \le k \le \operatorname{ind}(P) + \operatorname{ind}(Q)$.

If $PQ=0$ and $QP=0$, then $(P+Q)^D = P^D + Q^D$.

Lemma 2.2 (see [7]). *Let*

$$M_1 = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}, \qquad M_2 = \begin{pmatrix} B & C \\ 0 & A \end{pmatrix}, \tag{2.2}$$

where A, B are square matrices with $\operatorname{ind}(A) = r,\ \operatorname{ind}(B) = s$. *Then*

$$M_1^D = \begin{pmatrix} A^D & 0 \\ X & B^D \end{pmatrix}, \qquad M_2^D = \begin{pmatrix} B^D & X \\ 0 & A^D \end{pmatrix}, \tag{2.3}$$

where

$$X = \left(B^D\right)^2\left[\sum_{i=0}^{r-1}\left(B^D\right)^i C A^i\right]A^{\pi} + B^{\pi}\left(\sum_{i=0}^{s-1} B^i C\left(A^D\right)^i\right)\left(A^D\right)^2 - B^D C A^D. \tag{2.4}$$

Lemma 2.3 (see [8]). *Let $A = A_1A_2\cdots A_n$, $X = A_n^D A_{n-1}^D \cdots A_1^D$. Then $X = A^D$ if and only if $A_1, A_2, \ldots, A_n, A$ satisfy*

$$r\begin{pmatrix} (-1)^n A^{2k+1} & A^k\overline{E} \\ \overline{E}^T A^k & \overline{N} \end{pmatrix} = r\left(A^k\right) + r\left(A_1^k\right) + \cdots + r\left(A_n^k\right), \tag{2.5}$$

where $A_i \in C^{m\times m}$, $i = 1, 2, \ldots, n$, $\overline{E} = (0, \ldots, 0, I_m) \in C^{m\times m(n+1)}$ and

$$\overline{N} = \begin{pmatrix} & & A_1^{2k+1} & A_1^k \\ & \vdots & \vdots & \\ A_n^{2k+1} & A_n^k A_{n-1}^k & & \\ A_n^k & & & \end{pmatrix}, \tag{2.6}$$

with $k = \max\{\operatorname{ind}(A_i),\ \operatorname{ind}(A)\}$, $1 \le i \le n$.

Lemma 2.4 (see [9]). *Let $A \in C^{m\times n},\ B \in C^{m\times k},\ C \in C^{l\times n}$. Then*

$$r\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = r(B) + r(C) + r\left[\left(I_m - BB^{\dagger}\right)A\left(I_n - C^{\dagger}C\right)\right]. \tag{2.7}$$

3. Main Results

In this section, with rank equality constraints, we consider the Drazin inverse of block matrices.

Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A \in C^{m\times m}$ is invertible and $D - CA^{-1}B \in C^{n\times n}$ is singular. It is easy to verify that M can be decomposed as

$$M = \begin{pmatrix} I_m & 0 \\ CA^{-1} & I_n \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}\begin{pmatrix} I_m & A^{-1}B \\ 0 & I_n \end{pmatrix} = M_1M_2M_3. \tag{3.1}$$

Let

$$E = (0, 0, 0, I_{m+n}), \qquad N = \begin{pmatrix} & & M_1^{2k+1} & M_1^k \\ & M_2^{2k+1} & M_2^kM_1^k & \\ M_3^{2k+1} & M_3^kM_2^k & & \\ M_3^k & & & \end{pmatrix}, \tag{3.2}$$

where $k = \max\{\operatorname{ind}(M_2),\ \operatorname{ind}(M)\}$. According to Lemma 2.3, we have the following theorem.

Theorem 3.1. *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A \in C^{m\times m}$ is invertible and $D - CA^{-1}B \in C^{n\times n}$ is singular. If*

$$r\left(M^k\right) + r\begin{pmatrix} M_2^kM_1^{-1}F_{M^k} & 0 \\ 0 & E_{M^k}M_3^{-1}M_2^k \end{pmatrix} = r\left(M_2^k\right), \tag{3.3}$$

where $k = \max\{\operatorname{ind}(M_2),\ \operatorname{ind}(M)\}$, $E_{M^k} = I - M^k(M^k)^{\dagger}$, $F_{M^k} = I - (M^k)^{\dagger}M^k$, then M^D has the following form:

$$M^D = \begin{pmatrix} A^D + A^{-1}B\left(D - CA^{-1}B\right)^DCA^{-1} & -A^{-1}B\left(D - CA^{-1}B\right)^D \\ -\left(D - CA^{-1}B\right)^DCA^{-1} & \left(D - CA^{-1}B\right)^D \end{pmatrix}. \tag{3.4}$$

Proof. From Lemma 2.3 and (3.1), we know that if

$$r\begin{pmatrix} -M^{2k+1} & M^kE \\ E^TM^k & N \end{pmatrix} = r\left(M^k\right) + r\left(M_1^k\right) + r\left(M_2^k\right) + r\left(M_3^k\right), \tag{3.5}$$

where $k = \max\{\operatorname{ind}(M_2),\ \operatorname{ind}(M)\}$, then

$$\begin{aligned}M^D = M_3^{-1}M_2^D M_1^{-1} &= \begin{pmatrix} I_m & -A^{-1}B \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A^D & 0 \\ 0 & \left(D-CA^{-1}B\right)^D \end{pmatrix}\begin{pmatrix} I_m & 0 \\ -CA^{-1} & I_n \end{pmatrix} \\ &= \begin{pmatrix} A^D + A^{-1}B\left(D-CA^{-1}B\right)^D CA^{-1} & -A^{-1}B\left(D-CA^{-1}B\right)^D \\ -\left(D-CA^{-1}B\right)^D CA^{-1} & \left(D-CA^{-1}B\right)^D \end{pmatrix}.\end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned} r\begin{pmatrix} -M^{2k+1} & M^kE \\ E^TM^k & N \end{pmatrix} &= r\begin{pmatrix} -M^{2k+1} & 0 & 0 & 0 & M^k \\ 0 & 0 & 0 & M_1^{2k+1} & M_1^k \\ 0 & 0 & M_2^{2k+1} & M_2^kM_1^k & 0 \\ 0 & M_3^{2k+1} & M_3^kM_2^k & 0 & 0 \\ M^k & M_3^k & 0 & 0 & 0 \end{pmatrix} \\ &= r\begin{pmatrix} 0 & M^{k+1}M_3^k & 0 & 0 & M^k \\ 0 & 0 & 0 & M_1^{2k+1} & M_1^k \\ 0 & 0 & M_2^{2k+1} & M_2^kM_1^k & 0 \\ 0 & M_3^{2k+1} & M_3^kM_2^k & 0 & 0 \\ M^k & M_3^k & 0 & 0 & 0 \end{pmatrix} \\ &= r\begin{pmatrix} 0 & 0 & 0 & 0 & M^k \\ 0 & -M_1^kMM_3^k & 0 & M_1^{2k+1} & M_1^k \\ 0 & 0 & M_2^{2k+1} & M_2^kM_1^k & 0 \\ 0 & M_3^{2k+1} & M_3^kM_2^k & 0 & 0 \\ M^k & M_3^k & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.7}$$

Let

$$G = \left(0,0,0,M^k\right), \qquad J = \begin{pmatrix} 0 \\ 0 \\ 0 \\ M^k \end{pmatrix}, \qquad H = \begin{pmatrix} -M_1^kMM_3^k & 0 & M_1^{2k+1} & M_1^k \\ 0 & M_2^{2k+1} & M_2^kM_1^k & 0 \\ M_3^{2k+1} & M_3^kM_2^k & 0 & 0 \\ M_3^k & 0 & 0 & 0 \end{pmatrix}. \tag{3.8}$$

From Lemma 2.4, we have

$$
\begin{aligned}
r\begin{pmatrix} 0 & G \\ J & H \end{pmatrix} &= r\begin{pmatrix} H & J \\ G & 0 \end{pmatrix} = r(J) + r(G) + r\left[\left(I_{4m} - JJ^{\dagger}\right)H\left(I_{4m} - G^{\dagger}G\right)\right] \\
&= 2r\left(M^{k}\right) + r\left[\left(I_{4m} - JJ^{\dagger}\right)H\left(I_{4m} - G^{\dagger}G\right)\right].
\end{aligned}
\tag{3.9}
$$

Note that $J^{\dagger} = (0,0,0,(M^{k})^{\dagger})$, $G^{\dagger} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (M^{k})^{\dagger} \end{pmatrix}$. Then we get

$$
\begin{aligned}
&\left(I_{4m} - JJ^{\dagger}\right)H\left(I_{4m} - G^{\dagger}G\right) \\
&= \begin{pmatrix} -M_1^{k} M M_3^{k} & 0 & M_1^{2k+1} & M_1^{k}\left[I_{4m} - \left(M^{k}\right)^{\dagger} M^{k}\right] \\ 0 & M_2^{2k+1} & M_2^{k} M_1^{k} & 0 \\ M_3^{2k+1} & M_3^{k} M_2^{k} & 0 & 0 \\ \left[I_{4m} - M^{k}\left(M^{k}\right)^{\dagger}\right] M_3^{k} & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{3.10}
$$

Let $E_{M^k} = I_{4m} - M^{k}(M^{k})^{\dagger}$, $F = I_{4m} - (M^{k})^{\dagger} M^{k}$. Then $(I_{4m} - JJ^{\dagger})H(I_{4m} - G^{\dagger}G)$ can be rewritten as the following three matrix products:

$$
\begin{pmatrix} M_1^{k} & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & M_3^{k} & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}
\begin{pmatrix} -M & 0 & M_1 & F_M^{k} \\ 0 & M_2^{2k+1} & M_2^{k} & 0 \\ M_3 & M_2^{k} & 0 & 0 \\ E_M^{k} & 0 & 0 & 0 \end{pmatrix}
\begin{pmatrix} M_3^{k} & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & M_1^{k} & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}.
\tag{3.11}
$$

Since M_1, M_3 is nonsingular, then

$$
\begin{aligned}
&r\left[\left(I_{4m} - JJ^{\dagger}\right)H\left(I_{4m} - G^{\dagger}G\right)\right] \\
&\quad= r\begin{pmatrix} -M & 0 & M_1 & F_{M^k} \\ 0 & M_2^{2k+1} & M_2^{k} & 0 \\ M_3 & M_2^{k} & 0 & 0 \\ E_{M^k} & 0 & 0 & 0 \end{pmatrix} \\
&\quad= r\begin{pmatrix} -M & 0 & I_{m+n} & F_{M^k} \\ 0 & M_2^{2k+1} & M_2^{k} M_1^{-1} & 0 \\ I_{m+n} & M_3^{-1} M_2^{k} & 0 & 0 \\ E_{M^k} & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= r\begin{pmatrix} 0 & M_1M_2^{k+1} & I_{m+n} & F_{M^k} \\ 0 & M_2^{2k+1} & M_2^kM_1^{-1} & 0 \\ I_{m+n} & M_3^{-1}M_2^k & 0 & 0 \\ 0 & -E_{M^k}M_3^{-1}M_2^k & 0 & 0 \end{pmatrix} \\
&= r\begin{pmatrix} 0 & M_1M_2^{k+1} & I_{m+n} & F_{M^k} \\ 0 & 0 & 0 & -M_2^kM_1^{-1}F_{M^k} \\ I_{m+n} & M_3^{-1}M_2^k & 0 & 0 \\ 0 & -E_{M^k}M_3^{-1}M_2^k & 0 & 0 \end{pmatrix} \\
&= r\begin{pmatrix} 0 & 0 & I_{m+n} & 0 \\ 0 & 0 & 0 & -M_2^kM_1^{-1}F_{M^k} \\ I_{m+n} & 0 & 0 & 0 \\ 0 & -E_{M^k}M_3^{-1}M_2^k & 0 & 0 \end{pmatrix} \\
&= 2(m+n) + r\begin{pmatrix} M_2^kM_1^{-1}F_{M^k} & 0 \\ 0 & E_{M^k}M_3^{-1}M_2^k \end{pmatrix}.
\end{aligned} \tag{3.12}
$$

Thus, we have

$$
\begin{aligned}
&r\begin{pmatrix} -M^{2k+1} & M^kE \\ E^TM^k & N \end{pmatrix} \\
&\quad = r\begin{pmatrix} 0 & G \\ J & H \end{pmatrix} = 2r\left(M^k\right) + 2(m+n) + r\begin{pmatrix} M_2^kM_1^{-1}F_{M^k} & 0 \\ 0 & E_{M^k}M_3^{-1}M_2^k \end{pmatrix}.
\end{aligned} \tag{3.13}
$$

From the above equality and the condition (3.3), (3.5) is easily verified. □

Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A \in C^{m\times m}$, $D \in C^{n\times n}$. It is easy to verify that the matrix M can be decomposed as

$$
M = \begin{pmatrix} I_m & 0 \\ CA^D & I_n \end{pmatrix}\begin{pmatrix} A & A^\pi B \\ CA^\pi & S \end{pmatrix}\begin{pmatrix} I_m & A^DB \\ 0 & I_n \end{pmatrix} = A_1A_2A_3, \tag{3.14}
$$

where $S = D - CA^DB$ is the generalized Schur complement of A in M.

Let

$$N_1 = \begin{pmatrix} & & A_1^{2k+1} & A_1^k \\ & A_2^{2k+1} & A_2^k A_1^k & \\ A_3^{2k+1} & A_3^k A_2^k & & \\ A_3^k & & & \end{pmatrix}, \tag{3.15}$$

where $k = \max\{\operatorname{ind}(A_2), \operatorname{ind}(M)\}$. Then we have the following theorem.

Theorem 3.2. *If $AA^\pi B = 0$, $CA^\pi B = 0$, and the matrices A_1, A_2, A_3, M satisfy*

$$r\left(M^k\right) + r\begin{pmatrix} A_2^k A_1^{-1} F_{M^k} & 0 \\ 0 & E_{M^k} A_3^{-1} A_2^k \end{pmatrix} = r\left(A_2^k\right), \tag{3.16}$$

then,

$$M^D = \begin{pmatrix} A^D + A^\pi BXA^D + \left(A^\pi BS^D - A^D B\right)Y & A^\pi B\left(S^D\right)^2 - A^D BS^D \\ Y & S^D \end{pmatrix}, \tag{3.17}$$

where $S = D - CA^D B$, $Y = (S^D)^2[\sum_{i=0}^{\operatorname{ind}(A)-1}(S^D)^i CA^\pi A^i]A^\pi - S^D CA^D$.

Proof. From Lemma 2.3 and (3.14), we get that if

$$r\begin{pmatrix} -M^{2k+1} & M^k E \\ E^T M^k & N_1 \end{pmatrix} = r\left(M^k\right) + r\left(A_1^k\right) + r\left(A_2^k\right) + r\left(A_3^k\right), \tag{3.18}$$

then

$$M^D = A_3^{-1} A_2^D A_1^{-1} = \begin{pmatrix} I_m & -A^D B \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A & A^\pi B \\ CA^\pi & S \end{pmatrix}^D \begin{pmatrix} I_m & 0 \\ -CA^D & I_n \end{pmatrix}. \tag{3.19}$$

Similar to the proof of Theorem 3.1, we derive that the rank condition (3.18) can be simplified as (3.16).

Next, we will give the representation for A_2^D. Let

$$A_2 = \begin{pmatrix} A & 0 \\ CA^\pi & S \end{pmatrix} + \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix} = P + Q. \tag{3.20}$$

Since $AA^\pi B = 0$, $CA^\pi B = 0$, and $A^\pi A^\pi = A^\pi$, then $PQ = 0$. From Lemma 2.2, we get

$$Q^D = 0, \qquad P^D = \begin{pmatrix} A^D & 0 \\ X & S^D \end{pmatrix}, \quad \text{where } X = \left(S^D\right)^2 \left[\sum_{i=0}^{\operatorname{ind}(A)-1} \left(S^D\right)^i CA^\pi A^i\right] A^\pi. \tag{3.21}$$

From Lemma 2.1 and the fact $Q^i = 0,\ i \geq 2$, it follows that

$$\begin{aligned} A_2^D &= (P+Q)^D = \left(I_{m+n} - QQ^D\right)\left(I_{m+n} + QP^D\right)P^D \\ &= \left[I_{m+n} + \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^D & 0 \\ X & S^D \end{pmatrix}\right]\begin{pmatrix} A^D & 0 \\ X & S^D \end{pmatrix} \\ &= \begin{pmatrix} I_m + A^\pi BX & A^\pi BS^D \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A^D & 0 \\ X & S^D \end{pmatrix} \\ &= \begin{pmatrix} A^D + A^\pi B\left(XA^D + S^D X\right) & A^\pi B\left(S^D\right)^2 \\ X & S^D \end{pmatrix}. \end{aligned} \tag{3.22}$$

Substituting A_2^D in (3.19), the conclusion can be obtained. □

From Theorem 3.2, we can easily obtain the following corollaries.

Corollary 3.3. *If $A^\pi B = 0$ and the rank equality* (3.16) *hold, then*

$$M^D = \begin{pmatrix} A^D - A^D B\left(X - S^D CA^D\right) & -A^D BS^D \\ X - S^D CA^D & S^D \end{pmatrix}, \tag{3.23}$$

where S and X are the same as in Theorem 3.2.

Corollary 3.4. *If $A^\pi B = 0,\ CA^\pi = 0$, and the rank equality* (3.16) *hold, then*

$$M^D = \begin{pmatrix} A^D + A^D BS^D CA^D & -A^D BS^D \\ -S^D CA^D & S^D \end{pmatrix}, \tag{3.24}$$

where $S = D - CA^D B$.

Next, we will consider another decomposition of M involving the generalized Schur complement $S_B = C - DB^D A$.

Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A, D \in C^{m\times m}$. Then M can be decomposed as

$$M = \begin{pmatrix} I_m & 0 \\ DB^D & I_m \end{pmatrix}\begin{pmatrix} B^\pi A & B \\ S_B & DB^\pi \end{pmatrix}\begin{pmatrix} I_m & 0 \\ B^D A & I_m \end{pmatrix} = V_1 V_2 V_3, \tag{3.25}$$

where $S_B = C - DB^D A$ is the generalized Schur complement of B in M.

Let

$$E_1 = (0, 0, 0, I_{2m}), \qquad N_2 = \begin{pmatrix} & & V_1^{2k+1} & V_1^k \\ & V_2^{2k+1} & V_2^k V_1^k & \\ V_3^{2k+1} & V_3^k V_2^k & & \\ V_3^k & & & \end{pmatrix}, \tag{3.26}$$

where $k = \max\{\operatorname{ind}(V_2),\ \operatorname{ind}(M)\}$. Then we have the following theorem.

Theorem 3.5. *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A,\ D \in C^{m\times m}$. If $B^{\pi}A = 0$, $CB = DB^{D}AB$ and the matrices $V_1, V_2, V_3,\ M$ satisfy the following rank equality:*

$$r\left(M^k\right) + r\begin{pmatrix} V_2^k V_1^{-1} F_{M^k} & 0 \\ 0 & E_{M^k} V_3^{-1} V_2^k \end{pmatrix} = r\left(V_2^k\right), \tag{3.27}$$

then

$$M^D = \begin{pmatrix} BX^2(XS_B - DB^D) & BX^2 \\ (I - B^D ABX)X(XS_B - DB^D) & (I - B^D ABX)X \end{pmatrix}, \tag{3.28}$$

where $X = (DB^{\pi})^D$.

Proof. From Lemma 2.3 and (3.25), we get that if the following rank condition

$$r\begin{pmatrix} -M^{2k+1} & M^k E \\ E^T M^k & N_2 \end{pmatrix} = r\left(M^k\right) + r\left(V_1^k\right) + r\left(V_2^k\right) + r\left(V_3^k\right) \tag{3.29}$$

holds, then $M = V_3^{-1} V_2^D V_1^{-1}$. From the same method used in Theorem 3.1, we can verify that the above condition (3.29) can be reduced to (3.27).

Next, we will give the representation for V_2^D. For $B^{\pi}A = 0$, we write

$$V_2 = \begin{pmatrix} 0 & 0 \\ S_B & DB^{\pi} \end{pmatrix} + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = P_1 + Q_1, \tag{3.30}$$

where $S_B = C - DB^D A$.

From the condition $CB = DB^D AB$, we get $P_1 Q_1 = 0$, according to Lemma 2.2, we have

$$Q_1^D = 0, \qquad P_1^D = \begin{bmatrix} 0 & 0 \\ \left((DB^{\pi})^D\right)^2 S_B & (DB^{\pi})^D \end{bmatrix}. \tag{3.31}$$

Let $X = (DB^{\pi})^D$. By Lemma 2.1 and the fact $Q_1^i = 0,\ i \geq 2$, we get

$$\begin{aligned} V_2^D &= \left(I_{2m} - Q_1 Q_1^D\right)\left(I_{2m} + Q_1 P_1^D\right) P_1^D \\ &= \left[I_{2m} + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ X^2 S_B & X \end{pmatrix}\right]\begin{pmatrix} 0 & 0 \\ X^2 S_B & X \end{pmatrix} \\ &= \begin{pmatrix} I_m + BX^2 S_B & BX \\ 0 & I_m \end{pmatrix}\begin{pmatrix} 0 & 0 \\ X^2 S_B & X \end{pmatrix} = \begin{pmatrix} BX^3 S_B X & BX^2 \\ X^2 S_B & X \end{pmatrix}. \end{aligned} \tag{3.32}$$

Therefore, we get

$$
\begin{aligned}
M^D &= \begin{pmatrix} I_m & 0 \\ -B^D A & I_m \end{pmatrix} \begin{pmatrix} BX^3 S_B X & BX^2 \\ X^2 S_B & X \end{pmatrix} \begin{pmatrix} I_m & 0 \\ -DB^D & I_m \end{pmatrix} \\
&= \begin{pmatrix} BX^2 (XS_B - DB^D) & BX^2 \\ (I - B^D ABX) X (XS_B - DB^D) & (I - B^D ABX) X \end{pmatrix}.
\end{aligned}
\tag{3.33}
$$

□

Remark 3.6. In addition to the decompositions of M in (3.14) and (3.25), the matrix M also can be decomposed as other matrix products involving the generalized Schur complements $S_C = B - AC^D D$ or $S_D = A - BD^D C$. In these cases, new formulas for M^D would be derived by the method used in this paper.

4. Conclusion

In this paper, we mainly discuss the Drazin inverse of block matrices under rank equality constraints. Comparing with the existing results, it is obvious that our results have more strong restrictions, but the methods used in this paper are different from those in previous relevant paper.

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