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Research Article

Lattices Generated by Orbits of Subspaces under Finite Singular Orthogonal Groups II

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Let $\mathbb{F}_q^{(2\nu+\delta+l)}$ be a $(2\nu+\delta+l)$-dimensional vector space over the finite field $\mathbb{F}_q$. In this paper we assume that $\mathbb{F}_q$ is a finite field of odd characteristic, and $O_{2\nu+\delta+l,\,\Delta}(\mathbb{F}_q)$ the singular orthogonal groups of degree $2\nu+\delta+l$ over $\mathbb{F}_q$. Let $\mathcal{M}$ be any orbit of subspaces under $O_{2\nu+\delta+l,\,\Delta}(\mathbb{F}_q)$. Denote by $\mathcal{L}$ the set of subspaces which are intersections of subspaces in $\mathcal{M}$, where we make the convention that the intersection of an empty set of subspaces of $\mathbb{F}_q^{(2\nu+\delta+l)}$ is assumed to be $\mathbb{F}_q^{(2\nu+\delta+l)}$. By ordering $\mathcal{L}$ by ordinary or reverse inclusion, two lattices are obtained. This paper studies the questions when these lattices $\mathcal{L}$ are geometric lattices.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements, where q is an odd prime power. We choose a fixed nonsquare element z in $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$. Let $\mathbb{F}_q^{(2\nu+\delta+l)}$ be a $(2\nu+\delta+l)$-dimensional row vector space over the finite field $\mathbb{F}_q$, and let $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$ be one of the singular orthogonal groups of degree $2\nu+\delta+l$ over $\mathbb{F}_q$. There is an action of $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$ on $\mathbb{F}_q^{(2\nu+\delta+l)}$ defined as follows:

$$\begin{gathered}\mathbb{F}_q^{(2\nu+\delta+l)} \times O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q) \longrightarrow \mathbb{F}_q^{(2\nu+\delta+l)},\\ ((x_1,x_2,\ldots,x_{2\nu+\delta+l}),T) \longmapsto (x_1,x_2,\ldots,x_{2\nu+\delta+l})T.\end{gathered} \tag{1.1}$$

Let P be an m-dimensional subspace of $\mathbb{F}_q^{(2\nu+\delta+l)}$ $(1 \le m \le 2\nu+\delta+l)$, and $v_1, v_2, \ldots, v_m$ be a basis of P. Then, the $m \times (2\nu + \delta + l)$ matrix:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \tag{1.2}$$

is called a *matrix representation* of P. We usually denote a matrix representation of the m-dimensional subspace P still by P. The above action induces an action on the set of subspaces of $\mathbb{F}_q^{(2\nu+\delta+l)}$, that is, a subspace P is carried by $T \in O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$ into the subspace PT. The set of subspaces of $\mathbb{F}_q^{(2\nu+\delta+l)}$ is partitioned into orbits under $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$. Clearly, $\{0\}$ and $\{\mathbb{F}_q^{(2\nu+\delta+l)}\}$ are two trivial orbits. Let $\mathcal{M}$ be any orbit of subspaces under $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$. Denote the set of subspaces which are intersections of subspaces in $\mathcal{M}$ by $\mathcal{L}(\mathcal{M})$ and call $\mathcal{L}(\mathcal{M})$ the set of subspaces generated by $\mathcal{M}$. We agree that the intersection of an empty set of subspaces is $\mathbb{F}_q^{(2\nu+\delta+l)}$. Then, $\mathbb{F}_q^{(2\nu+\delta+l)} \in \mathcal{L}(\mathcal{M})$. Partially ordering $\mathcal{L}(\mathcal{M})$ by ordinary or reverse inclusion, we get two posets and denote them by $\mathcal{L}_O(\mathcal{M})$ and $\mathcal{L}_R(\mathcal{M})$, respectively. Clearly, for any two elements $P, Q \in \mathcal{L}_O(\mathcal{M})$,

$$P \wedge Q = P \cap Q, \qquad P \vee Q = \cap\{R \in \mathcal{L}_O(\mathcal{M}) : R \supseteq \langle P, Q\rangle\}, \tag{1.3}$$

where $\langle P, Q\rangle$ is a subspace generated by P and Q. Therefore, $\mathcal{L}_O(\mathcal{M})$ is a finite lattice.

Similarly, for any two elements $P, Q \in \mathcal{L}_R(\mathcal{M})$,

$$P \wedge Q = \cap\{R \in \mathcal{L}_R(\mathcal{M}) : R \supseteq \langle P, Q\rangle\}, \qquad P \vee Q = P \cap Q, \tag{1.4}$$

so $\mathcal{L}_R(\mathcal{M})$ is also a finite lattice. Both $\mathcal{L}_O(\mathcal{M})$ and $\mathcal{L}_R(\mathcal{M})$ are called the lattices generated by $\mathcal{M}$.

The results on the geometricity of lattices generated by subspaces in d-bounded distance-regular graphs can be found in Guo et al. [1]; on the geometricity and the characteristic polynomial of lattices generated by orbits of flats under finite affine-classical groups can be found in Wang and Feng [2], Wang and Guo [3]; on inclusion relations, the geometricity and the characteristic polynomial of lattices generated by orbits of subspaces under finite nonsingular classical groups and a characterization of subspaces contained in lattices can be found in Huo [4–6], Huo and Wan [7, 8]; on inclusion relations, the geometricity and the characteristic polynomial of lattices generated by orbits of subspaces under finite singular symplectic groups, singular unitary groups, and singular pseudosymplectic groups and a characterization of subspaces contained in lattices can be found in Gao and You [9–12]. In [13], the authors studied the various lattices $\mathcal{L}_O(\mathcal{M})$ and $\mathcal{L}_R(\mathcal{M})$ generated by different orbits $\mathcal{M}$ of subspaces under singular orthogonal group $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$. The study contents include the inclusion relations between different lattices, the characterization of subspaces contained in a given lattice $\mathcal{L}_R(\mathcal{M})$ (resp., $\mathcal{L}_O(\mathcal{M})$), and the characteristic polynomial of $\mathcal{L}_R(\mathcal{M})$. The purpose of this paper is to study the questions when $\mathcal{L}_R(\mathcal{M})$ (resp., $\mathcal{L}_O(\mathcal{M})$) are geometric lattices.

2. Preliminaries

In the following, we recall some definitions and facts on ordered sets and lattices (see [8, 14]).

Let A be a partially ordered set, and $a, b \in A$. We say that b *covers* a and write $a < \cdot\, b$, if $a < b$ and there exists no $c \in A$ such that $a < c < b$. An element $m \in A$ is called the *minimal element* if there exists no elements $a \in A$ such that $a < m$. If A has a unique minimal element, denote it by 0 and we say that A is a poset with 0.

Let A be a poset with 0 and $a \in A$. If all maximal ascending chains starting from 0 with endpoint a have the same finite length, this common length is called the *rank* $r(a)$ of a. If rank $r(a)$ is defined for every $a \in A$, A is said to have the rank function $r : A \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set consisting of all positive integers and 0.

A poset A is said to satisfy the *Jordan-Dedekind (JD) condition* if any two maximal chains between the same pair of elements of A have the same finite length.

Proposition 2.1 ([14, Proposition 2.1]). *Let A be a poset with 0. If A satisfies the JD condition then A has the rank function $r : A \rightarrow \mathbb{N}$ which satisfies*

(i) $r(0) = 0$,

(ii) $a < \cdot\, b \Rightarrow r(b) = r(a) + 1$.

Conversely, if A admits a function $r : A \rightarrow \mathbb{N}$ satisfying (i) and (ii), then A satisfies the JD condition with r as its rank function.

Let A be a poset with 0. An element $a \in A$ is called an atom of A if $0 < \cdot\, a$. A lattice L with 0 is called an atomic lattice (or a point lattice) if every element $a \in L \setminus \{0\}$ is a supremum of atoms, that is, $a = \sup\{b \in L \mid 0 < \cdot b \leq a\}$.

Definition 2.2 ([14, page 46]). A lattice L is called a *semimodular lattice* if for all $a, b \in L$,

$$a \wedge b < \cdot a \Longrightarrow b < \cdot a \vee b. \tag{2.1}$$

Proposition 2.3 ([14, Theorem 2.27]). *Let L be a lattice with 0. Then, L is a semimodular lattice if and only if L possesses a rank function r such that for all $x, y \in L$*

$$r(x \wedge y) + r(x \vee y) \leq r(x) + r(y). \tag{2.2}$$

Definition 2.4 ([14, page 52]). A lattice L is called a geometric lattice if it is

G_1' an atomic lattice,

G_2' a semimodular lattice,

G_3 without infinite chains in L.

According to Definition 2.2, Proposition 2.3, and Definition 2.4, we can obtain the following proposition.

Proposition 2.5. *Let L be a lattice with 0. Then, L is a geometric lattice if and only if*

G_1 *for every element $a \in L \setminus \{0\}$, $a = \sup\{b \in L \mid 0 < \cdot b \leq a\}$,*

G_2 *L possesses a rank function r and for all $x, y \in L$,* (2.2) *holds,*

G_3 *without infinite chains in L.*

Let

$$S_{2\nu+\delta,\Delta} = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & \Delta \end{pmatrix}, \qquad S_l = \begin{pmatrix} S & \\ & 0^{(l)} \end{pmatrix}, \tag{2.3}$$

where $S = S_{2\nu+\delta,\Delta}$, $\delta = 0, 1$, or 2, and

$$\Delta = \begin{cases} \phi, & \text{if } \delta = 0, \\ 1 \ or \ z, & \text{if } \delta = \ 1, \\ \begin{pmatrix} 1 & \\ & -z \end{pmatrix}, & \text{if } \delta = 2. \end{cases} \tag{2.4}$$

The set of all $(2\nu+\delta+l) \times (2\nu+\delta+l)$ nonsingular matrices T over $\mathbb{F}_q$ satisfying

$$TS_lT^t = S_l \tag{2.5}$$

forms a group which will be called the *singular orthogonal group* of degree $2\nu+\delta+l$, rank $2\nu+\delta$, and with definite part Δ over $\mathbb{F}_q$ and denoted by $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$. Clearly, $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$ consists of all $(2\nu+\delta+l) \times (2\nu+\delta+l)$ nonsingular matrices of the form:

$$T = \begin{matrix} \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} & \begin{matrix} 2\nu+\delta \\ l \end{matrix} \\ \begin{matrix} 2\nu+\delta & l \end{matrix} & \end{matrix}, \tag{2.6}$$

where $T_{11}ST_{11}^t = S$, and T_{22} is nonsingular.

Two $n \times n$ matrices A and B are called to be *cogredient* if there exists a nonsingular matrix P such that $PAP^t = B$.

An m-dimensional subspace P is said to be a *subspace of type* $(m, 2s+\gamma, s, \Gamma)$, if PS_lP^t is cogredient to $M(m, 2s+\gamma, s, \Gamma)$, where the matrix $M(m, 2s+\gamma, s, \Gamma)$, respectively, is as follows

$$\begin{aligned} M(m, 2s, s) &= \begin{pmatrix} 0 & I^{(s)} & \\ I^{(s)} & 0 & \\ & & 0^{(m-2s)} \end{pmatrix}, \quad \text{if } \gamma = 0, \\ M(m, 2s+1, s, 1) &= \begin{pmatrix} 0 & I^{(s)} & & \\ I^{(s)} & 0 & & \\ & & 1 & \\ & & & 0^{(m-2s-1)} \end{pmatrix}, \quad \text{if } \gamma = 1 \end{aligned} \tag{2.7}$$

or

$$M(m,2s+1,s,z)=\begin{pmatrix}0 & I^{(s)} & & \\ I^{(s)} & 0 & & \\ & & z & \\ & & & 0^{(m-2s-1)}\end{pmatrix},\quad \text{if } \gamma=1,$$
$$M(m,2s+2,s)=\begin{pmatrix}0 & I^{(s)} & & & \\ I^{(s)} & 0 & & & \\ & & 1 & & \\ & & & -z & \\ & & & & 0^{(m-2s-2)}\end{pmatrix},\quad \text{if } \gamma=2. \tag{2.8}$$

Let $e_1, e_2, \ldots, e_{2\nu+\delta}, e_{2\nu+\delta+1}, \ldots, e_{2\nu+\delta+l}$ be a basis of $\mathbb{F}_q^{(2\nu+\delta+l)}$, where

$$e_i=(0,\ldots,0,1,0,\ldots,0), \tag{2.9}$$

1 is in the ith position. Denote by E the l-dimensional subspace of $\mathbb{F}_q^{(2\nu+\delta+l)}$ generated by $e_{2\nu+\delta+1}, e_{2\nu+\delta+2}, \ldots, e_{2\nu+\delta+l}$. An m-dimensional subspace P is called a *subspace of type* $(m,2s+\gamma,s,\Gamma,k)$ if

(i) P is a subspace of type $(m,2s+\gamma,s,\Gamma)$,

(ii) $\dim(P\cap E)=k$.

Denote the set of all subspaces of type $(m,2s+\gamma,s,\Gamma,k)$ in $\mathbb{F}_q^{(2\nu+\delta+l)}$ by $\mathcal{M}(m,2s+\gamma,s,\Gamma,k;\ 2\nu+\delta+l,\Delta)$. By [15, Theorem 6.28], we know that $\mathcal{M}(m,2s+\gamma,s,\Gamma,k;\ 2\nu+\delta+l,\Delta)$ is nonempty if and only if

$$k\le l,$$
$$2s+\gamma\le m-k\le\begin{cases}\nu+s+\min\{\delta,\gamma\}, & \\ \text{if } \gamma\ne\delta \text{ or } \gamma=\delta \text{ and } \Gamma=\Delta, & \\ \nu+s, & \\ \text{if } \gamma=\delta=1 \text{ and } \Gamma\ne\Delta,\end{cases} \tag{2.10}$$

or

$$\min\{l,m-2s-\gamma\}\ge k\ge\begin{cases}\max\{0,m-\nu-s-\min\{\delta,\gamma\}\}, & \\ \text{if } \gamma\ne\delta \text{ or } \gamma=\delta \text{ and } \Gamma=\Delta, & \\ \max\{0,m-\nu-s\}, & \\ \text{if } \gamma=\delta=1 \text{ and } \Gamma\ne\Delta.\end{cases} \tag{2.11}$$

Moreover, if $\mathcal{M}(m,2s+\gamma,s,\Gamma,k;\ 2\nu+\delta+l,\Delta)$ is nonempty, then it forms an orbit of subspaces under $O_{2\nu+\delta+l,\Delta}(\mathbb{F}_q)$. Let $\mathcal{L}(m,2s+\gamma,s,\Gamma,k;\ 2\nu+\delta+l,\Delta)$ denote the set of subspaces which are intersections of subspaces in $\mathcal{M}(m,2s+\gamma,s,\Gamma,k;2\nu+\delta+l,\Delta)$, where we make the convention that the intersection of an empty set of subspaces of $\mathbb{F}_q^{(2\nu+\delta+l)}$ is assumed to be $\mathbb{F}_q^{(2\nu+\delta+l)}$. Partially ordering $\mathcal{L}(m,2s+\gamma,s,\Gamma,k;2\nu+\delta+l,\Delta)$ by ordinary or reverse inclusion, we get two finite lattices and denote them by $\mathcal{L}_O(m,2s+\gamma,s,\Gamma,k;2\nu+\delta+l,\Delta)$ and $\mathcal{L}_R(m,2s+\gamma,s,\Gamma,k;2\nu+\delta+l,\Delta)$, respectively.

The case $\mathcal{L}_R(m-l,2s+\gamma,s,\Gamma;\ 2\nu+\delta,\Delta)$ has been discussed in [8]. So, we only discuss the case $0\le k<l$ in this paper.

By [13], we have the following results.

Theorem 2.6. *Let $2\nu+\delta+l>m\ge 1$, $0\le k<l$, assume that $(m,2s+\gamma,s,\Gamma,k)$ satisfies conditions* (2.10) *and* (2.11)*. Then,*

$$\mathcal{L}_R(m,2s+r,s,\Gamma,k;\ 2\nu+\delta+l,\Delta)\supset\mathcal{L}_R(m_1,2s_1+\gamma_1,s_1,\Gamma_1,k_1;\ 2\nu+\delta+l,\Delta) \tag{2.12}$$

if and only if

$$\begin{gathered} k_1\le k<l,\\ 2(m-k)-2(m_1-k_1)\ge\begin{cases}(2s+\gamma)-(2s_1+\gamma_1)+|\gamma-\gamma_1|\ge 2|\gamma-\gamma_1|,\\ if\ \gamma_1\neq\gamma\ or\ \gamma_1=\gamma\ and\ \Gamma_1=\Gamma,\\ (2s+\gamma)-(2s_1+\gamma_1)+2\ge 4,\\ if\ \gamma_1=\gamma=1\ and\ \Gamma_1\neq\Gamma.\end{cases}\end{gathered} \tag{2.13}$$

Theorem 2.7. *Let $2\nu+\delta+l>m\ge 1$, $0\le k<l$. Assume that $(m,2s+\gamma,s,\Gamma,k)$ satisfies condition* (2.10)*, then $\mathcal{L}_R(m,2s+\gamma,s,\Gamma,k;\ 2\nu+\delta+l,\Delta)$ consists of $\mathbb{F}_q^{(2\nu+\delta+l)}$ and all the subspaces of type $(m_1,2s_1+\gamma_1,s_1,\Gamma_1,k_1)$, where $(m_1,2s_1+\gamma_1,s_1,\Gamma_1,k_1)$ satisfies condition* (2.13)*.*

Theorem 2.8. *Let $2\nu+\delta+l>m\ge 1$, $0\le k<l$, and $(m,2s+\gamma,s,\Gamma,k)$ satisfy*

$$2s+\gamma\le m-k\le\begin{cases}\nu+s+\min\{\delta,\gamma\},\\ if\ \gamma\neq\delta\ or\ \gamma=\delta\ and\ \Gamma=\Delta,\\ \nu+s,\\ if\ \gamma=\delta=1\ and\ \Gamma\neq\Delta.\end{cases} \tag{2.14}$$

For any $X\in\mathcal{L}_O(m,2s+\gamma,s,\Gamma,k;\ 2\nu+\delta+l,\Delta)$, define

$$r(X)=\begin{cases}\dim X, & if\ X\neq\mathbb{F}_q^{(2\nu+\delta+l)},\\ m+1, & if\ X=\mathbb{F}_q^{(2\nu+\delta+l)},\end{cases} \tag{2.15}$$

then $r:\mathcal{L}_O(m,2s+\gamma,s,\Gamma,k;\ 2\nu+\delta+l,\Delta)\to\mathbb{N}$ is a rank function of the lattice $\mathcal{L}_O(m,2s+\gamma,s,\Gamma,k;\ 2\nu+\delta+l,\Delta)$.

Theorem 2.9. *Let* $2\nu+\delta+l > m \geq 1$, $0 \leq k < l$, *and* $(m, 2s+\gamma, s, \Gamma, k)$ *satisfy* (2.14). *For any* $X \in \mathcal{L}_R(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, *define*

$$r'(X) = \begin{cases} m+1-\dim X, & \text{if } X \neq \mathbb{F}_q^{(2\nu+\delta+l)}, \\ 0, & \text{if } X = \mathbb{F}_q^{(2\nu+\delta+l)}, \end{cases} \tag{2.16}$$

then $r' : \mathcal{L}_R(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta) \to \mathbb{N}$ *is a rank function of the lattice* $\mathcal{L}_R(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$.

3. The Geometricity of Lattices $\mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k; 2\nu+\delta+l, \Delta)$

Theorem 3.1. *Let* $2\nu+\delta+l > m \geq 1, 0 \leq k < l$, *assume that* $(m, 2s+\gamma, s, \Gamma, k)$ *satisfies conditions* (2.10) *and* (2.11). *Then*

(i) *each of* $\mathcal{L}_O(k+1, 0, 0, \phi, k;\ 2\nu+\delta+l, \Delta)$ *and* $\mathcal{L}_O(k+1, 1, 0, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ ($\Gamma = 1$ *or* z) *is a finite geometric lattice, when* $k = 0$, *and is a finite atomic lattice, but not a geometric lattice when* $0 < k < l$;

(ii) *when* $2 \leq m-k \leq 2\nu+\delta-1$, $\mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ *is a finite atomic lattice, but not a geometric lattice.*

Proof. By Theorem 2.8, the rank function of $\mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ is defined by formula (2.15), we will show the condition G_1 of Proposition 2.5 holds for $\mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$. $\{0\} \in \mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ and it is the minimal element, so all 1-dim subspaces in $\mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ are atoms of $\mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$.

Let $U \in \mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta) \setminus \{\{0\}, \mathbb{F}_q^{(2\nu+\delta+l)}\}$, by Theorem 2.7, U is a subspace of type $(m_1, 2s_1+\gamma_1, s_1, \Gamma_1, k_1)$ and satisfies condition (2.13). If $m_1 = 1$, then U is an atom of $\mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$. Assume $m_1 \geq 2$, then

$$US_lU^t = \left[S_{2s_1+\gamma_1, \Gamma_1}, 0^{(m_1-k_1-2s_1-\gamma_1)}, 0^{(k_1)}\right], \tag{3.1}$$

where $\Gamma_1 = \phi, (1), (z)$, or $[1, -z]$.

Let U_i be an ith $(1 \leq i \leq m_1)$ row vector of U, then $\langle U_i \rangle$ is a subspace of type $(1, 0, 0, \phi, 0), (1, 1, 0, 1, 0), (1, 1, 0, z, 0)$, or $(1, 0, 0, 0, 1)$, and $\langle U_i \rangle \subset U$. By Theorem 2.7, we know $\langle U_i \rangle \in \mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, so $\langle U_i \rangle$ is an atom of $\mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, and $U = \vee_{i=1}^{m_1} \langle U_i \rangle$, hence, U is a union of atoms in $\mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$. Since $|\mathcal{M}(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)| \geq 2$, there exist $W_1, W_2 \in \mathcal{M}(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta), W_1 \neq W_2$, such that $\mathbb{F}_q^{(2\nu+\delta+l)} = W_1 \vee W_2$. W_1, W_2 are unions of atoms in $\mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, hence, $\mathbb{F}_q^{(2\nu+\delta+l)}$ is a union of atoms in $\mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, therefore, G_1 holds.

In the following, we prove (i) and (ii).

The Proof of (i). We only prove the formula (2.2) holds for $\mathcal{L}_O(k+1, 1, 0, \Gamma, k;\ 2\nu+\delta+l, \Delta)$. The other can be obtained in the similar way. We consider two cases:

(a) $k = 0$. $\mathcal{L}_O(k+1, 1, 0, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ consists of $\mathbb{F}_q^{(2\nu+\delta+l)}$, $\{0\}$ and subspaces of type $(1, 1, 0, \Gamma, 0)$. Let $U, W \in \mathcal{L}_O(1, 1, 0, \Gamma, 0;\ 2\nu+\delta+l, \Delta)$, if U, W are $\mathbb{F}_q^{(2\nu+\delta+l)}$, $\{0\}$, respectively, then

$U \vee V = \mathbb{F}_q^{(2\nu+\delta+l)}, U \wedge W = \{0\}$, so $r(U \vee W) + r(U \wedge W) = r(U) + r(W)$. If $U = W$ is $\{0\}$ or $\mathbb{F}_q^{(2\nu+\delta+l)}$, the other is a subspace of type $(1, 1, 0, \Gamma, 0)$, then $U \wedge W$ is $\{0\}$ or subspace of type $(1, 1, 0, \Gamma, 0)$, $U \vee W$ is a subspace of type $(1, 1, 0, \Gamma, 0)$ or $\mathbb{F}_q^{(2\nu+\delta+l)}$, so $r(U \vee W) + r(U \wedge W) = r(U) + r(W)$. If U and W are subspaces of type $(1, 1, 0, \Gamma, 0)$, then $U \wedge W = \{0\}, U \vee W = \mathbb{F}_q^{(2\nu+\delta+l)}$, so $r(U \vee W) + r(U \wedge W) = r(U) + r(W)$.

Hence, (2.2) holds and $\mathcal{L}_O(k+1, 1, 0, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ is a finite geometric lattice when $k = 0$.

(b) $0 < k < l$. Let $U = \langle e_1 + (\Gamma/2)e_{\nu+1}\rangle$, $W = \langle e_{s+1} + (\Gamma/2)e_{\nu+s+1}\rangle$, where $s \leq \nu - 1$, then $U, W \in \mathcal{L}_O(k+1, 1, 0, \Gamma, k;\ 2\nu+\delta+l, \Delta)$. When $q = 3(\text{mod } 4)$ or $q = 1(\text{mod } 4)$, then -1 is a nonsquare element or a square element, respectively. Thus, $[\Gamma, \Gamma]$ is cogredient to either $[1, -z]$ or $S_{2\cdot 1}$, and $\langle U, W\rangle$ is a subspace of type $(2, 2, 0, \Gamma, 0)$, where $\Gamma = [1, -z]$, or a subspace of type $(2, 2, 1, \phi, 0)$. So $\langle U, W\rangle \notin \mathcal{L}_O(k+1, 1, 0, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, and we have $U \vee W = \mathbb{F}_q^{(2\nu+\delta+l)}$, $U \wedge W = \{0\}$. By the definition of rank function, $r(U \vee W) = k+1+1 = k+2, r(U \wedge W) = 0$, $r(U) = r(W) = 1$, we have $r(U \vee W) + r(U \wedge W) = k+2 > r(U) + r(W) = 2$.

Hence, $\mathcal{L}_O(k+1, 1, 0, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ is a finite atomic lattice, but not a geometric lattice when $0 < k < l$.

The Proof of (ii). We will show there exist $U, W \in \mathcal{L}_O(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ such that the formula (2.2) does not hold. As to $\gamma = 0, 1$, or 2, we only show the proof of $\gamma = 1$, others can be obtained in the similar way. We distinguish the following three cases.

(a) $\delta = 0$, or $\delta = 1$, $\Gamma \neq \Delta$. Then, the formula (2.10) is changed into $2s+1 \leq m-k \leq \nu+s$. Let $\sigma = \nu + s - m + k$, we distinguish the following two subcases.

(a.1) $m-k-2s-1 \geq 1$. From $m-k-2s-1 \geq 1$ and $m-k \leq \nu+s$, we have $s+2 \leq \nu$. Let

$$U = \begin{pmatrix} I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \Gamma/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(\sigma_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 \end{pmatrix}, \tag{3.2}$$
$$\begin{matrix} s & 1 & 1 & \sigma_1 & \sigma & s & 1 & 1 & \sigma_1 & \sigma & k & l-k \end{matrix}$$
$$W = \left\langle e_{s+2} + \left(\frac{\Gamma}{2}\right)e_{\nu+s+2}\right\rangle,$$

where $\sigma_1 = m-k-2s-2$, then U is a subspace of type $(m-1, 2s+1, s, \Gamma, k)$, W is a subspace of type $(1, 1, 0, \Gamma, 0)$. When $q = 3(\text{mod } 4)$ or $q = 1(\text{mod } 4)$, then -1 is a nonsquare element or a square element, respectively, thus $[\Gamma, \Gamma]$ is cogredient to either $[1, -z]$ or $S_{2\cdot 1}$, and $\langle U, W\rangle$ is a subspace of type $(m, 2s+2, s, \Gamma, k)$ or type $(m, 2(s+1), s+1, \phi, k)$. Consequently, $U, W \in \mathcal{L}_O(m, 2s+1, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, $\langle U, W\rangle \notin \mathcal{L}_O(m, 2s+1, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$. Thus, we have $U \vee W = \mathbb{F}_q^{(2\nu+\delta+l)}$, $U \wedge W = \{0\}$, $r(U \vee W) = m+1, r(U \wedge W) = 0$, $r(U) = m-1$, $r(W) = 1$. Then,

$$r(U \vee W) + r(U \wedge W) = m+1 > r(U) + r(W) = m-1+1 = m. \tag{3.3}$$

(a.2) $m-k-2s-1=0$. From $2\le m-k\le 2\nu+\delta-1$, we have $s+1\le\nu,\ s\ge 1$. Let

$$U=\begin{pmatrix} I^{(s-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \Gamma/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 \end{pmatrix}, \qquad (3.4)$$
$$\begin{matrix} s-1 & 1 & 1 & \sigma & s & 1 & \sigma & k & l-k \end{matrix}$$
$$W=\left\langle e_{s+1}-\left(\frac{\Gamma}{2}\right)e_{\nu+s+1}\right\rangle,$$

then U is a subspace of type $(m-1,2(s-1)+1,s-1,\Gamma,k)$, W is a subspace of type $(1,1,0,-\Gamma,0)$, $\langle U,W\rangle$ is a subspace of type $(m,2s,s,\phi,k)$. Consequently, $U,W\in\mathcal{L}_O(m,2s+1,s,\Gamma,k;\ 2\nu+\delta+l,\Delta)$, $\langle U,W\rangle\notin\mathcal{L}_O(m,2s+1,s,\Gamma,k;\ 2\nu+\delta+l,\Delta)$. Thus, we have $U\vee W=\mathbb{F}_q^{(2\nu+\delta+l)}$, $U\wedge W=\{0\}$, $r(U\vee W)=m+1$, $r(U\wedge W)=0$, $r(U)=m-1$, $r(W)=1$. Then,

$$r(U\vee W)+r(U\wedge W)=m+1>r(U)+r(W)=m-1+1=m. \qquad (3.5)$$

Therefore, there exist $U,W\in\mathcal{L}_O(m,2s+1,s,\Gamma,k;\ 2\nu+\delta+l,\Delta)$ such that formula (2.2) does not hold.

(b) $\delta=1,\Gamma=\Delta$. Then, the formula (2.10) is changed into $2s+1\le m-k\le\nu+s+1$. Let $\sigma=\nu+s-m+k+1$, we distinguish the following two subcases.

(b.1) $m-k-2s-1\ge 1$. From $m-k-2s-1\ge 1$, and $2\le m-k\le 2\nu$, we have $s+1\le\nu$. Let

$$U=\begin{pmatrix} I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & I^{(\sigma_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 \end{pmatrix}, \qquad (3.6)$$
$$\begin{matrix} s & 1 & \sigma_1 & \sigma & s & 1 & \sigma_1 & \sigma & 1 & k & l-k \end{matrix}$$
$$W=\left\langle e_{s+1}+\left(\frac{\Delta}{2}\right)e_{\nu+s+1}\right\rangle,$$

where $\sigma_1=m-k-2s-2$, then U is a subspace of type $(m-1,2s+1,s,\Delta,k)$, W is a subspace of type $(1,1,0,\Delta,0)$. When $q=3(\text{mod }4)$ or $q=1(\text{mod }4)$, similar to the proof of the case (a.1), $\langle U,W\rangle$ is a subspace of type $(m,2s+2,s,\Gamma,k)$ or $(m,2(s+1),s+1,\phi,k)$. Consequently, $U,W\in\mathcal{L}_O(m,2s+1,s,\Delta,k;\ 2\nu+1+l,\Delta)$, $\langle U,W\rangle\notin\mathcal{L}_O(m,2s+1,s,\Delta,k;\ 2\nu+1+l,\Delta)$, and the formula (2.2) does not hold.

(b.2) $m-k-2s-1=0$. From $2 \le m-k \le 2\nu$, we have $s+1 \le \nu$. Let

$$U = \begin{pmatrix} I^{(s-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 \end{pmatrix},$$
$$\begin{matrix} s-1 & 1 & 1 & \sigma-1 & s & 1 & \sigma-1 & 1 & k & l-k \end{matrix}$$
$$W = \left\langle e_{s+1} + \left(\frac{\Delta}{2}\right) e_{\nu+s+1} \right\rangle, \tag{3.7}$$

then U is a subspace of type $(m-1, 2(s-1)+1, s-1, \Delta, k)$, W is a subspace of type $(1,1,0,\Delta,0)$, when $q=3(\text{mod } 4)$ or $q=1(\text{mod } 4)$, $\langle U, W\rangle$ is subspace of type $(m, 2(s-1)+2, s-1, \Gamma, k)$ or $(m, 2s, s, \phi, k)$. Similar to the proof of the case (a.1), the formula (2.2) does not hold for U and W.

(c) $\delta = 2$. Then, the formula (2.10) is changed into $2s+1 \le m-k \le \nu+s+1$. Let $\sigma = \nu+s-m+k+1$, we distinguish the following two subcases.

(c.1) $m-k-2s-1 \ge 1$. From $m-k-2s-1 \ge 1$, and $m-k \le 2\nu+1$, we have $s+1 \le \nu$. Let

$$U = \begin{pmatrix} I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & y & 0 & 0 \\ 0 & 0 & I^{(\sigma_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 \end{pmatrix},$$
$$\begin{matrix} s & 1 & \sigma_1 & \sigma & s & 1 & \sigma_1 & \sigma & 1 & 1 & k & l-k \end{matrix}$$
$$W = \left\langle e_{s+1} + \left(\frac{\Gamma}{2}\right) e_{\nu+s+1} \right\rangle, \tag{3.8}$$

where $\sigma_1 = m-k-2s-2$ and $x^2 - zy^2 = \Gamma$, then U is a subspace of type $(m-1, 2s+1, s, \Gamma, k)$, W is a subspace of type $(1,1,0,\Gamma,0)$. But when $q=3(\text{mod } 4)$ or $q=1(\text{mod } 4)$, similar to the proof of the case (a.1), $\langle U, W\rangle$ is a subspace of type $(m, 2s+2, s, \Gamma, k)$ or $(m, 2(s+1), s+1, \phi, k)$. Consequently, $U, W \in \mathcal{L}_O(m, 2s+1, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, $\langle U, W\rangle \notin \mathcal{L}_O(m, 2s+1, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, and the formula (2.2) does not hold.

(c.2) $m-k-2s-1=0$. From $2 \le m-k \le 2\nu+1$, we have $s \ge 1$ and $m \ge 3$. We choose (a,b) and (c,d) being two linearly independent solutions of the equation $x^2 - zy^2 = \Gamma$. Let

$$U = \begin{pmatrix} I^{(s-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0 \end{pmatrix},$$
$$\begin{matrix} s-1 & 1 & \sigma & s & \sigma & 1 & 1 & k & l-k \end{matrix}$$
$$W = \langle c e_{2\nu+1} + d e_{2\nu+2} \rangle, \tag{3.9}$$

then U is a subspace of type $(m-1, 2(s-1)+1, s-1, \Gamma, k)$, W is a subspace of type $(1, 1, 0, \Gamma, 0)$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \\ & -z \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t, \tag{3.10}$$

because det $A = -(ad-bc)^2 z$, hence, A is cogredient to $[1, -z]$. Then,

$$\begin{pmatrix} U \\ W \end{pmatrix} S_l \begin{pmatrix} U \\ W \end{pmatrix}^t \tag{3.11}$$

is cogredient to

$$\left[S_{2(s-1)+2,\Delta}, o^{(m-k-2s)}, o^{(k)}\right]. \tag{3.12}$$

Therefore, $\langle U, W \rangle$ is a subspace of type $(m, 2(s-1)+2, s-1, \Gamma, k)$. Similar to the proof of the case (a.2), the formula (2.2) does not hold for U and W. □

4. The Geometricity of Lattices $\mathcal{L}_R(m, 2s+\gamma, s, \Gamma, k; 2\nu+\delta+l, \Delta)$

Theorem 4.1. *Let* $2\nu+\delta+l > m \geq 1$, $0 \leq k < l$, *assume that* $(m, 2s+\gamma, s, \Gamma, k)$ *satisfies conditions* (2.10) *and* (2.11). *Then,*

(i) *each of* $\mathcal{L}_R(k+1, 0, 0, \phi, k;\ 2\nu+\delta+l, \Delta)$, $\mathcal{L}_R(k+1, 1, 0, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ $(\Gamma = 1$ *or* $z)$ *and* $\mathcal{L}_R(2\nu+\delta+k-1, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ *is a finite geometric lattice when* $k = 0$, *and is a finite atomic lattice, but not a geometric lattice when* $0 < k < l$;

(ii) *when* $2 \leq m-k \leq 2\nu+\delta-2$, $\mathcal{L}_R(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ *is a finite atomic lattice, but not a geometric lattice.*

Proof. By Theorem 2.9, the rank function of $\mathcal{L}_R(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ is defined by formula (2.16), $\mathbb{F}_q^{(2\nu+\delta+l)}$ is the minimal element of $\mathcal{L}_R(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, all subspaces of type $(m, 2s+\gamma, s, \Gamma, k)$ in $\mathcal{L}_R(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ are atoms of $\mathcal{L}_R(m, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$.

The Proof of (i). By [8], $\mathcal{L}_R(k+1, 0, 0, \phi, k;\ 2\nu+\delta+l, \Delta)$, $\mathcal{L}_R(k+1, 1, 0, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, and $\mathcal{L}_R(2\nu+\delta+k-1, 2s+\gamma, s, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ are finite geometric lattices when $k = 0$; in the following, we will show that they are finite atomic lattices, but not geometric lattices when $0 < k < l$.

(a) Let

$$\begin{aligned} U &= \langle e_{\nu+1}, e_{2\nu+\delta+1}, e_{2\nu+\delta+2}, \ldots, e_{2\nu+\delta+k} \rangle, \\ W &= \langle e_1, e_{2\nu+\delta+2}, e_{2\nu+\delta+3}, \ldots, e_{2\nu+\delta+k+1} \rangle. \end{aligned} \tag{4.1}$$

Then, both U and W are subspaces of type $(k+1, 0, 0, \phi, k)$, and $U \cap W = \langle e_{2\nu+\delta+2}, e_{2\nu+\delta+3}, \ldots, e_{2\nu+\delta+k} \rangle$, $\langle U, W \rangle$ is a subspace of type $(k+3, 2, 1, \phi, k+1)$. Consequently,

$\langle U, W\rangle \notin \mathcal{L}_R(k+1, 0, 0, \phi, k;\ 2\nu+\delta+l, \Delta)$, $r'(U \wedge W) = r'(\mathbb{F}_q^{(2\nu+\delta+l)}) = 0$, $r'(U \vee W) = r'(U \cap W) = k+2-(k-1) = 3$, $r'(U) = r'(W) = k+2-(k+1) = 1$. Thus,

$$r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W). \tag{4.2}$$

That is, (2.2) does not hold for U and W. Hence, $\mathcal{L}_R(k+1, 0, 0, \phi, k;\ 2\nu+\delta+l, \Delta)$ are not geometric lattices when $0 < k < l$.

(b) Let

$$\begin{aligned} U &= \left\langle e_1 + \left(\frac{\Gamma}{2}\right) e_{\nu+1}, e_{2\nu+\delta+1}, e_{2\nu+\delta+2}, \ldots, e_{2\nu+\delta+k} \right\rangle, \\ W &= \left\langle e_{s+1} + \left(\frac{\Gamma}{2}\right) e_{\nu+s+1}, e_{2\nu+\delta+2}, e_{2\nu+\delta+3}, \ldots, e_{2\nu+\delta+k+1} \right\rangle. \end{aligned} \tag{4.3}$$

Then, both U and W are subspaces of type $(k+1, 1, 0, \Gamma, k)$, and $U \cap W = \langle e_{2\nu+\delta+2}, e_{2\nu+\delta+3}, \ldots, e_{2\nu+\delta+k}\rangle$, $\langle U, W\rangle$ is a subspace of type $(k+3, 2, 0, \Gamma, k+1)$ or $(k+3, 2, 1, \phi, k+1)$ when $q = 3(\text{mod } 4)$ or $q = 1(\text{mod } 4)$. Consequently, $\langle \mathrm{U}, W\rangle \notin \mathcal{L}_R(k+1, 1, 0, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, $r'(U \wedge W) = r'(\mathbb{F}_q^{(2\nu+\delta+l)}) = 0$, $r'(U \vee W) = r'(U \cap W) = k+2-(k-1) = 3$, $r'(U) = r'(W) = k+2-(k+1) = 1$. Thus,

$$r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W). \tag{4.4}$$

That is, (2.2) does not hold for U and W. Hence, $\mathcal{L}_R(k+1, 1, 0, \Gamma, k;\ 2\nu+\delta+l, \Delta)$ are not geometric lattices when $0 < k < l$.

(c) From the condition (2.10), the following hold.

(i) If $\gamma = \delta = 1, \Gamma \neq \Delta$, then $2\nu+\delta-1 \le \nu+s$, that is, $\nu \le s, \nu = s$, hence $2\nu+1 \le 2\nu$, and it is a contradiction.

(ii) If $\gamma = \delta, \Gamma = \Delta$, then $2\nu+\delta-1 \le \nu+\mathrm{s}+\delta$, that is, $\nu-1 \le s$, hence $s = \nu$, or $s = \nu-1$. When $s = \nu$, from $2s+\gamma \le 2\nu+\delta-1$, we obtain $2\nu+\delta \le 2\nu+\delta-1$, and it is a contradiction. When $s = \nu-1$, we have $2\nu+\delta-2 \le 2\nu+\delta-1$. That is, in this situation, $\nu-1 = s$ holds.

(iii) If $\gamma \neq \delta$, then $2\nu+\delta-1 \le \nu+s+\min\{\delta, \gamma\} \le \nu+s+\delta$, that is, $\nu-1 \le s$, hence $s = \nu$, or $s = \nu-1$. When $s = \nu$, we have $2\nu+\gamma \le 2\nu+\delta-1$, then $\gamma \le \delta-1$. When $s = \nu-1$, we have $2\nu+\gamma-2 \le 2\nu+\delta-1$, then $\gamma-1 \le \delta$.

From the discussion above, we know that

(c.1) If $s = \nu$, then $\gamma \le \delta-1$, and we have $\delta = 1$, $\gamma = 0$; $\delta = 2$, $\gamma = 0$, and $\delta = 2$, $\gamma = 1$ three possible cases. For $\mathcal{L}_R(2\nu+\delta+k-1, 2\nu+\gamma, \nu, \Gamma, k;\ 2\nu+\delta+l, \Delta)$, here we just give the proof of the case $\delta = 2$, $\gamma = 1$, others can be obtained in the similar way. We choose (a, b) and (c, d) being two linearly independent solutions of the equation $x^2 - zy^2 = \Gamma$. Let

$$U = \begin{pmatrix} I^{(\nu)} & 0 & 0 & 0 & 0 & 0 \\ 0 & I^{(\nu)} & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(k)} & 0 \end{pmatrix}, \quad \begin{matrix} \nu & \nu & 1 & 1 & k & l-k \end{matrix}$$
$$W = \begin{pmatrix} 0 & 0 & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{matrix} \nu & \nu & 1 & 1 & k & l-k-1 & 1 \end{matrix} \tag{4.5}$$

then U is a subspace of type $(2\nu + k + 1, 2\nu + 1, \nu, \Gamma, k)$, W is a subspace of type $(2, 1, 0, \Gamma, 1)$, and $\langle U, W\rangle$ is a subspace of type $(2\nu + k + 3, 2\nu + 2, \nu, \Gamma, k + 1)$. Consequently, $U, W \in \mathcal{L}_R(2\nu + k + 1, 2\nu + 1, \nu, \Gamma, k;\ 2\nu + \delta + l, \Delta)$, $\langle U, W\rangle \notin \mathcal{L}_R(2\nu + k + 1, 2\nu + 1, \nu, \Gamma, k;\ 2\nu + \delta + l, \Delta)$. Thus, we have $U \vee W = \{0\}$, $U \wedge W = \mathbb{F}_q^{(2\nu+\delta+l)}$, $r'(U \vee W) = r'(U \cap W) = 2\nu + k + 2$, $r'(U \wedge W) = 0$, $r'(U) = 2\nu + k + 2 - 2\nu - k - 1 = 1$, $r'(W) = 2\nu + k + 2 - 2 = 2\nu + k$. Then,

$$r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W). \tag{4.6}$$

That is, (2.2) does not hold for U and W. Hence, $\mathcal{L}_R(2\nu + k + 1, 2\nu + 1, \nu, 1, k;\ 2\nu + \delta + l, \Delta)$ are not geometric lattices when $0 < k < l$.

(c.2) If $s = \nu - 1$, then we have $\gamma \neq \delta,\ \gamma - 1 \leq \delta$; or $\gamma = \delta, \Gamma = \Delta$. As to $\mathcal{L}_R(2\nu + \delta + k - 1, 2(\nu - 1) + \gamma, \nu - 1, \Gamma, k;\ 2\nu + \delta + l, \Delta)$, we consider $\delta = 0$, $\delta = 1$, and $\delta = 2$ three cases. Here we just give the proof of the case $\delta = 1$, and we also discuss the following three subcases:

(c.2.1) $\delta = 1, \gamma = 0$. For $\mathcal{L}_R(2\nu + k, 2(\nu - 1), \nu - 1, \phi, k;\ 2\nu + \delta + l, \Delta)$, let

$$U = \begin{pmatrix} I^{(\nu-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I^{(\nu)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(k)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{matrix} \nu-1 & 1 & \nu & 1 & k & l-k-1 & 1 \end{matrix}$$
$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{matrix} \nu-1 & 1 & \nu & 1 & k & l-k-1 & 1 \end{matrix} \tag{4.7}$$

then U is a subspace of type $(2\nu+k, 2(\nu-1), \nu-1, \phi, k+1)$, W is a subspace of type $(2, 1, 0, \Delta, 0)$, and $\langle U, W\rangle$ is a subspace of type $(2\nu + k + 2, 2\nu + 1, \nu, \Delta, k + 1)$. If $\nu = 1$, then $s = 0$, and as to W, from the condition (2.10), we obtain $2 \leq 1$, that is, it is a contradiction. Consequently, $\nu \geq 2$, and $U, W \in \mathcal{L}_R(2\nu + k, 2(\nu - 1), \nu - 1, \phi, k;\ 2\nu + \delta + l, \Delta)$, $\langle U, W\rangle \notin \mathcal{L}_R(2\nu + k, 2(\nu - 1), \nu - 1, \phi, k;\ 2\nu + \delta + l, \Delta)$. Thus, we have $U \vee W = \{0\}$, $U \wedge W = \mathbb{F}_q^{(2\nu+\delta+l)}$, $r'(U \vee W) = r'(U \cap W)$

$= 2\nu + k + 1$, $r'(U \wedge W) = 0$, $r'(U) = 2\nu + k + 1 - 2\nu - k = 1$, $r'(W) = 2\nu + k + 1 - 2 = 2\nu + k - 1$. Then,

$$r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W). \tag{4.8}$$

That is, (2.2) does not hold for U and W. Hence, $\mathcal{L}_R(2\nu+k, 2(\nu-1), \nu-1, \phi, k;\ 2\nu+\delta+l, \Delta)$ are not geometric lattices when $0 < k < l$.

(c.2.2) $\delta = 1, \gamma = 1, \Gamma = \Delta$. For $\mathcal{L}_R(2\nu + k, 2(\nu-1) + 1, \nu - 1, \Delta, k;\ 2\nu + \delta + l, \Delta)$, let

$$U = \begin{pmatrix} I^{(\nu-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I^{(\nu)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(k)} & 0 & 0 \end{pmatrix}, \\ \begin{matrix} \nu-1 & 1 & \nu & 1 & k & l-k-1 & 1 \end{matrix} \tag{4.9}$$

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \\ \begin{matrix} \nu-1 & 1 & \nu & 1 & k & l-k-1 & 1 \end{matrix}$$

then U is a subspace of type $(2\nu+k, 2(\nu-1)+1, \nu-1, \Delta, k)$, W is a subspace of type $(2, 1, 0, \Delta, 0)$, and $\langle U, W\rangle$ is a subspace of type $(2\nu + k + 2, 2\nu + 1, \nu, \Delta, k + 1)$. Consequently, $U, W \in \mathcal{L}_R(2\nu + k, 2(\nu-1)+1, \nu-1, \Delta, k;\ 2\nu+\delta+l, \Delta)$, $\langle U, W\rangle \notin \mathcal{L}_R(2\nu+k, 2(\nu-1)+1, \nu-1, \Delta, k;\ 2\nu+\delta+l, \Delta)$. Thus, we have $U \vee W = \{0\}$, $U \wedge W = \mathbb{F}_q^{(2\nu+\delta+l)}$, $r'(U\vee W) = r'(U \cap W) = 2\nu+k+1$, $r'(U\wedge W) = 0$, $r'(U) = 2\nu + k + 1 - 2\nu - k = 1$, $r'(W) = 2\nu + k + 1 - 2 = 2\nu + k - 1$. Then,

$$r'(U \wedge W) + r'(U \vee W) > r'(U) + r'(W). \tag{4.10}$$

That is, (2.2) does not hold for U and W. Hence, $\mathcal{L}_R(2\nu + k, 2(\nu - 1) + 1, \nu - 1, \Delta, k;\ 2\nu + \delta + l, \Delta)$ are not geometric lattices when $0 < k < l$.

(c.2.3) $\delta = 1, \gamma = 2$. See the proof of the Theorem 7 in [12].

The Proof of (ii). Let $U \in \mathcal{M}(m, 2s + \gamma, s, \Gamma, k;\ 2\nu + \delta + l, \Delta)$, then

$$US_lU^t = \left[\Lambda_1, 0^{m-k-2s-\gamma}, 0^{(k)}\right], \tag{4.11}$$

where $\Lambda_1 = S_{2s+\gamma,\Gamma}$. Hence, there exists a $(2\nu + \delta + l - m) \times (2\nu + \delta + l)$ matrix Z such that

$$\begin{pmatrix} U \\ Z \end{pmatrix} S_l \begin{pmatrix} U \\ Z \end{pmatrix}^t = \left[\Lambda_1, S_{2(m-k-2s-\gamma)}, \Lambda^*, 0^{(k)}, 0^{(l-k)}\right], \tag{4.12}$$

where Λ^* takes values in Table 1 as follows.

In Table 1 as follows $\sum_i = S_{2(\nu+s-m+k+i)}$, $i = 0, 1$, or 2.

As to $\delta = 0$; $\delta = 1, \Delta = 1$; $\delta = 1, \Delta = z$, and $\delta = 2$ four cases, we only show the proof of the case $\delta = 0$, others can be obtained in the similar way. We also distinguish the following three subcases.

Table 1

	$\delta=0$	$\delta=1,\Delta=1$	$\delta=1,\Delta=z$	$\delta=2$
$\gamma=0$	Σ_0	$[\Sigma_0,1]$	$[\Sigma_0,z]$	$[\Sigma_0,1,-z]$
$\gamma=1,\Gamma=1$	$[\Sigma_0,-1]$	Σ_1	$[\Sigma_0,-1,z]$	$[\Sigma_1,-z]$
$\gamma=1,\Gamma=z$	$[\Sigma_0,-z]$	$[\Sigma_0,1,-z]$	Σ_1	$[\Sigma_1,-1]$
$\gamma=2$	$[\Sigma_0,1,-z]$	$[\Sigma_1,z]$	$[\Sigma_1,1]$	Σ_2

(a) If $\gamma=0$, then $\Lambda_1=S_{2s},\Lambda^*=S_{2(\nu-m+k+s)}$. Let $u_1,u_2,\ldots,u_s,v_1,v_2,\ldots,v_s,u_{s+1},\ldots,u_{m-k-s},w_1,\ldots,w_k$ and $v_{s+1},\ldots,v_{m-k-s},u_{m-k-s+1},\ldots,u_\nu,v_{m-k-s+1},\ldots,v_\nu,w_{k+1},\ldots,w_l$ be row vectors of U and Z, respectively,

$$W=\langle v_{\nu-m+k+s+1},\ldots,v_{\nu-s},u_{\nu-s+1},\ldots,u_\nu,v_{\nu-s+1},\ldots,v_\nu,w_1,\ldots,w_k\rangle, \tag{4.13}$$

then $W\in\mathcal{M}(m,2s,s,\phi,k;\ 2\nu+l)$.

From $m-k\le 2\nu-2$, we know $s<\nu$. If $m-k=2s$, then $m-k-s=s<\nu$, so $u_\nu,v_\nu\notin U$. If $m-k>2s$, then $s<\nu-1$, so $v_{\nu-1},v_\nu\notin U$. In a word, $\dim\langle U,W\rangle\ge m+2,\dim(U\cap W)\le m-2$. That is, $U\wedge W=\mathbb{F}_q^{(2\nu+l)}$, $r'(U\wedge W)=0$, $r'(U\vee W)\ge m+1-(m-2)=3, r'(U)=r'(W)=m+1-m=1$. Consequently, $r'(U\wedge W)+r'(U\vee W)>r'(U)+r'(W)$.

(b) If $\gamma=1$, then $\Lambda_1=S_{2s+1,\Gamma},\Lambda^*=S_{2(\nu-m+k+s)+1,-\Gamma}$, and $\Gamma=(1)$ or (z). Let $u_1,u_2,\ldots,u_s,v_1,v_2,\ldots,v_s,\omega,u_{s+1},\ldots,u_{m-k-s-1},w_1,\ldots,w_k$ and $v_{s+1},\ldots,v_{m-k-s-1},u_{m-k-s},\ldots,u_{\nu-1},v_{m-k-s},\ldots,v_{\nu-1},\omega^*,w_{k+1},\ldots,w_l$ be row vectors of U and Z, respectively

$$\begin{aligned}W=\Big\langle &v_{\nu-m+k+s+1},\ldots,v_{\nu-s-1},u_{\nu-s},\ldots,u_{\nu-2},v_{\nu-s},\ldots,v_{\nu-2},\omega,\omega^*,\\ &\left(\frac{1}{2}\right)\Gamma u_{\nu-1}+v_{\nu-1},w_1,\ldots,w_k\Big\rangle,\end{aligned} \tag{4.14}$$

because $((1/2)\Gamma u_{\nu-1}+v_{\nu-1})S_{2\nu}((1/2)\Gamma u_{\nu-1}+v_{\nu-1})^t=\Gamma$, and

$$\begin{pmatrix}\left(\frac{1}{2}\right)\Gamma & \left(-\frac{1}{2}\right)\Gamma\\ 1 & 1\end{pmatrix}\begin{pmatrix}\omega\\ \omega^*\end{pmatrix}S_{2\nu}\begin{pmatrix}\omega\\ \omega^*\end{pmatrix}^t\begin{pmatrix}\left(\frac{1}{2}\right)\Gamma & \left(-\frac{1}{2}\right)\Gamma\\ 1 & 1\end{pmatrix}^t=S_{2\cdot 1}, \tag{4.15}$$

then $W\in\mathcal{M}(m,2s+1,s,\Gamma,k;\ 2\nu+l)$. From the conditions $2s+1\le m-k\le 2\nu-2$ and $m-k\le\nu+s$, we can obtain $m-k-s-1\le\nu-1$ and $s\le\nu-1$, hence $(1/2)\Gamma u_{\nu-1}+v_{\nu-1}\notin U$. Obviously, $\omega^*\notin U$. Similar to the proof of the case (a), $r'(U\wedge W)+r'(U\vee W)>r'(U)+r'(W)$.

(c) If $\gamma=2$, then $\Lambda_1=S_{2s+2,\Gamma},\Lambda^*=S_{2(\nu-m+k+s)+2,\Gamma}$, and $\Gamma=[1,-z]$. Let $u_1,u_2,\ldots,u_s,v_1,v_2,\ldots,v_s,\omega_1,\omega_2,u_{s+1},\ldots,u_{m-k-s-2},w_1,\ldots,w_k$ and $v_{s+1},\ldots,v_{m-k-s-2},u_{m-k-s-1},\ldots,u_{\nu-2},v_{m-k-s-1},\ldots,v_{\nu-2},\omega_1^*,\omega_2^*,w_{k+1},\ldots,w_l$ be row vectors of U and Z, respectively,

$$W=\langle v_{\nu-m+k+s+1},\ldots,v_{\nu-s-2},u_{\nu-s-1},\ldots,u_{\nu-2},v_{\nu-s-1},\ldots,v_{\nu-2},\omega_1^*,\omega_2^*,w_1,\ldots,w_k\rangle, \tag{4.16}$$

then $W\in\mathcal{M}(m,2s+2,s,\Gamma,k;\ 2\nu+l)$. Obviously, $\omega_1^*,\omega_2^*\notin U$. Similar to the proof of the case (a), $r'(U\wedge W)+r'(U\vee W)>r'(U)+r'(W)$.

From the discussion above, we know that when $2 \leq m - k \leq 2\nu - 2$, $\mathcal{L}_R(m, 2s + \gamma, s, \Gamma, k;\ 2\nu + l)$ is a finite atomic lattice, but not a geometric lattice. □

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