Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 384108, 18 pages
doi:10.1155/2012/384108

Research Article

Weak Convergence Theorems for Strictly Pseudocontractive Mappings and Generalized Mixed Equilibrium Problems

Jong Soo Jung

Department of Mathematics, Dong-A University, Busan 604-714, Republic of Korea

Correspondence should be addressed to Jong Soo Jung, jungjs@dau.ac.kr

Received 15 February 2012; Accepted 6 April 2012

Academic Editor: Yonghong Yao

We introduce a new iterative method for finding a common element of the set of fixed points of a strictly pseudocontractive mapping, the set of solutions of a generalized mixed equilibrium problem, and the set of solutions of a variational inequality problem for an inverse-strongly-monotone mapping in Hilbert spaces and then show that the sequence generated by the proposed iterative scheme converges weakly to a common element of the above three sets under suitable control conditions. The results in this paper substantially improve, develop, and complement the previous well-known results in this area.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $B : C \to H$ be a nonlinear mapping and $\varphi : C \to \mathbb{R}$ be a function, and Θ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

Then, we consider the following generalized mixed equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C, \tag{1.1}$$

which was introduced by Peng and Yao [1] recently. The set of solutions of the problem (1.1) is denoted by $\mathrm{GMEP}(\Theta, \varphi, B)$. Here some special cases of the problem (1.1) are stated as follows.

If $\varphi = 0$, then the problem (1.1) reduced the following generalized equilibrium problem (GEP) of finding $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x \rangle \geq 0, \quad \forall y \in C, \tag{1.2}$$

which was studied by S. Takahashi and M. Takahashi [2]. The set of solutions of the problem (1.2) is denoted by $\mathrm{GEP}(\Theta, B)$.

If $B = 0$, then the problem (1.1) reduces the following mixed equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C, \tag{1.3}$$

which was studied by Ceng and Yao [3] (see also [4]). The set of solutions of the problem (1.3) is denoted by $\mathrm{MEP}(\Theta, \varphi)$.

If $\varphi = 0$ and $B = 0$, then the problem (1.1) reduces the following equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of the problem (1.4) is denoted by $\mathrm{EP}(\Theta)$.

If $\varphi = 0$ and $\Theta(x, y) = 0$ for all $x, y \in C$, the problem (1.1) reduces the following variational inequality problem of finding $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of the problem (1.5) is denoted by $\mathrm{VI}(C, B)$.

The problem (1.1) is very general in the sense that it includes, as special cases, fixed point problems, optimization problems, variational inequality problems, minimax problems, Nash equilibrium problems in noncooperative games, and others; see, for example, [3, 5–7].

The class of pseudocontractive mappings is one of the most important classes of mappings among nonlinear mappings. We recall that a mapping $S : C \to H$ is said to be *k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{1.6}$$

Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, S is nonexpansive (i.e., $\|Sx - Sy\| \leq \|x - y\|$, $\forall x, y \in C$) if and only if S is 0-strictly pseudocontractive. The mapping S is also said to be pseudocontractive if $k = 1$ and S is said to be strongly pseudocontractive if there exists a constant $\lambda \in (0, 1)$ such that $S - \lambda I$ is pseudocontractive. Clearly, the class of k-strictly pseudocontractive mappings falls into the one between classes of nonexpansive mappings and pseudocontractive mappings. Also we remark that the class of strongly pseudocontractive mappings is independent of the class of k-strictly pseudocontractive mappings (see [8, 9]). Recently, many authors have been devoting the studies on the problems of finding fixed points to the class of pseudocontractive mappings; see, for example, [10–15] and the references therein.

Recently, in order to study the problems (1.1)–(1.5) coupled with the fixed point problem, many authors have introduced some iterative schemes for finding a common element of the set of the solutions of the problem (1.1)–(1.5) and the set of fixed points of a countable family of nonexpansive mappings and have studied strong convergence of the sequences generated by the proposed schemes; see [1–4, 16–18] and the references therein. Also we refer to [19–21] for the problems (1.1), (1.3), and (1.5) combined to the fixed point problem for nonexpansive semigroups and strictly pseudocontractrive mappings.

In this paper, inspired and motivated by [18, 22–27], we introduce a new iterative method for finding a common element of the set of fixed points of a k-strictly pseudocontractive mapping, the set of solutions of a generalized mixed equilibrium problem (1.1), and the set of solutions of the variational inequality problem (1.5) for an inverse-strongly monotone mapping in a Hilbert space. We show that, under suitable conditions, the sequence generated by the proposed iterative scheme converges weakly to a common element of the above three sets. The results in this paper can be viewed as an improvement and complement of the recent results in this direction.

2. Preliminaries and Lemmas

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. In the following, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. We denote by $F(T)$ the set of fixed points of the mapping T.

In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \tag{2.2}$$

for all $y \in C$. P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive and P_C satisfies

$$\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C(x)$ is characterized by the properties:

$$\begin{gathered} \|x-y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \\ u = P_C(x) \Longleftrightarrow \langle x - u, u - y\rangle \geq 0 \quad \forall x \in H,\ y \in C. \end{gathered} \tag{2.4}$$

In the context of the variational inequality problem for a nonlinear mapping F, this implies that

$$u \in \mathrm{VI}(C, F) \Longleftrightarrow u = P_C(u - \lambda F u) \quad \text{for any } \lambda > 0. \tag{2.5}$$

It is also well known that H satisfies the *Opial condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\| \tag{2.6}$$

holds for every $y \in H$ with $y \neq x$.

A mapping F of C into H is called α*-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle x - y, Fx - Fy\rangle \geq \alpha\|Fx - Fy\|^2, \quad \forall x, y \in C. \tag{2.7}$$

We know that if $F = I - T$, where T is a nonexpansive mapping of C into itself and I is the identity mapping of H, then F is $1/2$-inverse-strongly monotone and $\mathrm{VI}(C, F) = F(T)$. A mapping F of C into H is called *strongly monotone* if there exists a positive real number η such that

$$\langle x - y, Fx - Fy\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C. \tag{2.8}$$

In such a case, we say F is η-strongly monotone. If F is η-strongly monotone and κ-Lipschitzian, continuous, that is, $\|Fx - Fy\| \leq L\|x - y\|$ for all $x, y \in C$, then F is η/κ^2-inverse-strongly monotone. If F is an α-inverse-strongly monotone mapping of C into H, then it is obvious that F is $1/\alpha$-Lipschitzian. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I - \lambda F)x - (I - \lambda F)y\|^2 &= \|(x - y) - \lambda(Fx - Fy)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Fx - Fy\rangle + \lambda^2\|Fx - Fy\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Fx - Fy\|^2.\end{aligned} \tag{2.9}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda F$ is a nonexpansive mapping of C into H. The following result for the existence of solutions of the variational inequality problem for inverse-strongly monotone mappings was given in Takahashi and Toyoda [27].

Proposition 2.1. *Let C be a bounded closed convex subset of a real Hilbert space and let F be an α-inverse-strongly monotone mapping of C into H. Then,* $\mathrm{VI}(C, F)$ *is nonempty.*

A set-valued mapping $Q : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Qx$ and $g \in Qy$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $Q : H \to 2^H$ is *maximal* if the graph $G(Q)$ of Q is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping Q is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(Q)$ implies $f \in Qx$. Let F be an inverse-strongly monotone mapping of C

into H and let $N_C v$ be the *normal cone* to C at v, that is, $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0,$ for all $u \in C\}$, and define

$$Qv = \begin{cases} Fv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{2.10}$$

Then Q is maximal monotone and $0 \in Qv$ if and only if $v \in \mathrm{VI}(C, F)$; see [28, 29].

For solving the equilibrium problem for a bifunction $\Theta : C \times C \to \mathbb{R}$, let us assume that Θ and φ satisfy the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$,

(A2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$,

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y), \tag{2.11}$$

(A4) for each $x \in C, y \mapsto \Theta(x, y)$ is convex and lower semicontinuous,

(A5) for each $y \in C, x \mapsto \Theta(x, y)$ is weakly upper semicontiunuous,

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0, \tag{2.12}$$

(B2) C is a bounded set.

The following lemmas were given in [1, 5].

Lemma 2.2 (see [5]). *Let C be a nonempty closed convex subset of H and Θ a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\Theta(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.13}$$

Lemma 2.3 (see [1]). *Let C be a nonempty closed convex subset of H. Let Θ be a bifunction form $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A5) and $\varphi : C \to \mathbb{R}$ a proper lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\} \tag{2.14}$$

for all $z \in H$. Assume that either (B1) or (B2) holds. Then, the following hold:

(1) *for each $x \in H$, $T_r(x) \neq \emptyset$,*

(2) *T_r is single-valued,*

(3) T_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\left\|T_r x - T_r y\right\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \tag{2.15}$$

(4) $F(T_r) = \mathrm{MEP}(\Theta, \varphi)$,
(5) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex.*

We also need the following lemmas for the proof of our main results.

Lemma 2.4 (see [30]). *Let* H *be a real Hilbert space, let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 < a \leq \alpha_n \leq b < 1$ *for all* $n \geq 1$ *and let* $\{v_n\}$ *and* $\{w_n\}$ *be sequences in* H *such that, for some* c

$$\limsup_{n\to\infty}\|v_n\| \leq c, \qquad \limsup_{n\to\infty}\|w_n\| \leq c, \qquad \limsup_{n\to\infty}\|\alpha_n v_n + (1-\alpha_n)w_n\| = c. \tag{2.16}$$

Then $\lim_{n\to\infty}\|v_n - w_n\| = 0$.

Lemma 2.5 (see [27]). *Let* C *be a nonempty closed convex subset of a real Hilbert spaces* H *and let* $\{x_n\}$ *be a sequence in* H. *If*

$$\|x_{n+1} - x\| \leq \|x_n - x\|, \quad \forall x \in C, \ \forall n \geq 1, \tag{2.17}$$

then $\{P_C x_n\}$ *converges strongly to some* $z \in C$, *where* P_C *stands for the metric projection of* H *onto* C.

Lemma 2.6 (see [31]). *Let* H *be a Hilbert space,* C *a closed convex subset of* H. *If* T *is a* k*-strictly pseudocontractive mapping on* C, *then the fixed point set* $F(T)$ *is closed convex, so that the projection* $P_{F(T)}$ *is well defined.*

Lemma 2.7 (see [31]). *Let* H *be a Hilbert space,* C *a closed convex subset of* H, *and* $T : C \to H$ *a* k*-strictly pseudocontractive mapping. Define a mapping* $S : C \to H$ *by* $Sx = \lambda x + (1-\lambda)Tx$ *for all* $x \in C$. *Then, as* $\lambda \in [k, 1)$, S *is a nonexpansive mapping such that* $F(S) = F(T)$.

3. Main Results

In this section, we introduce a new iterative scheme for finding a common point of the set of fixed points of a k-strictly pseudocontractive mapping, the set of solutions of the problem (1.1), and the set of solutions of the problem (1.5) for an inverse-strongly monotone mapping.

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* Θ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)–(A5) and* $\varphi : C \to \mathbb{R}$ *a lower semicontinuous and convex function. Let* F, B *be two* α, β*-inverse-strongly monotone mappings of* C *into* H, *respectively. Let* T *be a* k*-strictly pseudocontractive mapping of* C *into itself for some* $k \in [0, 1)$ *such that*

$\Omega_1 := F(T) \bigcap \mathrm{GMEP}(\Theta, \varphi, B) \bigcap \mathrm{VI}(C, F) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and

$$\Theta(u_n, y) + \langle Bx_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = S(\alpha_n x_n + (1 - \alpha_n) P_C(u_n - \lambda_n F u_n)), \quad \forall n \geq 1, \tag{3.1}$$

where $S : C \to C$ is a mapping defined by $Sx = kx + (1 - k)Tx$ for $x \in C$, $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. Assume that $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, $\lambda_n \in [c, d] \subset (0, 2\alpha)$ and $r_n \in [e, f] \subset (0, 2\beta)$. Then $\{x_n\}$ and $\{u_n\}$ converge weakly to $z \in \Omega_1$, where $z = \lim_{n\to\infty} P_{\Omega_1}(x_n)$.

Proof. From now, we put $z_n = P_C(u_n - \lambda_n F u_n)$.

We divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $p \in \Omega_1 := F(T) \bigcap \mathrm{GMEP}(\Theta, \varphi, B) \bigcap \mathrm{VI}(C, F)$ and $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.3. Then, since $F(S) = F(T)$ by Lemma 2.7, $p = Sp$. Also, from (4) in Lemma 2.3 and (2.5), it follows that $p = T_{r_n}(p - r_n Bp)$ and $p = P_C(p - \lambda_n F p)$. From $z_n = P_C(u_n - \lambda_n F u_n)$ and the fact that P_C and $I - \lambda_n F$ are nonexpansive, it follows that

$$\|z_n - p\| \leq \|(I - \lambda_n F)u_n - (I - \lambda_n F)p\| \leq \|u_n - p\|. \tag{3.2}$$

Also, by $u_n = T_{r_n}(x_n - r_n B x_n) \in C$ and the β-inverse-strongly monotonicity of B, we have with $r_n \in (0, 2\beta)$,

$$\begin{aligned}
\|u_n - p\|^2 &\leq \|x_n - r_n B x_n - (p - r_n B p)\|^2 \\
&\leq \|x_n - p\|^2 - 2r_n \langle x_n - p, Bx_n - Bp \rangle + r_n^2 \|Bx_n - Bp\|^2 \\
&\leq \|x_n - p\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bp\|^2 \\
&\leq \|x_n - p\|^2,
\end{aligned} \tag{3.3}$$

that is, $\|u_n - p\| \leq \|x_n - p\|$, and so

$$\|z_n - p\| \leq \|x_n - p\|. \tag{3.4}$$

So, by using the convexity of $\|\cdot\|^2$, (3.2) and (3.3), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|S(\alpha_n x_n+(1-\alpha_n)z_n)-Sp\|^2\\ &\le \|\alpha_n(x_n-p)+(1-\alpha_n)(z_n-p)\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|z_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\left(\|x_n-p\|^2+r_n(r_n-2\beta)\|Bx_n-Bp\|^2\right)\\ &\le \|x_n-p\|^2+(1-b)e(f-2\beta)\|Bx_n-Bp\|^2\\ &\le \|x_n-p\|^2.\end{aligned} \tag{3.5}$$

So, there exists $r\in\mathbb{R}$ such that

$$r=\lim_{n\to\infty}\|x_n-p\|. \tag{3.6}$$

Therefore, $\{x_n\}$ is bounded, and so are $\{u_n\}$ and $\{z_n\}$ by (3.2) and (3.4). Moreover, from (3.5), it follows that

$$(1-b)e(2\beta-f)\|Bx_n-Bp\|^2\le\|x_n-p\|^2-\|x_{n+1}-p\|^2, \tag{3.7}$$

which implies that

$$\lim_{n\to\infty}\|Bx_n-Bp\|=0. \tag{3.8}$$

Step 2. We show that $\lim_{n\to\infty}\|x_n-u_n\|=0$. To this end, let $p\in\Omega_1$. Since T_{r_n} is firmly nonexpansive and $u_n=T_{r_n}(x_n-r_nBx_n)$, we have

$$\begin{aligned}\|u_n-p\|^2 &= \|T_{r_n}(x_n-r_nBx_n)-T_{r_n}(p-r_nBp)\|^2\\ &\le \langle T_{r_n}(x_n-r_nBx_n)-T_{r_n}(p-r_nBp),x_n-r_nBx_n-(p-r_nBp)\rangle\\ &= \langle x_n-r_nBx_n-(p-r_nBp),u_n-p\rangle\\ &= \frac{1}{2}\left\{\|u_n-p\|^2+\|x_n-r_nBx_n-(p-r_nBp)\|^2\right\}\\ &\quad-\frac{1}{2}\left\{\|x_n-r_nBx_n-(p-r_nBp)-(u_n-p)\|^2\right\}\\ &\le \frac{1}{2}\left\{\|u_n-p\|^2+\|x_n-p\|^2-\|x_n-u_n-r_n(Bx_n-Bp)\|^2\right\}\\ &\le \frac{1}{2}\Big\{\|u_n-p\|^2+\|x_n-p\|^2-\|x_n-u_n\|^2\\ &\qquad+2r_n\langle Bx_n-Bp,x_n-u_n\rangle-r_n^2\|Bx_n-Bp\|^2\Big\},\end{aligned} \tag{3.9}$$

and hence

$$\begin{aligned}\|u_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bp, x_n - u_n\rangle - r_n^2\|Bx_n - Bp\|^2 \\ &\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bp, x_n - u_n\rangle \\ &\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|Bx_n - Bp\|\|x_n - u_n\|.\end{aligned} \tag{3.10}$$

On the other hand, by using the convexity of $\|\cdot\|^2$, (3.2) and (3.10), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|S(\alpha_n x_n + (1-\alpha_n)z_n) - Sp\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\left[\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|Bx_n - Bp\|\|x_n - u_n\|\right],\end{aligned} \tag{3.11}$$

and hence

$$\begin{aligned}(1-b)\|x_n - u_n\|^2 &\le (1-\alpha_n)\|x_n - u_n\|^2 \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2r_n\|Bx_n - Bp\|\|x_n - u_n\| \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2f\|Bx_n - Bp\|M_1,\end{aligned} \tag{3.12}$$

where $M_1 = \sup\{\|x_n\| + \|u_n\| : n \ge 1\}$. Since $\lim_{n\to\infty}\|x_{n+1} - p\|^2 = \lim_{n\to\infty}\|x_n - p\|^2$ and $\lim_{n\to\infty}\|Bx_n - Bp\| = 0$ in (3.8), we obtain

$$\lim_{n\to\infty}\|x_n - u_n\| = 0 \tag{3.13}$$

and so is the limit of $\{Bx_n - Bu_n\}$ since B is Lipschitz.

Step 3. We show that $\lim_{n\to\infty}\|x_n - Sz_n\| = 0$. Indeed, let $p \in \Omega_1$ and set $r = \lim_{n\to\infty}\|x_n - p\|$. Being S nonexpansive and $F(T) = F(S)$, from (3.4) we can write

$$\|Sz_n - p\| \le \|z_n - p\| \le \|x_n - p\|, \tag{3.14}$$

and hence $\limsup_{n\to\infty}\|Sz_n - p\| \le r$. By (3.4), we also have

$$\begin{aligned}\limsup_{n\to\infty}\|\alpha_n(x_n - p) + (1-\alpha_n)(Sz_n - p)\| &= \limsup_{n\to\infty}[\alpha_n\|x_n - p\| + (1-\alpha_n)\|z_n - p\|] \\ &\le \limsup_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty}\|x_n - p\| = r.\end{aligned} \tag{3.15}$$

By Lemma 2.4, we obtain $\lim_{n\to\infty}\|Sz_n - x_n\| = 0$.

Step 4. We show that $\lim_{n\to\infty}\|u_n - z_n\| = 0$. Using $z_n = P_C(u_n - \lambda_n F u_n)$, $p = P_C(p - \lambda_n F p)$, we compute

$$
\begin{aligned}
\|z_n - p\|^2 &\leq \|(u_n - \lambda_n F u_n) - (p - \lambda_n F p)\|^2 \\
&\leq \|u_n - p\|^2 - 2\lambda_n \langle u_n - p, F u_n - F p\rangle + \lambda_n^2 \|F u_n - F p\|^2 \\
&\leq \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|F u_n - F p\|^2.
\end{aligned}
\tag{3.16}
$$

So, we get

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|S(\alpha_n x_n + (1-\alpha_n) z_n) - Sp\|^2 \\
&\leq \alpha\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\|F u_n - F p\|^2 \\
&= \|x_n - p\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\|F u_n - F p\|^2.
\end{aligned}
\tag{3.17}
$$

From conditions $\alpha_n \in [a,b] \subset (0,1)$ and $\lambda_n \in [c,d] \subset (0,2\alpha)$, it follows that

$$
\begin{aligned}
(1-b)c(2\alpha - d)\|F u_n - F p\|^2 &\leq (1-\alpha_n)\lambda_n(2\alpha - \lambda_n)\|F u_n - F p\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.
\end{aligned}
\tag{3.18}
$$

By $r = \lim_{n\to\infty}\|x_n - p\|$, we obtain

$$
\lim_{n\to\infty}\|F u_n - F p\| = 0. \tag{3.19}
$$

On the other hand, using $z_n = P_C(u_n - \lambda_n F u_n)$ and (2.3), we observe that

$$
\begin{aligned}
\|z_n - p\|^2 &= \|P_C(u_n - \lambda_n F u_n) - P_C(p - \lambda_n F p)\|^2 \\
&\leq \langle (u_n - \lambda_n F u_n) - (p - \lambda_n F p), z_n - p\rangle \\
&\leq \frac{1}{2}\left\{\|u_n - p\|^2 + \|z_n - p\|^2 - \|(u_n - z_n) - \lambda_n(F u_n - F p)\|^2\right\} \\
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|z_n - p\|^2 - \|u_n - z_n\|^2 \\
&\qquad + 2\lambda_n\langle u_n - z_n, F u_n - F p\rangle - \lambda_n^2\|F u_n - F p\|^2\Big\},
\end{aligned}
\tag{3.20}
$$

that is,

$$
\begin{aligned}
\|z_n - p\|^2 \leq{}& \|x_n - p\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\langle u_n - z_n, F u_n - F p\rangle \\
&- \lambda_n^2\|F u_n - F p\|^2.
\end{aligned}
\tag{3.21}
$$

Thus, by (3.21), we have

$$\begin{aligned}
\left\|x_{n+1}-p\right\|^2 &= \left\|S(\alpha_n x_n+(1-\alpha_n)z_n)-Sp\right\|^2 \\
&\le \alpha_n\left\|x_n-p\right\|^2+(1-\alpha_n)\left\|z_n-p\right\|^2 \\
&\le \alpha_n\left\|x_n-p\right\|^2+(1-\alpha_n)\left\|x_n-p\right\|^2-(1-\alpha_n)\|u_n-z_n\|^2 \\
&\quad +(1-\alpha_n)2\lambda_n\left\langle u_n-z_n, Fu_n-Fp\right\rangle-\lambda_n^2\left\|Fu_n-Fp\right\|^2,
\end{aligned} \tag{3.22}$$

which implies that

$$\begin{aligned}
(1-b)\|u_n-z_n\|^2 &\le (1-\alpha_n)\|u_n-z_n\|^2 \\
&\le \left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2+2d(1-a)M_2\left\|Fu_n-Fp\right\|,
\end{aligned} \tag{3.23}$$

where $M_2 = \sup\{\|z_n\|+\|u_n\| : n \ge 1\}$. From $\lim_{n\to\infty}\|Fu_n-Fp\| = 0$ in (3.19) and $r = \lim_{n\to\infty}\|x_n-p\|$, we conclude that $\lim_{n\to\infty}\|u_n-z_n\| = 0$.

Step 5. We show that $\lim_{n\to\infty}\|x_n-Sx_n\| = 0$. Indeed, since

$$\|Sx_n-x_n\| \le \|Sx_n-Su_n\|+\|Su_n-Sz_n\|+\|Sz_n-x_n\|, \tag{3.24}$$

by Step 2, Step 3, and Step 4, we have

$$\lim_{n\to\infty}\|x_n-Sx_n\| = 0. \tag{3.25}$$

Step 6. We show that any of its weak cluster point z of $\{x_n\}$ belongs in Ω_1. In this case, there exists a subsequence $\{x_{n_i}\}$ which converges weakly to z. By Step 2 and Step 4, without loss of generality, we may assume that $\{z_{n_i}\}$ converges weakly to $z \in C$. Since

$$\|Sz_n-z_n\| \le \|Sz_n-x_n\|+\|x_n-u_n\|+\|u_n-z_n\|, \tag{3.26}$$

from Step 2, Step 3, and Step 4, it follows that $\|Sz_n-z_n\| \to 0$ and $Sz_{n_i} \rightharpoonup z$.

We will show that $z \in \Omega_1$. First we show that $z \in F(S) = F(T)$. Assume that $z \notin F(S)$. Since $z_{n_i} \rightharpoonup z$ and $Sz \neq z$, by the Opial condition, we obtain

$$\begin{aligned}
\liminf_{i\to\infty}\|z_{n_i}-z\| &< \liminf_{i\to\infty}\|z_{n_i}-Sz\| \\
&\le \liminf_{i\to\infty}(\|z_{n_i}-Sz_{n_i}\|+\|Sz_{n_i}-Sz\|) \\
&\le \liminf_{i\to\infty}\|z_{n_i}-z\|,
\end{aligned} \tag{3.27}$$

which is a contradiction. Thus we have $z \in F(S) = F(T)$.

Next we prove that $z \in \mathrm{VI}(C,F)$. Let

$$Qv = \begin{cases} Fv + N_C v, & v \in C, \\ \emptyset, & v \notin C, \end{cases} \tag{3.28}$$

where $N_C v$ is normal cone to C at v. We have already known that in this case the mapping Q is maximal monotone, and $0 \in Qv$ if and only if $v \in \mathrm{VI}(C,F)$. Let $(v,w) \in G(Q)$. Since $w - Fv \in N_C v$ and $z_n \in C$, we have

$$\langle v - z_n, w - Fv \rangle \geq 0. \tag{3.29}$$

On the other hand, from $z_n = P_C(u_n - \lambda_n F u_n)$, we have

$$\langle v - z_n, z_n - (u_n - \lambda_n F u_n) \rangle \geq 0, \tag{3.30}$$

that is,

$$\left\langle v - z_n, \frac{z_n - u_n}{\lambda_n} + F u_n \right\rangle \geq 0. \tag{3.31}$$

Thus, we obtain

$$\begin{aligned} \langle v - z_{n_i}, w \rangle &\geq \langle v - z_{n_i}, Fv \rangle \\ &\geq \langle v - z_{n_i}, Fv \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} + F u_{n_i} \right\rangle \\ &= \langle v - z_{n_i}, Fv - F z_{n_i} \rangle + \langle v - z_{n_i}, F z_{n_i} - F u_{n_i} \rangle \\ &\quad - \left\langle v - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\geq \langle v - z_{n_i}, F z_{n_i} - F u_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle. \end{aligned} \tag{3.32}$$

Since $\|z_n - u_n\| \to 0$ in Step 4 and F is α-inverse-strongly monotone, it follows from (3.32) that

$$\langle v - z, w \rangle \geq 0, \quad \text{as } i \longrightarrow \infty. \tag{3.33}$$

Since Q is maximal monotone, we have $z \in Q^{-1}0$ and hence $z \in \mathrm{VI}(C,F)$.

Finally, we show that $z \in \mathrm{GMEP}(\Theta, \varphi, B)$. By $u_n = S_{r_n}(x_n - r_n B x_n)$, we know that

$$\Theta(u_n, y) + \langle B x_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.34}$$

It follows from (A2) that

$$\langle Bx_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \Theta(y, u_n), \quad \forall y \in C. \tag{3.35}$$

Hence

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + \varphi(y) - \varphi(u_{n_i}) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq \Theta(y, u_{n_i}), \quad \forall y \in C. \tag{3.36}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $\Theta(y_t, z) \leq 0$. So, from (3.36), we have

$$\begin{aligned}
\langle y_t - u_{n_i}, By_t\rangle &\geq \langle y_t - u_{n_i}, By_t\rangle - \varphi(y_t) + \varphi(u_{n_i}) - \langle y_t - u_{n_i}, Bx_{n_i}\rangle \\
&\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + \Theta(y_t, u_{n_i}) \\
&= \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle \\
&\quad - \varphi(y_t) + \varphi(u_{n_i}) - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + \Theta(y_t, u_{n_i}).
\end{aligned} \tag{3.37}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$ by Step 2, we have $\|Bu_{n_i} - Bx_{n_i}\| \to 0$ and $\|(u_{n_i} - x_{n_i})/r_{n_i}\| \leq \|(u_{n_i} - x_{n_i})/e\| \to 0$, that is, $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$. Also by $\|u_n - z_n\| \to 0$ in Step 4, we have $u_{n_i} \rightharpoonup z$. Moreover, from the inverse-strongly monotonicity of B, we have $\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \geq 0$. So, from (A4) and the weak lower semicontinuity of φ, if follows that

$$\langle y_t - z, By_t\rangle \geq -\varphi(y_t) + \varphi(z) + \Theta(y_t, z) \quad \text{as } i \longrightarrow \infty. \tag{3.38}$$

By (A1), (A4), and (3.38), we also obtain

$$\begin{aligned}
0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, z) + t\varphi(y_t) + (1-t)\varphi(z) - \varphi(y_t) \\
&\leq t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t - z, By_t\rangle \\
&= t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)t\langle y - z, By_t\rangle,
\end{aligned} \tag{3.39}$$

and hence

$$0 \leq \Theta(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - z, By_t\rangle. \tag{3.40}$$

Letting $t \to 0$ in (3.40), we have for each $y \in C$

$$\Theta(z, y) + \langle Bz, y - z \rangle + \varphi(y) - \varphi(z) \geq 0. \tag{3.41}$$

This implies that $z \in \mathrm{GMEP}(\Theta, \varphi, B)$. Therefore, we have $z \in \Omega_1$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z'$. Then, we have $z' \in \Omega_1$. If $z \neq z'$, from the Opial condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z\| &= \liminf_{i\to\infty} \|x_{n_i} - z\| < \liminf_{i\to\infty} \|x_{n_i} - z'\| \\
&= \lim_{n\to\infty} \|x_n - z'\| = \liminf_{j\to\infty} \left\|x_{n_j} - z'\right\| \\
&< \liminf_{j\to\infty} \left\|x_{n_j} - z\right\| = \lim_{n\to\infty} \|x_n - z\|.
\end{aligned} \tag{3.42}$$

This is a contradiction. So, we have $z = z'$. This implies that

$$x_n \rightharpoonup z \in \Omega_1. \tag{3.43}$$

Also from Step 2, it follows that $u_n \rightharpoonup z \in \Omega_1$.

Let $w_n = P_{\Omega_1}(x_n)$. Since $z \in \Omega_1$, we have

$$\langle x_n - w_n, w_n - z \rangle \geq 0. \tag{3.44}$$

Since $\|x_{n+1} - p\| \leq \|x_n - p\|$ for $p \in \Omega_1$, by Lemma 2.5, we have that $\{w_n\}$ converges strongly to some $z_0 \in \Omega_1$. Since $\{x_n\}$ converges weakly to z, we have

$$\langle z - z_0, z_0 - z \rangle \geq 0. \tag{3.45}$$

Therefore, we obtain

$$z = z_0 = \lim_{n\to\infty} P_{\Omega_1}(x_n). \tag{3.46}$$

This completes the proof. □

As direct consequences of Theorem 3.1, we also obtain the following new weak convergence theorems for the problems (1.2) and (1.3) and fixed point problem of a strict pseudocontractive mapping.

Corollary 3.2. *Let H, C, Θ, B, and F be as in Theorem 3.1. Let T be a k-strictly pseudocontractive mapping of C into itself for some $k \in [0,1)$ such that $\Omega_2 := F(T) \bigcap \mathrm{GEP}(\Theta, B) \bigcap \mathrm{VI}(C, F) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{gathered}
\Theta(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
x_{n+1} = S(\alpha_n x_n + (1 - \alpha_n) P_C(u_n - \lambda_n F u_n)), \quad \forall n \geq 1,
\end{gathered} \tag{3.47}$$

where $S : C \to C$ is a mapping defined by $Sx = kx + (1-k)Tx$ for $x \in C$, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0, \infty)$. Assume that $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,1)$, $\lambda_n \in [c,d] \subset (0, 2\alpha)$ and $r_n \in [e,f] \subset (0, 2\beta)$. Then $\{x_n\}$ and $\{u_n\}$ converge weakly to $z \in \Omega_2$, where $z = \lim_{n\to\infty} P_{\Omega_2}(x_n)$.

Proof. Putting $\varphi \equiv 0$ in Theorem 3.1, we obtain the desired result. □

Corollary 3.3. *Let H, C, Θ, and B be as in Corollary 3.2. Let T be a k-strictly pseudocontractive mapping of C into itself for some $k \in [0,1)$ such that $\Omega_3 := F(T) \bigcap \mathrm{GEP}(\Theta, B) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{gathered}
\Theta(u_n, y) + \langle Bx_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
x_{n+1} = S(\alpha_n x_n + (1 - \alpha_n) u_n), \quad \forall n \geq 1,
\end{gathered} \tag{3.48}$$

where $S : C \to C$ is a mapping defined by $Sx = kx + (1-k)Tx$ for $x \in C$, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0, \infty)$. Assume that $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,1)$ and $r_n \in [e,f] \subset (0, 2\beta)$. Then $\{x_n\}$ and $\{u_n\}$ converge weakly to $z \in \Omega_3$, where $z = \lim_{n\to\infty} P_{\Omega_3}(x_n)$.

Proof. Putting $F \equiv 0$ in Corollary 3.2, we obtain the desired result. □

Corollary 3.4. *Let H, C, Θ, φ, and F be as in Theorem 3.1. Let T be a k-strictly pseudocontractive mapping of C into itself for some $k \in [0,1)$ such that $\Omega_4 := F(T) \bigcap \mathrm{MEP}(\Theta, \varphi) \bigcap \mathrm{VI}(C, F) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{gathered}
\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
x_{n+1} = S(\alpha_n x_n + (1 - \alpha_n) P_C(u_n - \lambda_n F u_n)), \quad \forall n \geq 1,
\end{gathered} \tag{3.49}$$

where $S : C \to C$ is a mapping defined by $Sx = kx + (1-k)Tx$ for $x \in C$, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0, \infty)$. Assume that $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,1)$, $\lambda_n \in [c,d] \subset (0, 2\alpha)$ and $r_n \in [e,f] \subset (0, \infty)$. Then $\{x_n\}$ and $\{u_n\}$ converge weakly to $z \in \Omega_4$, where $z = \lim_{n\to\infty} P_{\Omega_4}(x_n)$.

Proof. Putting $B \equiv 0$ in Theorem 3.1, we obtain the desired result. □

Corollary 3.5. *Let H, C, Θ and φ be as in Theorem 3.1. Let T be a k-strictly pseudocontractive mapping of C into itself for some $k \in [0,1)$ such that $\Omega_5 := F(T) \bigcap \mathrm{MEP}(\Theta,\varphi) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{gathered}\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = S(\alpha_n x_n + (1-\alpha_n)u_n), \quad \forall n \geq 1,\end{gathered} \tag{3.50}$$

where $S : C \to C$ is a mapping defined by $Sx = kx + (1-k)Tx$ for $x \in C$, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$. Assume that $\{\alpha_n\} \subset [a,b]$ for some $a,b \in (0,1)$ and $r_n \in [e,f] \subset (0,\infty)$. Then $\{x_n\}$ and $\{u_n\}$ converge weakly to $z \in \Omega_5$, where $z = \lim_{n\to\infty} P_{\Omega_5}(x_n)$.

Proof. Putting $F \equiv 0$ in Corollary 3.4, we obtain the desired result. □

Remark 3.6.

(1) As a new result for a new iterative scheme, Theorem 3.1 develops and complements the corresponding results, which were obtained recently by many authors in references and others; for example, see [22–24, 26]. In particular, even though $F \equiv 0$ in Theorem 3.1, Theorem 3.1 develops and complements Theorem 3.1 of Ceng et al. [22] in the following aspects:

  (a) the iterative scheme (3.1) in Theorem 3.1 is a new one different from those in Theorem 3.1 of [22].

  (b) the equilibrium problem in Theorem 3.1 of [22] is extended to the case of generalized mixed equilibrium problem.

(2) We point out that our iterative schemes in Corollaries 3.2, 3.3, 3.4 and 3.5 are new ones different from those in the literature (see [22–24, 26] and others in references).

Acknowledgments

The author thanks the referees for their valuable comments and suggests, which improved the presentation of this paper, and for providing some recent related papers. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0003901).

References

[1] J.-W. Peng and J.-C. Yao, "A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems," *Taiwanese Journal of Mathematics*, vol. 12, no. 6, pp. 1401–1432, 2008.

[2] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis. Series A*, vol. 69, no. 3, pp. 1025–1033, 2008.

[3] L.-C. Ceng and J.-C. Yao, "A hybrid iterative scheme for mixed equilibrium problems and fixed point problems," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.

[4] Y. Yao, M. A. Noor, S. Zainab, and Y.-C. Liou, "Mixed equilibrium problems and optimization problems," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 1, pp. 319–329, 2009.

[5] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.
[6] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.
[7] S. D. Flåm and A. S. Antipin, "Equilibrium programming using proximal-like algorithms," *Mathematical Programming*, vol. 78, pp. 29–41, 1997.
[8] F. E. Browder, "Fixed-point theorems for noncompact mappings in Hilbert space," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 53, pp. 1272–1276, 1965.
[9] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.
[10] G. L. Acedo and H.-K. Xu, "Iterative methods for strict pseudo-contractions in Hilbert spaces," *Nonlinear Analysis. Series A*, vol. 67, no. 7, pp. 2258–2271, 2007.
[11] Y. J. Cho, S. M. Kang, and X. Qin, "Some results on k-strictly pseudo-contractive mappings in Hilbert spaces," *Nonlinear Analysis. Series A*, vol. 70, no. 5, pp. 1956–1964, 2009.
[12] J. S. Jung, "Strong convergence of iterative methods for k-strictly pseudo-contractive mappings in Hilbert spaces," *Applied Mathematics and Computation*, vol. 215, no. 10, pp. 3746–3753, 2010.
[13] J. S. Jung, "Some results on a general iterative method for k-strictly pseudo-contractive mappings," *Fixed Point Theory and Applications*, vol. 2011, 24 pages, 2011.
[14] G. Marino and H.-K. Xu, "Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 336–346, 2007.
[15] C. H. Morales and J. S. Jung, "Convergence of paths for pseudocontractive mappings in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 128, no. 11, pp. 3411–3419, 2000.
[16] J. S. Jung, "Strong convergence of composite iterative methods for equilibrium problems and fixed point problems," *Applied Mathematics and Computation*, vol. 213, no. 2, pp. 498–505, 2009.
[17] J. S. Jung, "A new iteration method for nonexpansive mappings and monotone mappings in Hilbert spaces," *Journal of Inequalities and Applications*, vol. 2010, Article ID 251761, 16 pages, 2010.
[18] J. S. Jung, "A general composite iterative method for generalized mixed equilibrium problems, variational inequality problems and optimization problems," *Journal of Inequalities and Applications*, vol. 2011, article 51, 2011.
[19] T. Chamnarnpan and P. Kumam, "A new iterative method for a common solution of a fixed points for pseudo-contractive mappings and variational inequalities," *Fixed Point Theory and Applications*. In press.
[20] P. Katchang and P. Kumam, "A system of mixed equilibrium problems, a general system of variational inequaliity problems for relaxed cocoercive, and fixed point problems for nonexpansive semigroups and strictly pseudo-contractive mappings," *Journal of Applied Mathematics*, vol. 2012, Article ID 414831, 36 pages, 2012.
[21] P. Kumam, U. Hamphries, and P. Katchang, "Common solutions of generalized mixed equilibrium problems, variational inclusions, and common fixed points for nonexpansive semigroups and strictly pseudocontractive mappings," *Journal of Applied Mathematics*, vol. 2011, Article ID 953903, 28 pages, 2011.
[22] L.-C. Ceng, S. Al-Homidan, Q. H. Ansari, and J.-C. Yao, "An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings," *Journal of Computational and Applied Mathematics*, vol. 223, no. 2, pp. 967–974, 2009.
[23] C. Jaiboon, P. Kumam, and U. W. Humphries, "Weak convergence theorem by an extragradient method for variational inequality, equilibrium and fixed point problems," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 32, no. 2, pp. 173–185, 2009.
[24] A. Moudafi, "Weak convergence theorems for nonexpansive mappings and equilibrium problems," *Journal of Nonlinear and Convex Analysis*, vol. 9, no. 1, pp. 37–43, 2008.
[25] S. Plubtieng and P. Kumam, "Weak convergence theorem for monotone mappings and a countable family of nonexpansive mappings," *Journal of Computational and Applied Mathematics*, vol. 224, no. 2, pp. 614–621, 2009.
[26] A. Tada and W. Takahashi, "Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem," *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 359–370, 2007.
[27] W. Takahashi and M. Toyoda, "Weak convergence theorems for nonexpansive mappings and monotone mappings," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 417–428, 2003.
[28] R. T. Rockafellar, "On the maximality of sums of nonlinear monotone operators," *Transactions of the American Mathematical Society*, vol. 149, pp. 75–88, 1970.

[29] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.

[30] J. Schu, "Weak and strong convergence to fixed points of asymptotically nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 153–159, 1991.

[31] H. Zhou, "Convergence theorems of fixed points for κ-strict pseudo-contractions in Hilbert spaces," *Nonlinear Analysis. Series A*, vol. 69, no. 2, pp. 456–462, 2008.