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Research Article

Positive Solution for a Class of Boundary Value Problems with Finite Delay

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We study a class of boundary value problems with equation of the form $x''(t)+f(t,x(t),x'(t-\tau))=0$. Some sufficient conditions for existence of positive solution are obtained by using the Krasnoselskii fixed point theorem in cones.

1. Introduction

Because of applications in physics, bioscience, epidemiology and so on, most literature about differential equations with delay has focused on existence of periodic solution, oscillation, and so on. In recent years, there has been increasing interest in boundary value problems for differential equations with delay, see for example [1–8]. By using fixed-point theorems, the authors discussed the existence of one solution or twin, triple solutions for differential equations of the forms

$$\begin{gathered} x''(t)+f(t,x(t),x(t-\tau))=0, \quad 0<t<1, \\ x''(t)+f\big(t,x(t),x(t-\tau),x'(t)\big)=0, \quad 0<t<1, \\ x''(t)+f\big(t,x(t),x'(t-1)\big)=0, \quad 0<t<1, \end{gathered} \tag{1.1}$$

with boundary value conditions

$$\begin{gathered} x(t)=\xi(t), \quad t\in[-\tau,0] \text{ or } t\in[-1,0], \\ x(1)=0. \end{gathered} \tag{1.2}$$

Here $\xi(t)$ may be identically 0 in $[-\tau,0]$ or $[-1,0]$.

Especially, Shu et al. investigated some sufficient conditions for the existence of triple solutions for the following boundary value problem in [1] by using Avery-Peterson fixed point theorem:

$$\begin{gathered} x''(t) + f\left(t, x(t), x(t-\tau), x'(t), x'(t-\tau)\right) = 0, \quad 0 < t < 1, \\ x(t) = \xi(t), \quad t \in [-\tau, 0], \\ x(1) = 0. \end{gathered} \tag{1.3}$$

For such a boundary value problem with delay in the first-order derivative, the differentiability of $x(t)$ at $t = 0$ is very important. In fact, a series of problems arise when we discuss boundary value problems like (1.3) without differentiability of $x(t)$ at $t = 0$.

In this paper, we study the following boundary value problem with finite delay:

$$\begin{gathered} x''(t) + p(t) f\left(t, x(t), x'(t-\tau)\right) = 0, \quad 0 < t < 1,\ 0 \le \tau < \frac{1}{2}, \\ x'(t) = \xi(t), \quad t \in [-\tau, 0], \\ x(1) = 0, \end{gathered} \tag{1.4}$$

where $f : [0,1] \times [0,+\infty) \times R \to [0,+\infty)$ and $p : (0,1) \to [0,+\infty)$ are both continuous; $\int_0^1 (1-t)p(t)dt < \infty$ and $p(t)$ is not identically 0 on any subinterval of $(0,1)$; $\xi : [-\tau, 0] \to R$ is continuous and $\xi(0) = 0$.

With some semilinear and superlinear hypotheses about f, we provide some sufficient conditions for existence of positive solution. The following lemma is fundamental in the proofs of our results.

Lemma 1.1 (Krasnoselskii fixed point theorem [9]). *Let E be a Banach space and $K \subset E$ a cone. Assume Ω_1 and Ω_2 are bounded open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that one of the following holds:*

(i) $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$;

(ii) $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$, *and* $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Main Results

In this section, we consider the existence of at least one positive solution to boundary value problem (1.4). Firstly, we present some necessary definitions and notes.

Definition 2.1. A function x is said to be a positive solution of the boundary value problem (1.4) if it satisfies the following conditions:

(i) $x \in C^2(0,1) \cap C^1[0,1]$;

(ii) $x(t) > 0$, $t \in (0,1)$; $x'(0) = x(1) = 0$;

(iii) $x''(t) + p(t) f(t, x(t), \xi(t-\tau)) = 0$ holds for $t \in (0, \tau]$ and $x''(t) + p(t) f(t, x(t), x'(t-\tau)) = 0$ holds for $t \in [\tau, 1]$.

If x is a positive solution of (1.4), we can easily check that it can be expressed by

$$\begin{aligned} x(t) &= \int_0^{\tau} G(t,s)p(s)f(s,x(x),\xi(s-\tau))ds \\ &\quad + \int_{\tau}^{1} G(t,s)p(s)f\big(s,x(x),x'(s-\tau)\big)ds, \quad t\in[0,1], \end{aligned} \tag{2.1}$$

where $G(t,s)=\min\{1-t,1-s\}$.

Let $E=C[0,1]\cap C^1[0,1-\tau]$ with norm $\|\cdot\|$ given by

$$\|x\| = \max\left\{\max_{t\in[0,1]}|x(t)|, \max_{t\in[0,1-\tau]}\left|x'(t)\right|\right\}. \tag{2.2}$$

Obviously, $(E,\|\cdot\|)$ is a Banach space.

Now we define a cone $K\subset E$ as $K=\{x\in E \mid x'(0)=x(1)=0;\ x \text{ is concave on } [0,1]\}$. Then for all $x\in K$, $\|x\|=\max\{x(0),|x'(1-\tau)|\}$, and we have the following lemma.

Lemma 2.2. *If $x\in K$, then $|x'(t)|\le x(t)/(1-t)$, $t\in[0,1-\tau]$.*

This lemma follows from the concavity of x in a straightforward way, and we omit the details of the proof.

Now we define a map T on K by

$$Tx(t) = \int_0^{\tau} G(t,s)p(s)f(s,x(x),\xi(s-\tau))ds + \int_{\tau}^{1} G(t,s)p(s)f\big(s,x(x),x'(s-\tau)\big)ds, \tag{2.3}$$

where $G(t,s)=\min\{1-t,1-s\}$. It shows straightforward show the following lemma.

Lemma 2.3. *$T: K \to K$ is completely continuous.*

Our aim now is to show that T has a fixed point. We introduce the following notations:

$$\delta = \min\left\{\int_0^1 (1-s)p(s)ds, \int_0^{1-\tau} p(s)ds\right\}, \qquad \gamma = \min_{t\in[-\tau,0]}\xi(t), \qquad \mu = \max_{t\in[-\tau,0]}\xi(t). \tag{2.4}$$

Theorem 2.4. *Suppose the following conditions are satisfied:*

(H1) $\exists r>0$, $f(t,x,y)\le r\delta^{-1}$, $(t,x,y)\in[0,1]\times[0,r]\times[\min\{-r,\gamma\},\max\{0,\mu\}]$;

(H2) $\exists R>r\tau^{-1}$, $f(t,x,y)\ge R/\int_{\tau}^{1-\tau}p(s)ds$, $(t,x,y)\in[\tau,1-\tau]\times[R\tau,R]\times[-R,0]$.

Then boundary value problem (1.4) *has at least one positive solution.*

Proof. Let $\Omega_1=\{x\in E \mid \|x\|<r\}$, then for all $x\in K\cap\partial\Omega_1$, $\|x\|=\max\{x(0),|x'(1-\tau)|\}=r$, we will show $\|Tx\|=\max\{Tx(0),|(Tx)'(1-\tau)|\}\le r$.

With (H1), we have

$$Tx(0) = \int_0^1 (1-s)p(s)f\big(s, x(s), x'(s-\tau)\big)ds \le r\delta^{-1} \cdot \int_0^1 (1-s)p(s)ds \le r,$$
$$\big|(Tx)'(1-\tau)\big| = \int_0^{1-\tau} p(s)f\big(s, x(s), x'(s-\tau)\big)ds \le r\delta^{-1} \cdot \int_0^{1-\tau} p(s)ds \le r. \tag{2.5}$$

Equation (2.5) provides that for all $x \in K \cap \partial\Omega_1$, $\|Tx\| \le r$.

On the other hand, let $\Omega_2 = \{x \in E \mid \|x\| < R\}$, then for all $x \in K \cap \partial\Omega_2$, $\|x\| = \max\{x(0), |x'(1-\tau)|\} = R$.

If $x(0) = R$, then $|x'(1-\tau)| \le R$. The concavity of x provides $x(1-\tau) \ge R\tau$. Therefore, for all $t \in [0, 1-\tau]$, $x(t) \in [R\tau, R]$, $x'(t) \in [-R, 0]$.

If $|x'(1-\tau)| = R$, then $x(0) \le R$, and Lemma 2.2 provides $x(1-\tau) \ge |x'(1-\tau)|\tau = R\tau$. Therefore, for all $t \in [0, 1-\tau]$, $x(t) \in [R\tau, R]$, $x'(t) \in [-R, 0]$.

In summary, for all $x \in K \cap \partial\Omega_2$, $x(t) \in [R\tau, R]$ and $x'(t) \in [-R, 0]$ both hold for $t \in [0, 1-\tau]$. Then conditions (H2) guarantee

$$\begin{aligned}\big|(Tx)'(1-\tau)\big| &= \int_0^{1-\tau} p(s)f\big(s, x(s), x'(s-\tau)\big)ds,\\ &\ge \int_\tau^{1-\tau} p(s)f\big(s, x(s), x'(s-\tau)\big)ds,\\ &\ge \frac{R}{\int_\tau^{1-\tau} p(s)ds} \cdot \int_\tau^{1-\tau} p(s)ds = R.\end{aligned} \tag{2.6}$$

That is, for all $x \in K \cap \partial\Omega_2$, $\|Tx\| = \max\{Tx(0), |(Tx)'(1-\tau)|\} \ge \|x\| = R$. Then, Lemmas 2.3 and 1.1 guarantee the existence of a positive solution to boundary value problem (1.4). □

Corollary 2.5. *Suppose the following conditions are satisfied:*

(H3) $\lim_{x\to 0+}(f(t,x,y)/x < \delta^{-1})$ *for* $t \in [0,1]$ *and* $|y| \le \max\{1, \max_{t\in[-\tau,0]}|\xi(t)|\}$;

(H4) $\lim_{x\to+\infty}\inf_{y\in[-x\tau^{-1},0]}(f(t,x,y)/x) > 1/\tau\int_\tau^{1-\tau} p(s)ds$ *for* $t \in [\tau, 1-\tau]$.

Then boundary value problem (1.4) *has at least one positive solution.*

Proof. By (H3), there should exist a constant $0 < r' < \tau\mu$, $f(t,x,y) \le x\delta^{-1} < r'\delta^{-1}$ for $t \in [0,1]$, $0 < x < r'$ and $|y| \le \max|\xi(t)|$. Let $r = r'$, then (H1) is satisfied.

By (H4), there should exist a constant $R' > r$, so that $f(t,x,y) \ge x/\tau\int_\tau^{1-\tau} p(s)ds$ for $t \in [\tau, 1-\tau]$, $x > R'$ and $y \in [-x\tau^{-1}, 0]$. Let $R = R'\tau^{-1}$, then (H2) is satisfied. □

Theorem 2.6. *Suppose the following conditions are satisfied:*

(H5) $\exists r > 0$, $f(t,x,y) \ge r/\int_\tau^{1-\tau} p(s)ds$, $(t,x,y) \in [\tau, 1-\tau] \times [r\tau, r] \times [-r, 0]$;

(H6) $\lim_{x\to+\infty}\sup_{y\in[-x\tau^{-1},\max\{0,\mu\}]}(f(t,x,y)/x) < \lambda\delta^{-1}$ *for* $t \in [0,1]$, *where* $0 < \lambda < 1$;

(H7) $\lim_{y\to-\infty}(f(t,x,y)/|y|) < \lambda\delta^{-1}$ *for* $t \in [0,1]$ *and any fixed* $x \ge 0$.

Then boundary value problem (1.4) *has at least one positive solution.*

Proof. Let $\Omega_1 = \{x \in E \mid \|x\| < r\}$, then for all $x \in K \cap \partial\Omega_1$, $\max\{x(0), |x'(1-\tau)|\} = r$ and for all $t \in [0, 1-\tau]$, $x(t) \in [r\tau, r]$, $x'(t) \in [-r, 0]$. With (H5), we have

$$\left|(Tx)'(1-\tau)\right| = \int_0^{1-\tau} p(s) f\left(s, x(s), x'(s-\tau)\right) ds \geq \int_\tau^{1-\tau} p(s) f\left(s, x(s), x'(s-\tau)\right) ds \geq r, \quad (2.7)$$

that is, $\|Tx\| \geq \|x\| = r$, for all $x \in K \cap \partial\Omega_1$.

On the other hand, we need two steps to show that $\exists R > 0$ and $\Omega_2 = \{x \in E \mid \|x\| < R\}$ such that for all $x \in K \cap \partial\Omega_2$, $\|Tx\| \leq \|x\|$.

Step 1. (H6) guarantees $\exists R' > 0$, for all $x > R'$, $f(t, x, y) < x\delta^{-1}$ for $t \in [0, 1]$ and $y \in [-x\tau^{-1}, \mu]$.

Let $R > R'\tau^{-1}$, $\Omega_2 = \{x \in E \mid \|x\| < R\}$. For all $x \in K \cap \partial\Omega_2$, we have $R' < R\tau \leq x(t) \leq R$, $x'(t) \in [-R, 0]$, $t \in [0, 1-\tau]$. Then by (H6), we have

$$\left|(Tx)'(1-\tau)\right| = \int_0^{1-\tau} p(s) f\left(s, x(s), x'(s-\tau)\right) ds < \lambda R. \quad (2.8)$$

Step 2. For all $x \in K \cap \partial\Omega_2$, the concavity of x implies $x(t) \geq x(0)(1-t) \geq R\tau(1-t)$, that is, $t \geq 1 - x(t)R^{-1}\tau^{-1}$. Suppose $x(t_x) = R'$, then $t_x \geq 1 - R'R^{-1}\tau^{-1}$.

At the same time, (H7) guarantees there exists $M > 0$, for all $y < -M$, $f(t, x, y) \leq |y|\lambda\delta^{-1}$.

For all $x \in K \cap \Omega_2$, define $t'_x \in [\tau, 1]$ as follows:

(I) if $x'(1-\tau) \geq -M$, then $t'_x = 1$;

(II) if $x'(1-\tau) < -M$, then there should exist $t_0 \in [\tau, 1]$, $x'(t_0 - \tau) = -M$. $t_0 \leq t_x$, let $t'_x = t_x$; $t_0 > t_x$, let $t'_x = t_0$.

So for all $x \in K \cap \Omega_2$, interval $[0, 1]$ can be divided into three parts: $t \in [0, t_x]$, $R' \leq x(t) \leq R$; $t \in [t_x, t'_x]$, $0 \leq x(t) \leq R'$, $-M \leq x'(t-\tau) \leq 0$; $t \in [t'_x, 1]$, $0 \leq x(t) \leq R'$, $x'(t-\tau) < -M$.

$$\begin{aligned} Tx(0) &= \int_0^1 (1-s) p(s) f\left(s, x(s), x'(s-\tau)\right) ds \\ &< \int_0^{t_x} (1-s) p(s) \lambda\delta^{-1} x(s) ds + \max_{(t,x,y) \in \left[1 - R'R^{-1}\tau^{-1}, 1\right] \times [0, R'] \times [-M, 0]} f(t, x, y) \cdot \int_{t_x}^{t'_x} (1-s) p(s) ds \end{aligned}$$

$$
\begin{aligned}
&+\int_{t'_x}^{1}(1-s)p(s)\cdot\lambda\delta^{-1}\left|x'(s-\tau)\right|ds \\
&\leq \lambda R\delta^{-1}\int_0^{t_x}(1-s)p(s)+\max_{(t,x,y)\in\left[1-R'R^{-1}\tau^{-1},1\right]\times[0,R']\times[-M,0]} f(t,x,y)\cdot\int_{t_x}^{t'_x}(1-s)p(s)ds \\
&\quad+\lambda R\delta^{-1}\int_{t'_x}^{1}(1-s)p(s)\cdot\left|x'(s-\tau)\right|ds \\
&\leq \lambda R\delta^{-1}\int_0^{1}(1-s)p(s)ds+\max_{(t,x,y)\in\left[1-R'R^{-1}\tau^{-1},1\right]\times[0,R']\times[-M,0]} f(t,x,y)\cdot\int_{t_x}^{t'_x}(1-s)p(s)ds \\
&\leq \lambda R+\max_{(t,x,y)\in\left[1-R'R^{-1}\tau^{-1},1\right]\times[0,R']\times[-M,0]} f(t,x,y)\cdot\int_{t_x}^{t'_x}(1-s)p(s)ds.
\end{aligned}
\tag{2.9}
$$

Obviously, let $R \rightarrow +\infty$, then $t_x, t'_x \rightarrow 1$, the second part of (2.9) will tend to 0. So we can find $R > 0$, and $\Omega_2 = \{x \in E \mid \|x\| < R\}$, for all $x \in K \cap \partial\Omega_2$, $Tx(0) \leq \|x\| = R$. Together with (2.8), we have for all $x \in K \cap \partial\Omega_2$, $\|Tx\| \leq \|x\|$. Then by Lemma 1.1, we can complete the proof. □

Corollary 2.7. *Suppose (H6) and (H7) are all satisfied, moreover,*

(H8) $\lim_{x\to 0+}(f(t,x,y)/x) > 1/\tau\int_\tau^{1-\tau} p(s)ds$ *for* $t \in [\tau, 1-\tau]$ *and* $|y| \leq 1$.

Then boundary value problem (1.4) *has at least one positive solution.*

Proof. By (H8), there should exist $r' > 0$ such that $f(t,x,y) \geq x/\tau\int_\tau^{1-\tau} p(s)ds$ for $t \in [\tau, 1-\tau]$, $x \in [0, r']$ and $|y| \leq 1$. Let $r = r'\tau^{-1}$, then (H5) is satisfied. □

3. Examples

Example 3.1. Consider the following boundary value problem:

$$
\begin{gathered}
x''(t)+(1-t)^{-1}\cdot e^t x^2(t)\left(1+\left(x'\left(t-\frac{1}{3}\right)\right)^2\right)=0, \quad t\in(0,1), \\
x'(t)=\sin t, \quad t\in\left[-\frac{1}{3},0\right], \\
x(1)=0.
\end{gathered}
\tag{3.1}
$$

Here $p(t) = (1-t)^{-1}$, $f(t,x,y) = e^t x^2(1+y^2)$, $\xi(t) = \sin t$. We can check that (H3) and (H4) are all satisfied. Then Corollary 2.5 implies that (3.1) has at least one positive solution.

Example 3.2. Consider boundary value problem:

$$\begin{gathered} x''(t) + (1-t)^{-1} \cdot e^t x^{1/3}(t)\left(1 + \left(x'\left(t - \frac{1}{3}\right)\right)^{2/3}\right) = 0, \quad t \in (0,1), \\ x'(t) = \sin t, \quad t \in \left[-\frac{1}{3}, 0\right], \\ x(1) = 0. \end{gathered} \tag{3.2}$$

Here $p(t) = (1-t)^{-1}$, $f(t,x,y) = e^t x^{1/3}(1+y^{2/3})$, $\xi(t) = \sin t$. We can check that (H6), (H7), and (H8) are all satisfied and so Corollary 2.7 implies that (3.2) has at least one positive solution.

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