Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 379848, 12 pages
doi:10.1155/2012/379848

Research Article

The Core and Nucleolus in a Model of Information Transferal

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Received 29 May 2012; Accepted 28 August 2012

Academic Editor: Marco H. Terra

Galdeano et al. introduced the so-called information market game involving n identical firms acquiring a new technology owned by an innovator. For this specific cooperative game, the nucleolus is determined through a characterization of the symmetrical part of the core. The nonemptiness of the (symmetrical) core is shown to be equivalent to one of each, super additivity, zero-monotonicity, or monotonicity.

1. Introduction of the Information Market Game

Consider the following problem [1]. Besides n firms with identical characteristics, there exists an agent called the innovator, having relevant information for the firms. The innovator is not going to use the information for himself, but this information can be sold to the firms. Any firm that decides to acquire the new information (e.g., a new technology) is supposed to make use of the information. The n potential users of the information are the same before and after the innovator offers the new technology. The firms acquiring the information will be better than before obtaining it, while their utilities are computed under a conservator point of view, assuming that for any uninformed firm, the probability of making the right decision can be described by a *binomial probability distribution*, being $0 \leq p \leq 1$ the uniform probability of having success. The probability that k among n firms take the right decision is given by $\binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$, and hence, the *expected aggregated utility* of k firms having success is given by $k \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \cdot u_k$. Here $u_k \geq 0$ represents the utility if k firms make a right decision. Throughout the paper, the utility function is monotonic decreasing because when the number of firms taking the right decision increases, each firm receives a lower utility level, that is, $u_{k+1} \leq u_k$ for all $k \geq 1$ (not necessarily normalized in that $u_1 = 1$).

This information trading problem has been modeled by Galdeano et al. [1] as a cooperative game $\langle N, v\rangle$ in characteristic function form, where the set of firms $N = \{1, 2, \ldots, n+1\}$ consists of the innovator 1, having new information, and the users $2, 3, \ldots, n+1$, who could be willing to buy the new information. Throughout the paper, the size (or cardinality) of any coalition $S \subseteq N$ is denoted by $s, 0 \leq s \leq n+1$. In case coalition S contains the innovator, then its worth $v(S)$ in the so-called information market game equals $(s-1)\cdot u_n$ because any member of S, different from the innovator, took the right decision rewarding the expected utility u_n since the $n-s$ uninformed firms outside S are assumed to take right decisions too.

Definition 1.1. The $(n+1)$-person information market game $\langle N, v\rangle$ in characteristic function form is given by $v(\emptyset) = 0$ and on the one hand (cf. [1]),

$$v(S) = (s-1)\cdot u_n \quad \forall S \subseteq N \text{ with } 1 \in S \text{ and on the other,} \tag{1.1}$$

$$v(S) = f_n(s) = \sum_{j=1}^{s} j \cdot \binom{s}{j} \cdot p^j \cdot (1-p)^{s-j} \cdot u_{n-s+j} \quad \forall S \subseteq N,\ S \neq \emptyset,\ 1 \notin S. \tag{1.2}$$

If the innovator is not a member of coalition S, each one of k successful users rewards an expected utility the amount of $\binom{s}{k}\cdot p^k \cdot (1-p)^{s-k}\cdot u_{n-s+k}$ by assumption of the uninformed users outside S taking the right decisions. Particularly, the information market game satisfies $v(\{1\}) = 0$, and $v(\{i\}) = f_n(1) = p\cdot u_n$ for all $i \in N$, $i \neq 1$. Furthermore, $v(N) = n\cdot u_n$, $v(N\setminus\{i\}) = (n-1)\cdot u_n$ for all $i \in N$, $i\neq 1$, whereas $v(N\setminus\{1\}) = f_n(n)$. Consequently, the *marginal contributions* $b_i^v = v(N) - v(N\setminus\{i\})$, $i \in N$, are given by $b_i^v = u_n$ for all $i \in N$, $i \neq 1$, whereas $b_1^v = n\cdot u_n - f_n(n)$. It is left to the reader to verify

$$v(N) - v(S) = \sum_{i\in N\setminus S} [v(N) - v(N\setminus\{i\})] \quad \forall S \subseteq N \text{ with } 1 \in S. \tag{1.3}$$

The case $p = 1$ yields $v(S) = s\cdot u_n$ for all $S \subseteq N\setminus\{1\}$ and so, it concerns the inessential (additive) game corresponding with the vector $(0, u_n, u_n, \ldots, u_n) \in \mathbb{R}^{n+1}$. The case $p = 0$ yields zero worth to all coalitions not containing the innovator and so, it concerns the so-called big boss game [2] (with the innovator acting as the big boss). We summarize the main result(s) of Galdeano et al. [1].

Theorem 1.2. *For the $(n+1)$-person information market game $\langle N, v\rangle$ of the form* (1.1)-(1.2)*, the following three statements are equivalent.*

(i) *Zero-monotonicity, that is,*

$$v(S\cup\{i\}) \geq v(S) + v(\{i\}) \quad \forall i \in N,\ S \subseteq N\setminus\{i\}, \tag{1.4}$$

(ii) $s\cdot u_n \geq f_n(s)$ *for all* $1 \leq s \leq n$,

(iii) *(cf. [1, Theorem 2, page 25])*

$$\frac{u_n}{u_1} \geq \frac{p\cdot(1-p)^{n-2}}{1 + p\cdot(1-p)^{n-2}} \quad \text{applied to the normalization } u_1 = 1. \tag{1.5}$$

Besides their study of zero-monotonicity, Galdeano et al. determine the Shapley value of the information market game (cf. Theorem 4, page 27) and compare the Shapley value with the equilibrium outcome (cf. Theorem 7, page 29) in the noncooperative model analyzed by [3]. The main goal of the current paper is to determine the nucleolus of the information market game and for that purpose, we explore and characterize the symmetrical part of the core, provided nonemptiness of the core.

2. Properties of the Information Market Game

This section reports properties of the characteristic function for the information market game. In fact, we claim the equivalence of three game properties (called super-additivity, zero-monotonicity, and monotonicity). The proof of their equivalence is based on the monotonic increasing average profit function for coalitions not containing the innovator, that is, $f_n(s)/s \leq f_n(s+1)/(s+1)$ for all $1 \leq s \leq n-1$. This significant property has not been discovered before and allows us to report an equivalence theorem, which sharpens the previous Theorem 1.2.

Definition 2.1. Generally speaking, a cooperative game $\langle N, v\rangle$ in characteristic function form is said to be *super-additive, zero-monotonic, and monotonic,* respectively, if its characteristic function v satisfies $v(\emptyset) = 0$ and

(i) $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$ (super-additivity).

(ii) $v(S) + v(\{i\}) \leq v(S \cup \{i\})$ for all $i \in N$ and all $S \subseteq N \setminus \{i\}$ (zero-monotonicity).

(iii) $v(S) \leq v(T)$ for all $S, T \subseteq N$ with $S \subseteq T$ (monotonicity).

Theorem 2.2. *For the* $(n+1)$*-person information market game* $\langle N, v\rangle$ *of the form* (1.1)-(1.2), *the following four statements are equivalent:*

$$\textit{Super-additivity} \Longleftrightarrow \textit{Zero-monotonicity} \Longleftrightarrow \textit{Monotonicity} \Longleftrightarrow \frac{f_n(n)}{n} \leq u_n. \tag{2.1}$$

Obviously, super-additivity implies zero-monotonicity and in turn, zero-monotonicity implies monotonicity (for nonnegative games). The proof of the Equivalence Theorem 2.2 will be based on the fundamental lemma concerning the monotonicity of averaging the profit function $f_n(s)$ *of the form* (1.2).

Lemma 2.3. *The average function given by* $f_n(s)/s = \sum_{j=1}^{s}\binom{s-1}{j-1} \cdot p^j \cdot (1-p)^{s-j} \cdot u_{n-s+j}$ *satisfies*

(i) $f_n(s)/s \leq f_n(s+1)/(s+1)$ *for all* $1 \leq s \leq n-1$,

(ii) $f_n(s+t) \geq f_n(s) + f_n(t)$ *for all* $1 \leq s,\ t \leq n-1$ *with* $s+t \leq n$.

Proof of Lemma 2.3. Let $1 \leq s \leq n-1$. Concerning the case $s = 1$, note that $f_n(1) = p \cdot u_n$ as well as $f_n(2) = 2 \cdot p \cdot (1-p) \cdot u_{n-1} + 2 \cdot p^2 \cdot u_n$ and so, the inequality $f_n(2) \geq 2 \cdot f_n(1)$ holds due to the fact $(1-p)\cdot u_{n-1}+p\cdot u_n \geq u_n$. Generally speaking, the proof is based on the combinatorial relationship $\binom{s}{j-1}=\binom{s-1}{j-1}+\binom{s-1}{j-2}$ for all $2\leq j\leq s$ and proceeds as follows:

$$\begin{aligned}
\frac{f_n(s+1)}{s+1} &= \sum_{j=1}^{s+1}\binom{s}{j-1}\cdot p^j\cdot(1-p)^{s+1-j}\cdot u_{n-s-1+j}\\
&= p\cdot(1-p)^s\cdot u_{n-s}\\
&\quad + p^{s+1}\cdot u_n+\sum_{j=2}^{s}\left[\binom{s-1}{j-1}+\binom{s-1}{j-2}\right]\cdot p^j\cdot(1-p)^{s+1-j}\cdot u_{n-s-1+j}\\
&= p\cdot(1-p)^s\cdot u_{n-s}+\sum_{j=2}^{s}\binom{s-1}{j-1}\cdot p^j\cdot(1-p)^{s+1-j}\cdot u_{n-s-1+j}\\
&\quad + p^{s+1}\cdot u_n+\sum_{j=2}^{s}\binom{s-1}{j-2}\cdot p^j\cdot(1-p)^{s+1-j}\cdot u_{n-s-1+j}\\
&= p\cdot(1-p)^s\cdot u_{n-s}+\sum_{j=2}^{s}\binom{s-1}{j-1}\cdot p^j\cdot(1-p)^{s+1-j}\cdot u_{n-s-1+j}\\
&\quad + p^{s+1}\cdot u_n+\sum_{k=1}^{s-1}\binom{s-1}{k-1}\cdot p^{k+1}\cdot(1-p)^{s-k}\cdot u_{n-s+k}\\
&= \sum_{j=1}^{s}\binom{s-1}{j-1}\cdot p^j\cdot(1-p)^{s-j}\cdot\left[(1-p)\cdot u_{n-s-1+j}+p\cdot u_{n-s+j}\right]\\
&\geq \sum_{j=1}^{s}\binom{s-1}{j-1}\cdot p^j\cdot(1-p)^{s-j}\cdot u_{n-s+j}=\frac{f_n(s)}{s},
\end{aligned}\tag{2.2}$$

where the relevant inequality holds because the monotonic decreasing sequence $(u_k)_{k\in\mathbb{N}}$ satisfies $(1-p)\cdot u_{n-s-1+j}+p\cdot u_{n-s+j}\geq u_{n-s+j}$ for all $1\leq j\leq s$. This proves part (i). Concerning part (ii), suppose without loss of generality, $1\leq s\leq t\leq n-1$ with $s+t\leq n$. By applying part (i) twice, we obtain

$$f_n(s+t)\geq(s+t)\cdot\frac{f_n(t)}{t}=f_n(t)+s\cdot\frac{f_n(t)}{t}\geq f_n(t)+f_n(s).\tag{2.3}$$

□

Proof of Theorem 2.2. The super-additivity condition for disjoint, nonempty coalitions $S,T\subseteq N\setminus\{1\}$ (not containing the innovator 1) reduces to $f_n(s+t)\geq f_n(s)+f_n(t)$, whose inequality holds by Lemma 2.3(ii). For disjoint, nonempty coalitions $S,T\subseteq N$ with $1\in T$, $1\notin S$, it holds that $v(S\cup T)-v(T)=(s+t-1)\cdot u_n-(t-1)\cdot u_n=s\cdot u_n=v(S\cup\{1\})$ and so, the corresponding super-additivity condition reduces to $v(S)\leq v(S\cup\{1\})$ or equivalently, $f_n(s)\leq s\cdot u_n$ for all $1\leq s\leq n$. By Lemma 2.3(i), it is necessary and sufficient that $f_n(n)/n\leq u_n$. This proves the equivalence super-additivity $\Leftrightarrow f_n(n)/n\leq u_n$.

The zero-monotonicity condition for coalitions S containing the innovator is redundant (since $u_n \geq p \cdot u_n$). Among coalitions S not containing the innovator, the zero-monotonicity condition reduces to either $f_n(s+1) \geq f_n(s) + f_n(1)$, whose inequality holds by Lemma 2.3(ii), or $s \cdot u_n \geq f_n(s)$. As before, it is necessary and sufficient that $u_n \geq f_n(n)/n$.

Finally, note that the monotonicity condition requires $v(S) \leq v(S \cup \{1\})$ for all $S \subsetneq N \setminus \{1\}$, $S \neq \emptyset$, or equivalently, $f_n(s) \leq s \cdot u_n$ for all $1 \leq s \leq n$. □

3. The Core of the Information Market Game

Generally speaking, marginal contributions of players are well known as upper bounds for pay-offs according to core allocations, that is, $x_i \leq v(N) - v(N \setminus \{i\})$ for all $i \in N$ and all $\vec{x} \in \mathrm{CORE}(N, v)$. Throughout the paper, given a pay-off vector $\vec{x} = (x_i)_{i \in N} \in \mathbb{R}^{n+1}$ and a coalition $S \subseteq N$, we denote $\vec{x}(S) = \sum_{i \in S} x_i$, where $\vec{x}(\emptyset) = 0$. The core allocations are selected through *efficiency and group rationality*. The core, however, is a set-valued solution concept, which fails to satisfy the symmetry property in that users of the same type receive identical pay-offs according to core allocations. In order to determine the single-valued solution concept called nucleolus [4], being some symmetrical core allocation, our main goal is to investigate the symmetrical part of the core.

Definition 3.1. (i)

$$\mathrm{CORE}(N, v) = \left\{ \vec{x} \in \mathbb{R}^{n+1} \mid \vec{x}(N) = v(N),\ \vec{x}(S) \geq v(S)\ \forall S \subseteq N \right\}. \tag{3.1}$$

(ii) The symmetrical core allocations require equal pay-offs to users, that is,

$$\mathrm{SymCORE}(N, v) = \{ \vec{x} = (x_i)_{i \in N} \in \mathrm{CORE}(N, v) \mid x_2 = x_3 = \cdots = x_n = x_{n+1} \}. \tag{3.2}$$

Lemma 3.2. (i) *Any game $\langle N, v \rangle$ with a nonempty core, $\mathrm{CORE}\,(N, v) \neq \emptyset$, satisfies $v(N) \geq v(N \setminus \{i\}) + v(\{i\})$ for all $i \in N$.*

(ii) *In case $p = 1$, the core of the information market game is a singleton such that $\mathrm{CORE}\,(N, v) = \{(0, u_n, u_n, \ldots, u_n)\}$.*

(iii) *In case $0 \leq p < 1$, if the information market game possesses a nonempty core, then $b_1^v \geq 0$, or equivalently, $n \cdot u_n \geq f_n(n)$.*

(iv) *If $\vec{x} = (x_i)_{i \in N}$ satisfies $\vec{x}(N) = v(N)$ as well as $x_i \leq v(N) - v(N \setminus \{i\})$ for all $i \in N$, $i \neq 1$, then the core constraints $\vec{x}(S) \geq v(S)$ are redundant for all coalitions $S \subseteq N$ with $1 \in S$.*

Proof . (i) Choose $\vec{x} \in \mathrm{CORE}(N, v)$ if core is nonempty. Clearly, by (3.1), for all $i \in N$,

$$v(N) = \vec{x}(N) = \vec{x}(N \setminus \{i\}) + x_i \geq v(N \setminus \{i\}) + x_i \geq v(N \setminus \{i\}) + v(\{i\}). \tag{3.3}$$

(ii) In case $p = 1$, then the core-constraints $v(\{i\}) \leq x_i \leq v(N) - v(N \setminus \{i\})$ reduce to $p \cdot u_n \leq x_i \leq u_n$ and so, $x_i = u_n$ for all $\vec{x} \in \mathrm{CORE}(N, v)$, and all $i \in N$, $i \neq 1$. Consequently, by efficiency, $x_1 = 0$. The resulting vector $(0, u_n, u_n, \ldots, u_n)$ does indeed satisfy all the core constraints.

(iii) In case $0 \leq p < 1$, apply part (i) to the information market game to conclude that $b_1^v = v(N) - v(N \setminus \{1\}) \geq v(\{1\}) = 0$ and so, $b_1^v \geq 0$, or equivalently, $n \cdot u_n \geq f_n(n)$.

(iv) Under the given circumstances, $1 \in S$, together with (1.3), we derive the following:

$$\vec{x}(S) = v(N) - \vec{x}(N \setminus S) \geq v(N) - \sum_{i \in N \setminus S} [v(N) - v(N \setminus \{i\})] = v(S). \tag{3.4}$$

□

Theorem 3.3. *For the* $(n+1)$*-person information market game* $\langle N, v \rangle$ *of the form* (1.1)-(1.2) *with* $0 \leq p < 1$*, the following five statements are equivalent.*

(i) *The core is non-empty,* CORE $(N, v) \neq \emptyset$.

(ii) *The symmetrical core is non-empty,* SymCORE $(N, v) \neq \emptyset$.

(iii) $b_1^v \geq 0$.

(iv) $f_n(n)/n \leq u_n$.

(v) {*Super-additivity, Zero-monotonicity, Monotonicity*}.

The implication $(i) \Rightarrow (iii)$ is due to Lemma 3.2(iii). Notice the equivalences $(iii) \Leftrightarrow (iv)$ as well as $(iv) \Leftrightarrow (v)$. The implication $(ii) \Rightarrow (i)$ is trivial. It remains to show the implication $(iv) \Rightarrow (ii)$, the proof of which will be postponed till Section 4.

Remark 3.4. The significant condition $f_n(n)/n \leq u_n$ is equivalent to $g_n(p) \leq g_n(1)$, where the function $g_n : [0,1] \rightarrow \mathbb{R}$ is defined by

$$g_n(p) = p \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot p^k \cdot (1-p)^{n-1-k} \cdot u_{k+1} \quad \forall 0 \leq p \leq 1. \tag{3.5}$$

Note that p is treated as a variable and that the function satisfies $g_n(1) = u_n$. It is known that any function of the form $g(p) = p^a \cdot (1-p)^b$ is monotonic increasing on the interval $[0, a/(a+b)]$ and monotonic decreasing on the interval $[a/(a+b), 1]$ such that its maximum is attained by $p = a/(a+b)$ at level $g(a/(a+b)) = (a^a \cdot b^b)/(a+b)^{a+b}$. In our framework, the function $g_n(p)$ is composed as the sum of n functions, each of one is monotonic increasing on the subinterval $[0, (k+1)/n]$ and monotonic decreasing on the subinterval $[(k+1)/n, 1]$ such that its maximum value equals $((k+1)^{k+1} \cdot (n-1-k)^{(n-1-k)})/n^n$. On the final interval $[(n-1)/n, 1]$, all the components are monotonic decreasing, except for the very last component given by $u_n \cdot p^n$. Further investigation about the graph of the function $g_n(p)$ is desirable.

4. The Nucleolus of the Information Market Game

A direct consequence of Lemma 3.2(iv) and Lemma 2.3(i) is the following characterization of the symmetrical part of the core.

Corollary 4.1. (i) *A symmetrical pay-off vector of the form* $\vec{x}(\alpha) = (n \cdot (u_n - \alpha), \alpha, \alpha, \ldots, \alpha) \in \mathbb{R}^{n+1}$ *is a core allocation if and only if* $\alpha \leq u_n$ *and* $s \cdot \alpha \geq f_n(s)$ *for all* $1 \leq s \leq n$*, or equivalently,*

$$\frac{f_n(s)}{s} \leq \alpha \leq u_n, \quad \text{where } \frac{f_n(s)}{s} = \sum_{j=1}^{s} \binom{s-1}{j-1} \cdot p^j \cdot (1-p)^{s-j} \cdot u_{n-s+j}. \tag{4.1}$$

(ii) *A symmetrical pay-off vector*

$$(n \cdot (u_n - \alpha), \alpha, \alpha, \ldots, \alpha) \in \text{SymCORE}\ (N, v) \quad \textit{iff}\ \frac{f_n(n)}{n} \leq \alpha \leq u_n, \tag{4.2}$$

where

$$\frac{f_n(n)}{n} = \sum_{j=1}^{n} \binom{n-1}{j-1} \cdot p^j \cdot (1-p)^{n-j} \cdot u_j = p \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot p^k \cdot (1-p)^{n-1-k} \cdot u_{k+1}. \tag{4.3}$$

Definition 4.2. (i) Define the excess of coalition $S \subseteq N$, $S \neq \emptyset$, at pay-off vector $\vec{x}$ in any cooperative game $\langle N, v \rangle$ by $e^v(S, \vec{x}) = v(S) - \vec{x}(S)$. Notice that all the excesses of coalitions at core allocations are nonpositive.

(ii) The *excess vector* $\theta(\vec{x}) \in \mathbb{R}^{2^n-1}$ at pay-off vector $\vec{x}$ in any n-person game $\langle N, v \rangle$ has as its coordinates the excesses $e^v(S, \vec{x})$, $S \subseteq N$, $S \neq \emptyset$, arranged in nonincreasing order.

(iii) The nucleolus [4] of a cooperative game $\langle N, v \rangle$ is the unique pay-off vector $\vec{y}$ of which the excess vector $\theta(\vec{y})$ satisfies the lexicographic order $\theta(\vec{y}) \leq_L \theta(\vec{x})$ for any pay-off vector $\vec{x}$ satisfying efficiency and individual rationality (i.e., $\vec{x}(N) = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$).

(iv) The surplus $s_{ij}^v(\vec{x})$ of a player $i \in N$ over another player $j \in N$ at pay-off vector $\vec{x}$ in any cooperative game $\langle N, v \rangle$ is given by the maximal excess among coalitions containing player i, but not containing player j. That is,

$$s_{ij}^v(\vec{x}) = \max\left[e^v(S, \vec{x}) S \subseteq N, i \in S, j \notin S\right]. \tag{4.4}$$

For the purpose of the determination of the nucleolus of the information market game, the next lemma reports the maximal excess levels at symmetrical pay-off vectors $\vec{x}(\alpha) = (n \cdot (u_n - \alpha), \alpha, \alpha, \ldots, \alpha) \in \mathbb{R}^{n+1}$.

Lemma 4.3. *For the* $(n+1)$*-person information market game* $\langle N, v \rangle$ *of the form* (1.1)-(1.2)*, it holds that:*

(i) $e^v(S, \vec{x}(\alpha)) = -(n+1-s) \cdot (u_n - \alpha)$ *for all* $S \subseteq N$ *with* $1 \in S$*. In case* $\alpha \leq u_n$*, then the maximal excess among nontrivial coalitions containing player* 1 *equals* $\alpha - u_n$ *attained at* n*-person coalitions of the form* $N \setminus \{i\}$, $i \neq 1$,

(ii) $e^v(S, \vec{x}(\alpha)) = f_n(s) - s \cdot \alpha$ *for all* $S \subseteq N$, $S \neq \emptyset$, *with* $1 \notin S$*. In case* $f_n(n)/n \leq \alpha$*, there is no general conclusion about the maximal excess among coalitions not containing player* 1.

Proof. (i) For all $S \subseteq N$ with $1 \in S$, it holds that

$$\begin{aligned} e^v(S, \vec{x}(\alpha)) &= v(S) - \vec{x}(\alpha)(S) = (s-1) \cdot u_n - [n \cdot u_n - n \cdot \alpha + (s-1) \cdot \alpha] \\ &= -(n+1-s) \cdot (u_n - \alpha). \end{aligned} \tag{4.5}$$

Under the additional assumption $\alpha \leq u_n$, we obtain $-(n+1-s) \cdot (u_n - \alpha) \leq -(u_n - \alpha)$, that is, the maximum is attained for n-person coalitions of the form $N \setminus \{i\}$, $i \neq 1$, (provided $S \neq N$). On the other, for all $S \subseteq N$, $S \neq \emptyset$, with $1 \notin S$, it holds $e^v(S, \vec{x}(\alpha)) = v(S) - \vec{x}(\alpha)(S) = f_n(s) - s \cdot \alpha$. □

Theorem 4.4. *Suppose that the symmetrical core of the* $(n+1)$*-person information market game is nonempty, that is,* $u_n \ge f_n(n)/n$*. Let* $1 \le t \le n$ *be a maximizer in that*

$$\frac{f_n(t)+u_n}{t+1} \ge \frac{f_n(s)+u_n}{s+1} \quad \forall 1 \le s \le n. \tag{4.6}$$

Let $\overline{\alpha} = (f_n(t)+u_n)/(t+1)$ *and* $\vec{x}(\overline{\alpha}) = (n\cdot(u_n-\overline{\alpha}), \overline{\alpha}, \overline{\alpha}, \dots, \overline{\alpha}) \in \mathbb{R}^{n+1}$.

(i) *Then the pay-off vector* $\vec{x}(\overline{\alpha})$ *belongs to the symmetrical core in that* $f_n(n)/n \le \overline{\alpha} \le u_n$.

(ii) *The nucleolus of the* $(n+1)$*-person information market game equals* $\vec{x}(\overline{\alpha})$.

Proof . Suppose $n\cdot u_n \ge f_n(n)$. The following equivalences hold:

$$\overline{\alpha} \le u_n \quad \text{iff } \frac{f_n(t)+u_n}{t+1} \le u_n \text{ iff } f_n(t) \le t\cdot u_n \text{ iff } \frac{f_n(t)}{t} \le u_n. \tag{4.7}$$

By Lemma 2.3(i), the latter inequality holds since $f_n(t)/t \le f_n(n)/n \le u_n$. So, on the one hand, $\overline{\alpha} \le u_n$. On the other, from (4.6) applied to $s=n$ as well as the assumption $u_n \ge f_n(n)/n$, it follows that:

$$\overline{\alpha} = \frac{f_n(t)+u_n}{t+1} \ge \frac{f_n(n)+u_n}{n+1} \ge \frac{f_n(n)+f_n(n)/n}{n+1} = \frac{f_n(n)}{n}. \tag{4.8}$$

(ii) From part (i) and Lemma 4.3(i), on the one hand, we derive the following:

$$\begin{aligned} s_{12}^v(\vec{x}(\overline{\alpha})) &= \max[e^v(S,\vec{x}(\overline{\alpha})) \mid S \subseteq N, 1 \in S, 2 \notin S] \\ &= \max[-(n+1-s)\cdot(u_n-\overline{\alpha}) \mid 1 \le s \le n] \\ &= -(u_n-\overline{\alpha}) \text{ and on the other,} \\ s_{21}^v(\vec{x}(\overline{\alpha})) &= \max[e^v(S,\vec{x}(\overline{\alpha})) \mid S \subseteq N, 2 \in S, 1 \notin S] \\ &= \max\left[f_n(s) - s\cdot\overline{\alpha} \mid 1 \le s \le n\right] = \overline{\alpha} - u_n, \end{aligned} \tag{4.9}$$

where the latter equality is due to the choice of $\overline{\alpha}$. The equality $s_{12}^v(\vec{y}) = s_{21}^v(\vec{y})$ for $\vec{y} = \vec{x}(\overline{\alpha})$ suffices to conclude that the nucleolus is given by $\vec{x}(\overline{\alpha})$. Notice that $-s_{12}^v(\vec{x}(\overline{\alpha})) = u_n - \overline{\alpha}$ represents the maximal bargaining range within the core by transferring money from player 1 to player 2 starting at core allocation $\vec{x}(\overline{\alpha})$ while remaining in the core. By Lemma 3.2(iv), recall the redundancy of core constraints induced by coalitions containing player 1, so no lower bound for core allocations to player 1.

If the worth of any coalition not containing player 1 is zero (for instance, the big boss games), that is, $f_n(s) = 0$ for all $1 \le s \le n$, then Theorem 4.4 applies with $t=1$, $\overline{\alpha} = u_n/2$, yielding the nucleolus to simplify to $(u_n/2)\cdot(n,1,1,\dots,1)$. Thus, the nucleolus pay-off to the big boss equals the aggregate pay-off to all the users. □

Remark 4.5. Concerning the case $t=n$.

Recall that $b_1^v = n\cdot u_n - f_n(n)$ as well as $b_i^v = u_n$ for all $i \in N$, $i \ne 1$. Thus, the case $t=n$ yields $\overline{\alpha} = (f_n(n)+u_n)/(n+1) = u_n - b_1^v/(n+1) = b_i^v - b_1^v/(n+1)$ for all $i \in N$, $i \ne 1$. In other words, in this setting, the nucleolus coincides with the center of gravity of $n+1$ vectors given by $\vec{b}^v - \beta \cdot \vec{e}_i$, $i \in N$. Here $\beta = b_1^v$ and $\vec{e}_i$ is the ith standard vector in $\mathbb{R}^{n+1}$. Note that, for any $1 \le s \le n$, the underlying condition $(f_n(n)+u_n)/(n+1) \ge (f_n(s)+u_n)/(s+1)$ may be rewritten as

$$s \cdot f_n(n) - n \cdot f_n(s) + \left[f_n(n) - f_n(s)\right] \ge (n-s) \cdot u_n. \tag{4.10}$$

Remark 4.6. Inspired by the description of the nucleolus as given in Remark 4.5, we review a specific subclass of cooperative games with a similar conclusion concerning the nucleolus. A cooperative game $\langle N, v\rangle$ is said to be 1-*convex* if $v(\emptyset) = 0$ and its corresponding *gap function* g^v attains its minimum at the grand coalition N, that is, for every coalition $S \subseteq N$, $S \neq \emptyset$,

$$0 \le g^v(N) \le g^v(S), \quad \text{where } g^v(S) = \sum_{i\in S} b_i^v - v(S). \tag{4.11}$$

For 1-convex games, its nucleolus agrees with the center of gravity of the core, of which the extreme points are given by $\vec{b}^v - g^v(N) \cdot \vec{e}_i$, $i \in N$ [5].

The $(n+1)$-person information market game satisfies $b_i^v = u_n$ for all $i \in N$, $i \neq 1$, and so, its gap function g^v is given by $g^v(S) = b_1^v = n \cdot u_n - f_n(n)$ for all $S \subseteq N$ with $1 \in S$ and $g^v(S) = s \cdot u_n - f_n(s)$ otherwise. Consequently, the $(n+1)$-person information market game of the form (1.1)-(1.2) satisfies 1-convexity if and only if any slope $\Delta(f_n)(s) = (f_n(n) - f_n(s))/(n-s)$, $1 \le s \le n-1$, is bounded from below by the utility u_n in that $\Delta(f_n)(s) \ge u_n$, together with $\Delta(f_n)(0) \le u_n$ (provided $f_n(0) = 0$). Observe that the latter condition, together with Lemma 2.3(i), implies the validity of (4.10) with reference to the case $t = n$ of Theorem 4.4. To conclude, the 1-convexity property for $(n+1)$-person information market games is part of the case $t = n$ and the current procedure for the determination of the nucleolus agrees with the known approach being the center of gravity of the non-empty core.

Remark 4.7. A cooperative game $\langle N, v\rangle$ is said to be 2-*convex* [5] if $v(\emptyset) = 0$, and its corresponding gap function g^v satisfies

$$g^v(N) \le g^v(S) \quad \forall S \subseteq N \text{ with } s \ge 2, \tag{4.12}$$

$$g^v(\{i\}) \le g^v(N) \le g^v(\{i\}) + g^v(\{j\}) \quad \forall i, j \in N,\ i \neq j. \tag{4.13}$$

Recall $g^v(N) = g^v(\{1\}) = b_1^v$ and $g^v(\{i\}) = (1-p) \cdot u_n$ for all $i \neq 1$. Together with $b_1^v = n \cdot u_n - f_n(n)$, it follows that (4.13) reduces to $(1-p) \cdot u_n \le b_1^v \le 2 \cdot (1-p) \cdot u_n$ or equivalently,

$$(n - 2 + 2 \cdot p) \cdot u_n \le f_n(n) \le (n - 1 + p) \cdot u_n. \tag{4.14}$$

Consequently, the $(n+1)$-person information market game satisfies 2-convexity if and only if (4.14) holds as well as any slope $\Delta(f_n)(s)$, $2 \le s \le n-1$, is bounded from below by u_n. Particularly, (4.10) holds for all $2 \le s \le n-1$. Finally, it is left to the reader to derive from (4.14) the relevant inequality involving $s = 1$. That is,

$$\frac{f_n(n) + u_n}{n+1} \ge \frac{f_n(1) + u_n}{2} \quad \text{provided } n \ge 3,\ 0 \le p < 1,\ \text{where } f_n(1) = p \cdot u_n. \tag{4.15}$$

In summary, in the setting of Theorem 4.4, the case $t = n$ applies to $(n+1)$-person information market games, which are 2-convex. Particularly, the current procedure for the determination of the nucleolus agrees with the known approach valid for 2-convex games [6].

5. The Three-Person Information Market Game

The three-person information market game $\langle N, v\rangle$ (with $n = 2$) is given as shown in Table 1.

Note that $b_i^v = u_2$ for $i = 2, 3$, as well as $b_1^v = 2 \cdot u_2 - f_2(2)$, where $f_2(2) = 2 \cdot p \cdot [p \cdot u_2 + (1-p) \cdot u_1]$. Here $b_1^v \geq 0$ is a necessary and sufficient condition for nonemptiness of the core. The three-person information market game is 1-convex if, besides $b_1^v \geq 0$, one of the following equivalences hold:

$$b_1^v \leq (1-p) \cdot u_2 \Longleftrightarrow \frac{u_2}{u_1} \leq \frac{2 \cdot p}{2 \cdot p + 1} \Longleftrightarrow p \geq \frac{A}{2}, \quad \text{where } A = \frac{u_2}{u_1 - u_2}. \tag{5.1}$$

Its core is described by the constraints $x_1 + x_2 + x_3 = 2 \cdot u_2$ and $p \cdot u_2 \leq x_i \leq u_2$ for $i = 2, 3$, as well as $0 \leq x_1 \leq b_1^v$. The constraint $x_1 \geq 0$ is redundant, while the constraint $b_1^v \geq 0$ is a necessary and sufficient condition for nonemptiness of the core. We distinguish two cases concerning the core structure, depending on the location of the core constraint $x_1 = b_1^v$ with respect to the parallel line $x_1 = (1-p) \cdot u_2$. In case $b_1^v \leq (1-p) \cdot u_2$, then the core is a triangle with three vertices $(0, u_2, u_2)$, $(b_1^v, u_2 - b_1^v, u_2)$, and $(b_1^v, u_2, u_2 - b_1^v)$, representing the core of a 1-convex three-person game. Its nucleolus is given by the center of the core, that is $(b_1^v, u_2, u_2) - b_1^v/3 \cdot (1, 1, 1)$.

In case $b_1^v > (1-p) \cdot u_2$, then the core has five vertices $u_2 \cdot (0, 1, 1)$, $u_2 \cdot (1-p, 1, p)$, $u_2 \cdot (1-p, p, 1)$, $(b_1^v, p \cdot u_2, (2-p) \cdot u_2 - b_1^v)$, and $(b_1^v, (2-p) \cdot u_2 - b_1^v, p \cdot u_2)$ representing the core of a convex three-person game (with respect to its imputation set).

Concerning the condition (4.6), the following equivalences hold (provided $0 \leq p < 1$):

$$\frac{f_2(2) + u_2}{3} \geq \frac{f_2(1) + u_2}{2} \Longleftrightarrow \frac{u_2}{u_1} \leq \frac{4 \cdot p}{4 \cdot p + 1} \Longleftrightarrow p \geq \frac{A}{4}, \quad \text{where } A = \frac{u_2}{u_1 - u_2}. \tag{5.2}$$

According to the main Theorem 4.4, to conclude with, if $p \leq A/4$, then $t = 1$, $\overline{\alpha} = (f_2(1) + u_2)/2 = u_2/2 + (p \cdot u_2)/2$ and hence, the parametric representation of the nucleolus is given by $(u_2, u_2/2, u_2/2) + (u_2/2) \cdot (-2 \cdot p, p, p)$.

If $p \geq A/4$, then $t = 2$, $\overline{\alpha} = (f_2(2) + u_2)/3 = u_2 - b_1^v/3$, and hence, the parametric representation of the nucleolus is given by $(0, u_2, u_2) - (1/3) \cdot (-2 \cdot b_1^v, b_1^v, b_1^v)$.

If p varies upwards from zero till $A/4$, then the nucleolus starts at $(u_2, u_2/2, u_2/2)$ and moves with a speed scaled by $u_2/2$. If p varies downwards from 1 till $A/4$, then the nucleolus starts at $(0, u_2, u_2)$ and moves with a speed scaled by $b_1^v = 2 \cdot (1-p) \cdot [(1+p) \cdot u_2 - p \cdot u_1]$. Anyhow, the nucleolus moves by two different speeds from $(0, u_2, u_2)$ being the full core if $p = 1$ till $(u_2, u_2/2, u_2/2)$, being the center of the core if $p = 0$ with four vertices $(2 \cdot u_2, 0, 0)$, $(u_2, u_2, 0)$, $(u_2, 0, u_2)$, and $(0, u_2, u_2)$.

Table 1

Coalition	S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
Worth	$v(S)$	0	$p \cdot u_2$	$p \cdot u_2$	u_2	u_2	$f_2(2)$	$2 \cdot u_2$
Gap	$g^v(S)$	b_1^v	$(1-p) \cdot u_2$	$(1-p) \cdot u_2$	b_1^v	b_1^v	b_1^v	b_1^v

6. The Shapley Value of the Information Market Game

Theorem 6.1. *The Shapley value $Sh_1(N,v)$ of the innovator in the $(n+1)$-person information market game $\langle N,v\rangle$ equals the difference between one half of the aggregate pay-off and the average worth of coalitions not containing the innovator, that is,*

$$\begin{gathered} Sh_1(N,v) = \frac{n \cdot u_n}{2} - \frac{1}{n+1}\sum_{s=0}^{n} f_n(s) \quad \forall i \in N,\ i \neq 1, \\ Sh_i(N,v) = \frac{1}{n} \cdot [v(N) - Sh_1(N,v)] = \frac{u_n}{2} + \frac{1}{n \cdot (n+1)} \cdot \sum_{s=0}^{n} f_n(s). \end{gathered} \tag{6.1}$$

Proof. Put $f_n(0) = 0$. Using its classical formula [7], the Shapley value of the innovator 1 is determined as follows:

$$\begin{aligned} \mathrm{Sh}_1(N,v) &= \sum_{S \subseteq N\setminus\{1\}} \frac{s! \cdot (n-s)!}{(n+1)!} \cdot [v(S \cup \{1\}) - v(S)] \\ &= \sum_{S \subseteq N\setminus\{1\}} \frac{s! \cdot (n-s)!}{(n+1)!} \cdot v(S \cup \{1\}) - \sum_{S \subseteq N\setminus\{1\}} \frac{s! \cdot (n-s)!}{(n+1)!} \cdot v(S) \\ &= \sum_{S \subseteq N\setminus\{1\}} \frac{s! \cdot (n-s)!}{(n+1)!} \cdot s \cdot u_n - \sum_{S \subseteq N\setminus\{1\}} \frac{s! \cdot (n-s)!}{(n+1)!} \cdot f_n(s) \\ &= \sum_{s=0}^{n} \binom{n}{s} \cdot \frac{s! \cdot (n-s)!}{(n+1)!} \cdot s \cdot u_n - \sum_{s=0}^{n} \binom{n}{s} \cdot \frac{s! \cdot (n-s)!}{(n+1)!} \cdot f_n(s) \\ &= \sum_{s=0}^{n} \frac{s}{n+1} \cdot u_n - \sum_{s=0}^{n} \frac{f_n(s)}{n+1} = \frac{n \cdot u_n}{2} - \frac{1}{n+1} \cdot \sum_{s=0}^{n} f_n(s). \end{aligned} \tag{6.2}$$

□

Remark 6.2. The Shapley value $\mathrm{Sh}(N,v)$ is a symmetric allocation, which verifies the upper core bound u_n.

Indeed, by Lemma 3.2(i), it holds $f_n(n)/n \geq f_n(s)/s$ for all $1 \leq s \leq n$ and so,

$$\frac{1}{n \cdot (n+1)} \cdot \sum_{s=0}^{n} f_n(s) \leq \frac{1}{n \cdot (n+1)} \cdot \frac{f_n(n)}{n} \cdot \sum_{s=0}^{n} s = \frac{f_n(n)}{2 \cdot n} \leq \frac{u_n}{2}, \tag{6.3}$$

where the last inequality is due to the assumption $f_n(n) \leq n \cdot u_n$. Thus, $\mathrm{Sh}_i(N,v) \leq u_n$ for all $i \in N$, $i \neq 1$, whereas the Shapley value for users does not necessarily meet the lower core bound $f_n(n)/n$. For instance, for the three-person information market game (with $n = 2$ and $0 \leq p < 1$), the following equivalences hold:

$$\mathrm{Sh}_2(N, v) \geq \frac{f_2(2)}{2} \Longleftrightarrow \frac{u_2}{u_1} \geq \frac{4 \cdot p}{4 \cdot p + 3} \Longleftrightarrow p \leq \frac{3}{4} \cdot A, \tag{6.4}$$

where $A = u_2/(u_1 - u_2)$. By the super-additivity (or zero-monotonicity) of the information market game, its Shapley value satisfies individual rationality, that is, $\mathrm{Sh}_i(N, v) \geq v(\{i\})$ for all $i \in N$. To conclude, the Shapley value of the information market game is an imputation, but not necessarily a core allocation (in spite of the validity of the upper core bound for users).

7. Concluding Remarks

In this paper, we study the information market games, which have been recently introduced by Galdeano et al. [1]. In Section 3, we study the condition for the core to be not empty. We refer the reader to Section 4 where the nucleolus is determined through a characterization of the symmetrical part of the core. Furthermore, simple proof of the Shapley value of the information market game is given in Section 5.

Acknowledgment

The first author acknowledges financial support by the National Science Foundation of China (NSFC) through Grants nos. 71171163 and 71271171.

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