

Research Article

Noncompact Equilibrium Points and Applications

Zahra Al-Rumaih,¹ Souhail Chebbi,¹ and Hong Kun Xu²

¹ *Research Chair on Applied and Actuarial Mathematics, Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia*

² *Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan*

Correspondence should be addressed to Souhail Chebbi, schebbi@ksu.edu.sa

Received 16 February 2012; Accepted 28 March 2012

Academic Editor: Yonghong Yao

Copyright © 2012 Zahra Al-Rumaih et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We prove an equilibrium existence result for vector functions defined on noncompact domain and we give some applications in optimization and Nash equilibrium in noncooperative game.

1. Introduction

Let X be a subset of a vector space E and $f : X \times X \rightarrow \mathbb{R}$, with $f(x, x) = 0$. It is well known in the literature that a point \bar{x} satisfying the property:

$$f(\bar{x}, y) \geq 0, \quad \forall y \in X, \quad (1.1)$$

is called an *equilibrium point*. This notion of equilibrium plays an important role in various areas such as optimization, variational inequalities, and Nash equilibrium problems.

We recall that an equilibrium point in this formulation was first introduced and studied by Blum and Oettli [1], who, inspired by the very known work of Allen [2] and of Fan [3], proved the existence of an equilibrium point by using some hypothesis concerning continuity, convexity, compactness, and monotonicity.

Recently, many authors have investigated the existence of such equilibrium points in different context. In some references, different generalizations of monotonicity condition are used to prove the existence of equilibrium (see, e.g., [4–6]); while some other references studied this equilibrium problem under generalized convexity condition (see, e.g., [5]). The main objective of our work is to study this equilibrium problem by using a generalized coercivity-type condition.

In Section 3 of this paper, we prove the existence of equilibrium points when X is a noncompact subset of a Hausdorff real-topological vector space and f is a vector function that takes its values in another Hausdorff real topological vector space. The order on Y will be defined by a cone C . Formally, we obtain the existence of a point $\bar{x} \in X$, which will be called a *weak equilibrium point*, that satisfies the following condition:

$$f(\bar{x}, y) \notin -\text{int} C, \quad \forall y \in X, \quad (1.2)$$

where $\text{int} C$ denotes the interior of the cone C in Y . The existence of what we refer to as *equilibrium point*, $\bar{x} \in X$ satisfying the following condition:

$$f(\bar{x}, y) \notin -(C \setminus 0), \quad \forall y \in X, \quad (1.3)$$

is then deduced. The results that we obtain in this section generalize the corresponding results obtained in the classical formulation by Fan in [3, 7], Blum and Oettli in [1] as well as the corresponding results obtained in noncompact case by Tan and Tinh in [8].

In Section 4 and as applications, we prove the existence of saddle points for vector functions defined on noncompact domain. We also prove the existence of Nash equilibrium for an infinite set of players game in which every player has a noncompact strategy set and vector loss function.

2. Preliminaries

In this section, we will recall some notions, definitions, and some properties from the literature that will be used in the paper. Let E and Y be real Hausdorff topological vector spaces. Let $X \subset E$ be a nonempty closed convex subset and $C \subset Y$ a pointed convex closed cone. The cone C can define a partial order on Y , denoted by \leq , as follows: $x \leq y$ if and only if $y - x \in C$. We will write $x < y$ if and only if $y - x \in \text{int} C$, in the case $\text{int} C \neq \emptyset$.

We say that the cone C satisfies condition (*) if there is a pointed convex closed cone \tilde{C} such that $C \setminus \{0\} \subseteq \text{int} \tilde{C}$.

Let $f : X \rightarrow Y$ be a mapping. f is said to be *convex* (resp., *concave*) with respect to C if for all $x, y \in X$ and $\alpha \in [0, 1]$, the following condition is satisfied:

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y), \quad (2.1)$$

(resp.: $\alpha f(x) + (1 - \alpha)f(y) \leq f(\alpha x + (1 - \alpha)y$). It is clear that if C_1 and C_2 are two convex closed cones in Y with $C_1 \subseteq C_2$ and f is convex (resp., concave) with respect to C_1 , then f is also convex (resp., concave) with respect to C_2 .

The mapping f is said to be *lower semicontinuous*, in brief l.s.c. (resp., *upper semicontinuous*, in brief u.s.c.), at x_0 with respect to C if for any neighborhood V of $f(x_0)$ in Y , there exists a neighborhood U of x_0 in X such that

$$f(U \cap X) \subseteq V + C, \quad (2.2)$$

(resp. $f(U \cap X) \subseteq V - C$).

Note that following Lemma 2.11 in [8], if f is l.s.c. with respect to C , then the set $F = \{x \in X : f(x) \notin C\}$ is closed. The mapping f is said to be *continuous* with respect to C at a point x_0 in X if it is l.s.c. and u.s.c. with respect to C at this point.

The mapping f is said to be *monotone* with respect to C if for all $x, y \in X$, the following condition is satisfied:

$$f(x, y) + f(y, x) \leq 0. \quad (2.3)$$

In this paper, we will use the definition of coercing family borrowed from [9].

Definition 2.1. Consider a subset X of a topological vector space and a topological space Y . A family $\{(C_i, K_i)\}_{i \in I}$ of pair of sets is said to be coercing for a set-valued map $F : X \rightarrow Y$ if and only if the following hold.

- (i) For each $i \in I$, C_i is contained in a compact convex subset of X and K_i is a compact subset of Y .
- (ii) For each $i, j \in I$, there exists $k \in I$ such that $C_i \cup C_j \subseteq C_k$.
- (iii) For each $i \in I$, there exists $k \in I$ with $\bigcap_{x \in C_k} F(x) \subset K_i$.

Remark 2.2. Definition 2.1 can be reformulated by using the “dual” set-valued map $F^* : Y \rightarrow X$ defined for all $y \in Y$ by $F^*(y) = X \setminus F^{-1}(y)$. Indeed, a family $\{(C_i, K_i)\}_{i \in I}$ is coercing for F if and only if it satisfies conditions (i), (ii) of Definition 2.1 and the following one:

$$\forall i \in I, \exists k \in I, \quad \forall y \in Y \setminus K_i, F^*(y) \cap C_k \neq \emptyset. \quad (2.4)$$

Note that in case where the family is reduced to one element, condition (iii) of Definition 2.1 and in the sense of Remark 2.2 appeared first in this generality (with two sets K and C) in [10] and generalizes condition of Karamardian [11] and Allen [2]. Condition (iii) is also an extension of the coercivity condition given by Fan [7]. For other examples of set-valued maps admitting a coercing family that is not necessarily reduced to one element, see [9].

The following generalization of KKM principle obtained in [9] will be used in the proof of the main result of this paper.

Proposition 2.3. Let E be a Hausdorff topological vector space, Y a convex subset of E , X a nonempty subset of Y , and $F : X \rightarrow Y$ a KKM map with compactly closed values in Y (i.e., for all $x \in X$, $F(x) \cap C$ is closed for every compact set C of Y). If F admits a coercing family, then $\bigcap_{x \in X} F(x) \neq \emptyset$.

3. The Main Result

The main result of this paper is the following equilibrium theorem for vector valued maps.

Theorem 3.1. Let X be a nonempty closed convex subset of a Hausdorff topological vector space E , Y a Hausdorff topological vector space, and $f, g : X \times X \rightarrow Y$ be two functions satisfying the following conditions.

- (1) f is monotone function.

- (2) For any fixed $x \in X$, the function $f(x, \cdot) : X \rightarrow Y$ is convex, l.s.c. with respect to C on X .
- (3) $g(x, x) = 0$ for all $x \in X$.
- (4) For any fixed $y \in X$, the function $g(\cdot, y) : X \rightarrow Y$ is u.s.c. with respect to C on X .
- (5) For any fixed $x \in X$, the function $g(x, \cdot) : X \rightarrow Y$ is convex.
- (6) There exists a family $\{(C_i, K_i)\}_{i \in I}$ satisfying conditions (i) and (ii) of Definition 2.1 and the following one: For each $i \in I$, there exists $k \in I$ such that

$$\{x \in X : f(y, x) - g(x, y) \notin \text{int } C, \forall y \in C_k\} \subset K_i. \quad (3.1)$$

Then, there exists a point $\bar{x} \in X$ such that:

$$f(y, \bar{x}) - g(\bar{x}, y) \notin \text{int } C, \quad \forall y \in X. \quad (3.2)$$

Proof. For any $y \in X$, we consider the set-valued map:

$$F(y) = \{x \in X : f(y, x) - g(x, y) \notin \text{int } C\}. \quad (3.3)$$

We have the following.

- (i) For all $y \in X$, $F(y)$ is closed in X , then F has compactly closed values.
- (ii) Let $\{y_i : i \in I\}$ be a finite subset of X and $z \in \text{conv}\{y_i : i \in I\}$. We want to show that

$$\text{conv}\{y_i : i \in I\} \subset \bigcup_{i \in I} F(y_i). \quad (3.4)$$

By absurdity, suppose that $z = \sum_{i \in I} \lambda_i y_i$ with $\lambda_i \geq 0$, $\sum_{i \in I} \lambda_i = 1$ and $z \notin \bigcup_{i \in I} F(y_i)$, it means that for all $i \in I$, $f(y_i, z) - g(z, y_i) \in \text{int } C$,

$$\sum_{i \in I} \lambda_i (f(y_i, z) - g(z, y_i)) \in \text{int } C, \quad \forall i \in I. \quad (3.5)$$

By assumption (1) and (2), we obtain

$$\sum_{i \in I} \lambda_i f(y_i, z) \leq \sum_{i, j \in I} \lambda_i \lambda_j f(y_i, y_j) = \frac{1}{2} \sum_{i, j \in I} \lambda_i \lambda_j (f(y_i, y_j) + f(y_j, y_i)) \leq 0. \quad (3.6)$$

Further, it implies from assumptions (3) and (5) that

$$0 = g(z, z) \leq \sum_{i \in I} \lambda_i g(z, y_i). \quad (3.7)$$

It follows that

$$\sum_{i \in I} \lambda_i f(y_i, z) \leq \sum_{i \in I} \lambda_i g(z, y_i). \quad (3.8)$$

Or

$$\sum_{i \in I} \lambda_i (f(y_i, z) - g(z, y_i)) \in (-C). \quad (3.9)$$

We deduce that

$$\sum_{i \in I} \lambda_i (f(y_i, z) - g(z, y_i)) \in \text{int } C \cap (-C) = \emptyset, \quad (3.10)$$

so we have a contradiction.

(iii) Hypothesis (6) implies that the family $\{(C_i, K_i)\}_{i \in I}$ satisfies the following condition: for all $i \in I$, there exists $k \in I$ with

$$\bigcap_{y \in C_k} F(y) \subset K_i, \quad (3.11)$$

and hence it is a coercing family for F .

F satisfies all hypothesis of Proposition 2.3, so

$$\bigcap_{y \in X} F(y) \neq \emptyset. \quad (3.12)$$

Take \bar{x} in this intersection, then $f(y, \bar{x}) - g(\bar{x}, y) \notin \text{int } C$, for all $y \in X$. \square

Corollary 3.2. *Let $X, C, \{(C_i, K_i)\}_{i \in I}, f$ and g satisfy all assumptions of Theorem 3.1 and the additional following conditions.*

- (a) $f(x, x) = 0$ for all $x \in X$.
- (b) For any fixed $x, y \in X$, the function $k : t \in [0, 1] \rightarrow f(ty + (1-t)x, y)$ is u.s.c. with respect to C at $t = 0$.

Then, there exists a point $\bar{x} \in X$ such that

$$f(\bar{x}, y) + g(\bar{x}, y) \notin -\text{int } C, \quad \forall y \in X. \quad (3.13)$$

In addition, if C satisfies condition (*), then

$$f(\bar{x}, y) + g(\bar{x}, y) \notin -(C \setminus \{0\}), \quad \forall y \in X. \quad (3.14)$$

Proof. By Theorem 3.1, there exists $\bar{x} \in X$ with

$$f(y, \bar{x}) - g(\bar{x}, y) \notin \text{int } C, \quad \forall y \in X. \quad (3.15)$$

Since $\bar{x} \in K$, by applying Lemma 3.3 in [8], we obtain

$$f(\bar{x}, y) + g(\bar{x}, y) \notin -\text{int } C, \quad \forall y \in X. \quad (3.16)$$

This proves the first assertion of Corollary 3.2. Now, let C satisfy condition (*) and let \tilde{C} be a pointed convex closed cone in Y such that $C \setminus \{0\} \subset \text{int } \tilde{C}$. It is easy to see that X , \tilde{C} , $\{(C_i, K_i)\}_{i \in I}$, f , and g satisfy all assumptions of Theorem 3.1, by the first assertion, we have

$$f(\bar{x}, y) + g(\bar{x}, y) \notin -\text{int } \tilde{C}, \quad \forall y \in X. \quad (3.17)$$

Since $-(C \setminus \{0\}) \subset -\text{int } \tilde{C}$, it follows that

$$f(\bar{x}, y) + g(\bar{x}, y) \notin -(C \setminus \{0\}), \quad \forall y \in X. \quad (3.18)$$

□

Let K be a convex subset of X . The core of K relative to X , denoted by $\text{core}_X K$, is the set defined by $a \in \text{core}_X K$ if and only if $a \in K$ and $K \cap (a, y] \neq \emptyset$ for all $y \in X \setminus K$, where $(a, y] = \{x \in E : x = \lambda a + (1 - \lambda)y \text{ for } \lambda \in [0, 1)\}$.

The following result can be deduced from Theorem 3.1.

Corollary 3.3. *Let X , C , $\{(C_i, K_i)\}_{i \in I}$, f and g satisfy hypothesis (1–5) of Theorem 3.1 and the following condition*

(6') *There exists a nonempty convex compact subset K of X such that, for any $x \in K \setminus \text{core}_X K$, one can find a point $y \in \text{core}_X K$ such that*

$$f(x, y) + g(x, y) \leq 0. \quad (3.19)$$

Then, there exists a point $\bar{x} \in X$ such that

$$f(\bar{x}, y) + g(\bar{x}, y) \notin -\text{int } C, \quad \forall y \in X. \quad (3.20)$$

In addition, if C satisfies condition (), then*

$$f(\bar{x}, y) + g(\bar{x}, y) \notin -(C \setminus \{0\}), \quad \forall y \in X. \quad (3.21)$$

Proof. We just prove the first assertion. By taking for all $i \in I$, $C_i = K_i = K$, which is a convex compact set we can see that by using hypothesis (6') that F admits a coercing family in the sense of Remark 2.2. □

Remark 3.4. Note that if X is a compact convex subset of E , then condition (6) of Theorem 3.1 is automatically satisfied. Hence, Theorem 3.1 extends Theorem 1 in [1]. Corollary 3.2 extends also Lemma 3.2 in [8] obtained in the noncompact case and Corollary 3.3 corresponds to Theorem 3.1 in [1]. In case of real-valued function f , those results coincide with the corresponding results obtained in [3, 7].

4. Applications

Let I be a (possibly infinite) set of players. If each player $i \in I$ has a nonempty strategy subset X_i of a Hausdorff topological vector space and a loss function, $f^i : X = \prod_{i \in I} X_i \rightarrow Y$ depending on the strategies of n players. For $x = (x_i)_{i \in I} \in X$, we denote $x^{-i} = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots)$ and $(x^{-i}, y_i) = (x_1, x_2, \dots, x_{i-1}, y_i, x_{i+1}, \dots)$.

A point $\bar{x} = (\bar{x}_i) \in X$ is said to be a *weak Nash equilibrium* if and only if for all $i \in I$,

$$f_i(\bar{x}^{-i}, y_i) - f_i(\bar{x}) \notin \text{int } C \quad (4.1)$$

holds for all $y = (y_i)_{i \in I} \in X$.

A point $\bar{x} = (\bar{x}_i) \in X$ is said to be a *Nash equilibrium* if and only if for all $i \in I$,

$$f_i(\bar{x}^{-i}, y_i) - f_i(\bar{x}) \notin -(C \setminus \{0\}) \quad (4.2)$$

holds for all $y = (y_i)_{i \in I} \in X$.

Proposition 4.1. *Let X_i, f_i, X be as above. Assume that, for all $i \in I$, the following conditions are satisfied.*

- (1) f_i is continuous with respect to C .
- (2) For any fixed x^{-i} , the function $f_i(x^{-i}, \cdot)$ is convex.
- (3) There exists a family $\{(C_i, K_i)\}$ satisfying condition (a) and (b) of Definition 2.1 and the following one:

$$\left\{ x = (x_i)_{i \in I} \in X : \sum_{i \in I} (f_i(x^{-i}, y_i) - f_i(x)) \notin -\text{int } C, \forall y \in C_k \right\} \subset K_i. \quad (4.3)$$

Then, there exists a weak Nash equilibrium. In addition, if C satisfies condition (*), then there exists a Nash equilibrium.

Proof. Define $g : X \times X \rightarrow X$, for all $x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in X$, by

$$g(x, y) = \sum_{i \in I} (f_i(x^{-i}, y_i) - f_i(x)). \quad (4.4)$$

We can easily verify that $X, \{(C_i, K_i)\}_{i \in I}, f = 0$, and g as above satisfy all assumptions of Corollary 3.2. Applying the first part of this corollary, we conclude that there exists a point $\bar{x} = \bar{x}_{i \in I} \in X$ with

$$g(\bar{x}, y) \notin -\text{int } C \quad \forall y \in X, \quad (4.5)$$

or

$$\sum_{i \in I} (f_i(x^{-i}, y_i) - f_i(x)) \notin -\text{int } C \quad \forall y \in X. \quad (4.6)$$

If for $i \in I$, we choose $y \in X$ in such a way that $x^{-i} = y^{-i}$, then

$$g(x, y) = f_i(\bar{x}^{-i}, y_i) - f_i(\bar{x}) \notin -\text{int } C, \quad \forall y_i \in X_i. \quad (4.7)$$

This completes the proof of the first assertion of the proposition. The second assertion of Corollary 3.2 implies that

$$f_i(\bar{x}^{-i}, y_i) - f_i(\bar{x}) \notin -(C \setminus \{0\}) \quad \forall y_i \in X_i, \quad (4.8)$$

and this gives us the second assertion. \square

Let E_1 and E_2 be two Hausdorff topological vector spaces, X_1 and X_2 be two nonempty convex closed subsets of E_1 and E_2 , respectively. Let Y and C be as above and $T : X_1 \times X_2 \rightarrow Y$. A point $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ is called a *weak saddle point* of T with respect to C if

$$T(y_1, \bar{x}_2) - T(\bar{x}_1, y_2) \notin -\text{int } C \quad (4.9)$$

holds for all $(y_1, y_2) \in X_1 \times X_2$.

A point $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ is called a *saddle point* of f with respect to C if

$$f(y_1, \bar{x}_2) - f(\bar{x}_1, y_2) \notin -(C \setminus \{0\}) \quad (4.10)$$

holds for all $(y_1, y_2) \in X_1 \times X_2$.

Proposition 4.2. *Let X_1, X_2, Y and C be as above and $f : X_1 \times X_2 \rightarrow Y$. Assume that the following hold.*

- (1) *For any fixed $y \in X_2$, $f(\cdot, y)$ is a convex l.s.c. function with respect to C .*
- (2) *For any fixed $x \in X_1$, $f(x, \cdot)$ is a concave u.s.c. function with respect to C .*
- (3) *There exists a family $\{(C_i, K_i)\}$ satisfying condition (a) and (b) of Definition 2.1 and the following one. For each $i \in I$, there exists $k \in I$ such that*

$$\{x = (x_1, x_2) \in X_1 \times X_2 : f(y_1, x_2) - f(x_1, y_2) \notin -\text{int } C, \forall y = (y_1, y_2) \in C_k\} \subset K_i. \quad (4.11)$$

Then, there exists a weak saddle point of f . In addition, if C satisfies condition (), then there exists a saddle point of f .*

Proof. Consider the function $g : X \times X \rightarrow Y$, where $X = X_1 \times X_2$, defined for all $x = (x_1, x_2), y = (y_1, y_2) \in X_1 \times X_2$ by

$$g(x, y) = f(y_1, x_2) - f(x_1, y_2). \quad (4.12)$$

Apply the first part of Corollary 3.2 for g and the null function 0, we conclude that there exists a point $\bar{x} = (\bar{x}_1, \bar{x}_2) \in X$ with $g(\bar{x}, y) \notin -\text{int } C$, for all $y \in X$. This follows:

$$f(y_1, \bar{x}_2) - f(\bar{x}_1, y_2) \notin -\text{int } C, \quad \forall y = (y_1, y_2) \in X. \quad (4.13)$$

The second assertion of Corollary 3.2 implies that

$$f(y_1, \bar{x}_2) - f(\bar{x}_1, y_2) \notin -C \setminus \{0\}, \quad \forall y = (y_1, y_2) \in X, \quad (4.14)$$

and this completes the proof. \square

Acknowledgments

The third author extends his appreciation to the Deanship of Scientific Research at King Saud University for funding the work through the visiting professor program (VPP).

References

- [1] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.
- [2] G. Allen, "Variational inequalities, complementarity problems, and duality theorems," *Journal of Mathematical Analysis and Applications*, vol. 58, no. 1, pp. 1–10, 1977.
- [3] K. Fan, "A minimax inequality and applications," in *Inequalities*, O. Shisha, Ed., vol. 3, pp. 103–113, Academic Press, New York, NY, USA, 1972.
- [4] M. Bianchi, N. Hadjisavvas, and S. Schaible, "Vector equilibrium problems with generalized monotone bifunctions," *Journal of Optimization Theory and Applications*, vol. 92, no. 3, pp. 527–542, 1997.
- [5] M. Bianchi and S. Schaible, "Equilibrium problems under generalized convexity and generalized monotonicity," *Journal of Global Optimization*, vol. 30, no. 2–3, pp. 121–134, 2004.
- [6] Y. P. Fang and N. J. Huang, "Variational-like inequalities with generalized monotone mappings in Banach spaces," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 327–338, 2003.
- [7] K. Fan, "Some properties of convex sets related to fixed point theorems," *Mathematische Annalen*, vol. 266, no. 4, pp. 519–537, 1984.
- [8] N. X. Tan and P. N. Tinh, "On the existence of equilibrium points of vector functions," *Numerical Functional Analysis and Optimization*, vol. 19, no. 1–2, pp. 141–156, 1998.
- [9] H. Ben-El-Mechaiekh, S. Chebbi, and M. Florenzano, "A generalized KKM principle," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 2, pp. 583–590, 2005.
- [10] H. Ben-El-Mechaiekh, P. Deguire, and A. Granas, "Points fixes et coïncidences pour les applications multivoque (application de Ky fan)," *Comptes Rendus des Séances de l'Académie des Sciences. Série I*, vol. 295, no. 3, pp. 257–259, 1982.
- [11] S. Karamardian, "Generalized complementarity problem," *Journal of Optimization Theory and Applications*, vol. 8, pp. 161–168, 1971.