

## Research Article

# Approximate Solutions for Nonlinear Initial Value Problems Using the Modified Variational Iteration Method

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We have used the modified variational iteration method (MVIM) to find the approximate solutions for some nonlinear initial value problems in the mathematical physics, via the Burgers-Fisher equation, the Kuramoto-Sivashinsky equation, the coupled Schrodinger-KdV equations, and the long-short wave resonance equations together with initial conditions. The results of these problems reveal that the modified variational iteration method is very powerful, effective, convenient, and quite accurate to systems of nonlinear equations. It is predicted that this method can be found widely applicable in engineering and physics.

## 1. Introduction

Nonlinear partial differential equations are known to describe a wide variety of phenomena not only in physics, where applications extend over magnetofluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasma, but also in biology, chemistry, and several other fields. It is one of the important tasks in the study of the nonlinear partial differential equations to seek exact and explicit solutions. In the past several decades both mathematicians and physicists have made many attempts in this direction. Various methods for obtaining exact solutions to nonlinear partial differential equations have been proposed. Among these methods are the Bäcklund transformation method [1, 2], the Hirota's bilinear method [3], the inverse scattering transform method [4], extended tanh method [5–7], the Adomian-Pade approximation [8–10], the variational method [11–14], the variational iteration method [15, 16], the various Lindstedt-Poincare methods [17–20], the Adomian decomposition method [8, 21, 22], the  $F$ -expansion method

[23, 24], the Exp-function method [25–27] and others [28–35]. Zayed et al. [36] investigated the travelling wave solutions for nonlinear initial value problems using the homotopy perturbation method. The modified variational iteration method is the couples of the variational iteration method with the homotopy perturbation method. Recently Akbarzade and Langari [37] and Mei and Zhang [38] had used the modified variational iteration method for some nonlinear partial differential equations.

The main objective of the present paper is to use the modified variational iteration method (MVIM) for constructing the traveling wave solutions of the following nonlinear partial differential equations in mathematical physics:

(i) the nonlinear Burgers-Fisher equation [39]:

$$u_t - u_{xx} - uu_x - u(1 - u) = 0, \quad (1.1)$$

(ii) the nonlinear Kuramoto-Sivashinsky equation [40]:

$$u_t + auu_x + bu_{xx} + cu_{xxxx} = 0, \quad (1.2)$$

(iii) the nonlinear coupled Schrodinger KdV equations [41]:

$$\begin{aligned} iu_t - u_{xx} - uv &= 0, \\ v_t + v_{xxx} + 6vv_x - \left(|u|^2\right)_x &= 0, \end{aligned} \quad (1.3)$$

(iv) the nonlinear long-short wave resonance equations [42]:

$$\begin{aligned} iu_t + u_{xx} - uv &= 0, \\ v_t + \beta \left(|u|^2\right)_x &= 0, \end{aligned} \quad (1.4)$$

together with initial conditions, where  $a, b, c,$  and  $\beta$  are arbitrary constants while  $i = \sqrt{-1}$ . It is interesting to point out that (1.1) includes the convection term  $uu_x$  and the dissipation term  $u_{xx}$ . Equation (1.2) describes the fluctuations of the position of a flame front, the motion of a fluid going down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium. Equation (1.3) describe various processes in dusty plasma such as Langmuri, dust-acoustic wave and electromagnetic waves, while in (1.4)  $u$  is the envelope of the short wave and is a complex function, and  $v$  is the amplitude of the long wave which is a real function.

## 2. Basic Idea of He's Homotopy Perturbation Method

We illustrate the following nonlinear differential equation [43–54]:

$$A(u) - f(r) = 0, \quad r \in \Omega, \quad (2.1)$$

with the boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \quad (2.2)$$

where  $A$  is a general differential operator,  $B$  is a boundary operator,  $f(r)$  is an analytic function, and  $\Gamma$  is the boundary of the domain  $\Omega$ . Generally speaking, the operator  $A$  can be divided into two parts  $L$  and  $N$ , where  $L$  is linear but  $N$  is nonlinear. Therefore, (2.1) can be rewritten in the following form:

$$L(u) + N(u) - f(r) = 0. \quad (2.3)$$

By the homotopy technique, we construct a homotopy  $V(r, p) : \Omega \times [0, 1] \rightarrow R$  which satisfies

$$H(V, p) = (1 - p)[L(V) - L(u_0)] + p[A(V) - f(r)] = 0, \quad r \in \Omega, \quad (2.4)$$

or

$$H(V, p) = L(V) - L(u_0) + pL(u_0) + p(N(V) - f(r)) = 0, \quad r \in \Omega, \quad (2.5)$$

where  $p \in [0, 1]$  is an embedding parameter and  $u_0$  is an initial approximation of (2.1) which satisfies the boundary conditions (2.2). Obviously, from (2.4) and (2.5), we have

$$\begin{aligned} H(V, 0) &= L(V) - L(u_0) = 0, \\ H(V, 1) &= A(V) - f(r) = 0. \end{aligned} \quad (2.6)$$

The changing process of  $p$  from zero to unity is just that of  $V(r, p)$  from  $u_0(r)$  to  $u(r)$ . In topology, this is called the deformation but  $L(V) - L(u_0)$  and  $A(V) - f(r)$  are called the homotopies. According to the homotopy perturbation method, we can first use the embedding parameter " $p$ " as a small parameter and assume that (2.4) or (2.5) can be written as a power series in " $p$ " as follows:

$$V = V_0 + pV_1 + p^2V_2 + \dots. \quad (2.7)$$

Letting  $p \rightarrow 1$  in (2.7), the approximate solution of (2.3) takes the following form:

$$u = \lim_{p \rightarrow 1} V = V_0 + V_1 + V_2 + \dots. \quad (2.8)$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques.

### 3. Variational Iteration Method

Consider the following nonhomogeneous, nonlinear partial differential equation:

$$Lu(x, t) + N(u(x, t)) = f(x, t), \quad (3.1)$$

where  $L$  is a linear differential operator with respect to time,  $N$  is a nonlinear operator and  $f(x, t)$  is a given function.

According to the variational iteration method, we can construct correct functionals as follows:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\tau) [Lu_n(x, \tau) + N(\tilde{u}_n(x, \tau)) - f(x, \tau)] d\tau \quad (3.2)$$

which is variational iteration algorithm I, and  $\lambda$  is a general Lagrange multipliers. The variational iteration method can be identified optimally via variational theory [6, 7]. The second term on the right-hand side in (3.2) is called the corrections, the subscript  $n$  denotes the  $n$ th order approximation, and  $\tilde{u}_n$  is restricted variations. We can assume that the aforementioned correctional functionals are stationary (i.e.,  $\delta u_{n+1} = 0$ ), and then the Lagrange multipliers can be identified.

Now we can start with the given initial approximation and by the previous iteration formulas we can obtain the approximate solutions. He [55] has used the fractional iteration method to obtain the approximate solutions for nonlinear fractional differential equations.

### 4. The Modified Variational Iteration Method

To convey the basic idea of the variational homotopy perturbation method [2, 3], we consider the following general differential equation:

$$Lu + N(u) = f(x), \quad (4.1)$$

where  $L$  is a linear differential operator,  $N$  is a nonlinear operator and  $f(x, t)$  is an inhomogeneous term. According to the variational iteration method [4–13], we can construct a correct functional as follows:

$$u_{n+1} = u_n + \int_0^x \lambda(\tau) [Lu_n + N(\tilde{u}_n) - f(\tau)] d\tau, \quad (4.2)$$

where  $\lambda$  is a Lagrange multipliers, which can be identified optimally via variational theory [6, 7]. The subscripts  $n$  denote the  $n$ th approximation, and  $\tilde{u}_n$  is considered as a restricted variation. That is,  $\delta \tilde{u}_n = 0$  is called a correct functional. Now, we apply the homotopy perturbation method to (4.2):

$$\sum_{i=0}^{\infty} p^i u_i = u_0 + \int_0^x \lambda(\tau) \left\{ N \left( \sum_{i=0}^{\infty} p^i \tilde{u}_i \right) \right\} d\tau - \int_0^x \lambda(\tau) f(\tau) d\tau, \quad (4.3)$$

which is the variational iteration algorithm II and is formulated by the modified variational iteration method. The embedding parameter  $p \in [0, 1]$  can be considered as an expanding parameter [14–19].

The homotopy perturbation method uses the homotopy parameter  $p$  as an expanding parameter [14–19] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + \dots \quad (4.4)$$

If  $p \rightarrow 1$ , then (4.4) becomes the approximate solution of the following form:

$$u = u_0 + u_1 + u_2 + \dots \quad (4.5)$$

A comparison of like powers of  $p$  gives solutions of various orders.

The application of the Adomian polynomial is too complex so that we consider the variational iteration method and He's polynomial to calculate the approximate solutions (see, e.g., [56–60]).

## 5. Applications

In this section, we construct the approximate solutions for some nonlinear evolution equations in the mathematical physics, namely, the Burgers-Fisher equation (1.1), the Kuramoto-Sivashinsky equation (1.2), the coupled Schrodinger-KdV equations (1.3), and the long-short wave resonance equations (1.4) together with initial conditions by using the the modified variational iteration method. Applications of this method to similar equations can be found in [61–67].

### 5.1. Approximate Solution of Burgers-Fisher Equation with Initial Conditions Using Modified Variational Iteration Method

In this subsection, we use the MVIM to find the solution  $u(x, t)$  of an initial value problem consisting of the nonlinear Burgers-Fisher equation (1.1) and the following initial condition [39]:

$$V_0 = u(x, 0) = \frac{1}{2} \left[ 1 + \tanh\left(\frac{x}{4}\right) \right]. \quad (5.1)$$

This initial condition follows by setting  $t = 0$  in the following exact solution of (1.1):

$$u(x, t) = \frac{1}{2} \left( 1 + \tanh \left[ \frac{1}{4} \left( x + \frac{5}{2} t \right) \right] \right). \quad (5.2)$$

This exact solution has been derived by Wazwaz [39] using the tanh-coth method. To this end, we construct the modified variational iteration method for the nonlinear Burgers- Fisher equation (1.1) which satisfies

$$\sum_{i=0}^{\infty} p^i u_i = u(x, 0) + p \int_0^t \left[ \sum_{i=0}^{\infty} p^i u_{i,xx} + \left( \sum_{i=0}^{\infty} p^i u_i \right) \left( \sum_{i=0}^{\infty} p^i u_{i,x} \right) + \left( \sum_{i=0}^{\infty} p^i u_i \right) - \left( \sum_{i=0}^{\infty} p^i u_i \right)^2 \right] d\tau. \quad (5.3)$$

Comparing the different coefficient of like power of  $p$ , we have

$$\begin{aligned} p^0 : u(x, 0) &= \frac{1}{2} \left( 1 + \tanh \left[ \frac{1}{4} \left( x + \frac{5}{2} t \right) \right] \right), \\ p^1 : u_1(x, t) &= \int_0^t u_{0,xx}(x, \tau) d\tau + \int_0^t u_0(x, \tau) u_{0,x}(x, \tau) d\tau + \int_0^t u_0(x, \tau) d\tau - \int_0^t u_0^2(x, \tau) d\tau, \\ p^2 : u_2(x, t) &= \int_0^t u_{1,xx}(x, \tau) d\tau + \int_0^t u_0(x, \tau) u_{1,x}(x, \tau) d\tau + \int_0^t u_1(x, \tau) u_{0,x}(x, \tau) d\tau \\ &\quad + \int_0^t u_1(x, \tau) d\tau - 2 \int_0^t u_0(x, \tau) u_1(x, \tau) d\tau, \\ p^3 : u_3(x, t) &= \int_0^t u_{2,xx}(x, \tau) d\tau + \int_0^t u_0(x, \tau) u_{2,x}(x, \tau) d\tau + \int_0^t u_1(x, \tau) u_{1,x}(x, \tau) d\tau \\ &\quad + \int_0^t u_2(x, \tau) u_{0,x}(x, \tau) d\tau + \int_0^t u_2(x, \tau) d\tau + 2 \int_0^t u_0(x, \tau) u_2(x, \tau) d\tau \\ &\quad + \int_0^t u_1^2(x, \tau) d\tau, \end{aligned} \quad (5.4)$$

and so on. Consequently after some reduction with help of Maple or Mathematica, we get:

$$\begin{aligned} V_0(x, t) &= \frac{1}{2} \left[ 1 + \tanh \left( \frac{x}{4} \right) \right], \\ V_1(x, t) &= \frac{5t}{16} \operatorname{sech}^2 \left( \frac{x}{4} \right), \\ V_2(x, t) &= -\frac{25t^2}{128} \operatorname{sech}^2 \left( \frac{x}{4} \right) \tanh \left( \frac{x}{4} \right), \\ V_3(x, t) &= -\frac{375t^3}{3072} \operatorname{sech}^4 \left( \frac{x}{4} \right) + \frac{125t^3}{1536} \operatorname{sech}^2 \left( \frac{x}{4} \right). \end{aligned} \quad (5.5)$$

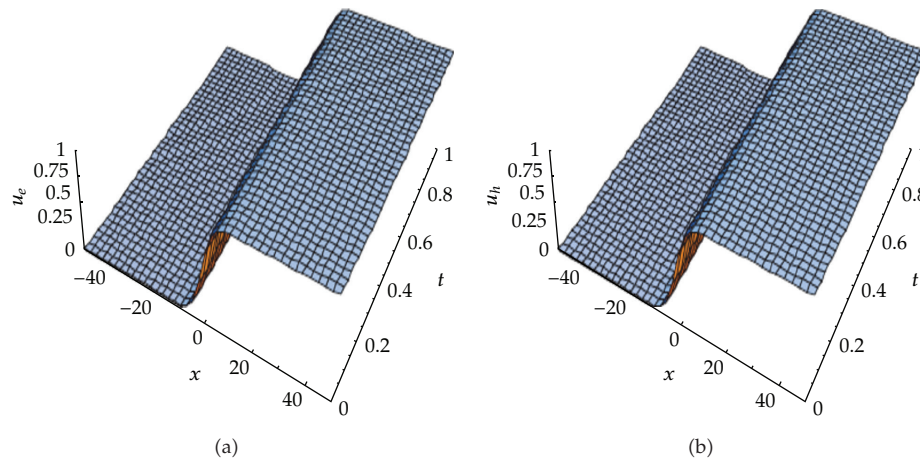
In this manner the other components can be obtained.

Substituting from (5.5) into (4.5), we obtain the following approximate solution of the initial value problem (1.1) and (5.1):

$$\begin{aligned} u(x, t) &= \frac{1}{2} \left[ 1 + \tanh \left( \frac{x}{4} \right) \right] + \frac{5t}{16} \operatorname{sech}^2 \left( \frac{x}{4} \right) - \frac{25t^2}{128} \operatorname{sech}^2 \left( \frac{x}{4} \right) \tanh \left( \frac{x}{4} \right) \\ &\quad - \frac{375t^3}{3072} \operatorname{sech}^4 \left( \frac{x}{4} \right) + \frac{125t^3}{1536} \operatorname{sech}^2 \left( \frac{x}{4} \right) + \dots \end{aligned} \quad (5.6)$$

**Table 1:** The approximate solution (5.6) in comparison with the exact solution (5.2) if  $t = 0.5$ .

$x$	$u_{\text{exact}}$	$u_{\text{VHP}}$	$ u_{\text{exact}} - u_{\text{VHP}} $
-50	$2.59461 \times 10^{-11}$	$2.58455 \times 10^{-11}$	$1.00576 \times 10^{-13}$
-40	$3.85074 \times 10^{-9}$	$3.83581 \times 10^{-9}$	$1.4929 \times 10^{-11}$
-30	$5.715 \times 10^{-7}$	$5.69285 \times 10^{-7}$	$2.21575 \times 10^{-9}$
-20	0.000084811	0.000084482	$3.28572 \times 10^{-7}$
-10	0.0124317	0.0123887	$4.29767 \times 10^{-5}$
0	0.651355	0.651164	$1.91128 \times 10^{-4}$
10	0.996406	0.996441	$3.44391 \times 10^{-5}$
20	0.999976	0.999976	$2.55845 \times 10^{-7}$
30	1	1	$1.7249 \times 10^{-9}$
40	1	1	$1.16228 \times 10^{-11}$
50	1	1	$7.81597 \times 10^{-14}$

**Figure 1:** The approximate solution (5.6) shown in (b) in comparison with the exact solution (5.2) shown in (a) if  $t = 0.5$ .

Note that if we expand the exact solution (5.2) in Taylor series near  $t = 0$ , we obtain the approximate solution (5.6). To demonstrate the convergence of the variational homotopy perturbation method, the results of the numerical example are presented and only few terms are required to obtain accurate solutions. The accuracy of the modified variational iteration method for the nonlinear Burgers-Fisher equation is controllable and absolute errors are very small with the present choice of  $x$  and  $t$ . These results are listed in Table 1. Both the exact solution (5.2) and the approximate solution (5.6) obtained for the first three approximations are plotted in Figure 1. There are no visible differences in diagrams. It is also evident that when more terms for the modified variational iteration method are computed, the numerical results get much more closer to the corresponding exact solution with the initial condition (5.1).

## 5.2. Approximate Solution of the Nonlinear Kuramoto-Sivashinsky Equation with Initial Conditions Using MVIM

In this subsection, we use the MVIM to find the solution  $u(x, t)$  of an initial-value problem consisting of the nonlinear Kuramoto-Sivashinsky equation (1.2) with the following initial condition [40]:

$$V_0 = u(x, 0) = \frac{15b\lambda}{19a} \left[ 2 + 3 \tanh\left(\frac{\lambda x}{2}\right) - \tanh^3\left(\frac{\lambda x}{2}\right) \right], \quad (5.7)$$

where  $\lambda = \sqrt{-b/19c}$ ,  $b/c < 0$ , and  $a, b$ , and  $c$  are constants. This initial condition follows by setting  $t = 0$  in the following exact solution of (1.2):

$$u(x, t) = \frac{15b\lambda}{19a} \left( 2 + 3 \tanh\left[\frac{\lambda}{2}\left(x - \frac{30b\lambda}{19}t\right)\right] - \tanh^3\left[\frac{\lambda}{2}\left(x - \frac{30b\lambda}{19}t\right)\right] \right). \quad (5.8)$$

This exact solution has been derived by Wazwaz [40] using the tanh method and the extended tanh method. Let us now apply the MVIM to the initial value problem (1.2) and (5.7). To this end, we construct an MVIM for the nonlinear Kuramoto-Sivashinsky equation (1.2) which satisfies

$$\sum_{i=0}^{\infty} p^i u_i = u(x, 0) - p \int_0^t \left[ a \left( \sum_{i=0}^{\infty} p^i u_i \right) \left( \sum_{i=0}^{\infty} p^i u_{i,x} \right) + b \sum_{i=0}^{\infty} p^i u_{i,xx} + c \left( \sum_{i=0}^{\infty} p^i u_{i,xxxx} \right) \right] d\tau. \quad (5.9)$$

Comparing the different coefficient of like power of  $p$ , we have

$$\begin{aligned} p^0 : u(x, 0) &= \frac{15b\lambda}{19a} \left[ 2 + 3 \tanh\left(\frac{\lambda x}{2}\right) - \tanh^3\left(\frac{\lambda x}{2}\right) \right], \\ p^1 : u_1(x, t) &= -a \int_0^t u_0(x, \tau) u_{0,x}(x, \tau) d\tau - b \int_0^t u_{0,xx}(x, \tau) d\tau - c \int_0^t u_{0,xxxx}(x, \tau) d\tau, \\ p^2 : u_2(x, t) &= -a \int_0^t u_0(x, \tau) u_{1,x}(x, \tau) d\tau - a \int_0^t u_1(x, \tau) u_{0,x}(x, \tau) d\tau - b \int_0^t u_{1,xx}(x, \tau) d\tau \\ &\quad - c \int_0^t u_{1,xxxx}(x, \tau) d\tau, \\ p^3 : u_3(x, t) &= -a \int_0^t u_0(x, \tau) u_{2,x}(x, \tau) d\tau - a \int_0^t u_1(x, \tau) u_{1,x}(x, \tau) d\tau - a \int_0^t u_2(x, \tau) u_{0,x}(x, \tau) d\tau \\ &\quad - b \int_0^t u_{2,xx}(x, \tau) d\tau - c \int_0^t u_{2,xxxx}(x, \tau) d\tau, \end{aligned} \quad (5.10)$$



**Table 2:** The approximate solution (5.12) in comparison with the exact solution (5.8) if  $b = -1$ ,  $a = 1$ ,  $\lambda = 0.1$ , and  $t = 0.1$ .

$x$	$u_{\text{exact}}$	$u_{\text{homotopy}}$	$ u_{\text{exact}} - u_{\text{homotopy}} $
-50	-0.0000423797	-0.0000423797	$2.12648 \times 10^{-13}$
-40	-0.000303737	-0.000303737	$1.41527 \times 10^{-12}$
-30	-0.00206958	-0.00206958	$7.86068 \times 10^{-12}$
-20	-0.0124247	-0.0124247	$2.60451 \times 10^{-11}$
-10	-0.0563527	0.0563527	$3.22282 \times 10^{-12}$
0	-0.158082	-0.158082	$7.76927 \times 10^{-11}$
10	-0.259668	-0.259668	$3.28948 \times 10^{-12}$
20	-0.303431	-0.303431	$2.6031 \times 10^{-11}$
30	-0.313732	-0.313732	$7.85122 \times 10^{-12}$
40	-0.315488	-0.315488	$1.41315 \times 10^{-12}$
50	-0.315747	-0.315747	$2.12275 \times 10^{-13}$

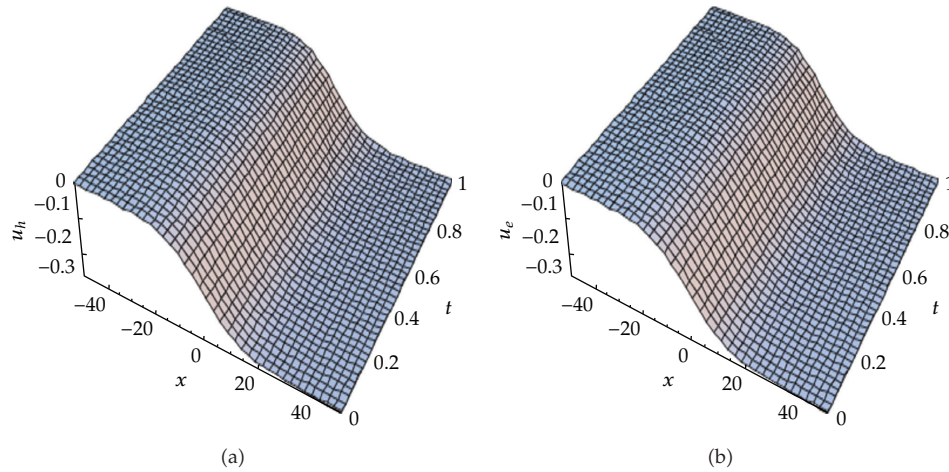
and so on. Consequently after some reduction with help of Maple or Mathematica, we get

$$\begin{aligned}
 V_0(x, t) &= \frac{15b\lambda}{19a} \left[ 2 + 3 \tanh\left(\frac{\lambda x}{2}\right) - \tanh^3\left(\frac{\lambda x}{2}\right) \right], \\
 V_1(x, t) &= -3\lambda t a \left(\frac{15b\lambda}{19a}\right)^2 \operatorname{sech}^4\left(\frac{\lambda x}{2}\right), \\
 V_2(x, t) &= -6\lambda^5 t^2 a^2 \left(\frac{15b\lambda}{19a}\right)^3 \operatorname{sech}^4\left(\frac{\lambda x}{2}\right) \tanh\left(\frac{\lambda x}{2}\right).
 \end{aligned} \tag{5.11}$$

and so on. Substituting from (5.11)-(5.12) into (4.5), we obtain the approximate solution of the initial value problem (1.2):

$$\begin{aligned}
 u(x, t) &= \frac{15b\lambda}{19a} \left[ 2 + 3 \tanh\left(\frac{\lambda x}{2}\right) - \tanh^3\left(\frac{\lambda x}{2}\right) \right] - 3\lambda t a \left(\frac{15b\lambda}{19a}\right)^2 \operatorname{sech}^4\left(\frac{\lambda x}{2}\right) \\
 &\quad - 6\lambda^5 t^2 a^2 \left(\frac{15b\lambda}{19a}\right)^3 \operatorname{sech}^4\left(\frac{\lambda x}{2}\right) \tanh\left(\frac{\lambda x}{2}\right) + \dots,
 \end{aligned} \tag{5.12}$$

which is in agreement with the exact solution (5.8) using Taylor series expansion near  $t = 0$ . The comparison between the exact solution (5.8) and the approximate solution (5.12) is shown in Table 2 and Figure 2. It seems that the errors are very small if  $b = -1$ ,  $a = 1$ ,  $\lambda = 0.1$ , and  $t = 0.1$ .



**Figure 2:** The approximate solution (5.12) shown in (b) in comparison with the exact solution (5.8) shown in (a) if  $b = -1$ ,  $a = 1$ ,  $\lambda = 0.1$ , and  $t = 0.1$ .

### 5.3. Approximate Solutions for the Nonlinear-Coupled Schrodinger-KdV Equations with Initial Conditions Using MVIM

In this subsection, we find the solutions  $u(x, t)$  and  $v(x, t)$  satisfying the nonlinear coupled Schrodinger-KdV equations (1.3) with the following initial conditions [41]:

$$\begin{aligned} u(x, 0) &= -c\alpha + 2ak \tanh(ikx), \\ v(x, 0) &= -2k^2 \operatorname{sech}^2(ikx), \end{aligned} \quad (5.13)$$

where  $k, \alpha$ , and  $c$  are arbitrary constants and  $\alpha = \sqrt{2k^2 + c/2}$ . These initial conditions follow by setting  $t = 0$  in the following exact solutions of (1.3):

$$u(x, t) = -c\alpha + 2ak \tanh(ik(x - ct)), \quad (5.14)$$

$$v(x, t) = -2k^2 \operatorname{sech}^2(ik(x - ct)). \quad (5.15)$$

These exact solutions have been derived by Zhang [41] using a direct algebraic approach. Let us now apply the MVIM to the initial value problem (1.3) and (5.13):

$$\begin{aligned} i \sum_{i=0}^{\infty} p^i u_i &= iu(x, 0) + p \int_0^t \left[ \sum_{i=0}^{\infty} p^i u_{i,xx} + \left( \sum_{i=0}^{\infty} p^i u_i \right) \left( \sum_{i=0}^{\infty} p^i v_i \right) \right] d\tau, \\ \sum_{i=0}^{\infty} p^i v_i &= v(x, 0) - p \int_0^t \left[ \sum_{i=0}^{\infty} p^i v_{i,xxx} + 6 \left( \sum_{i=0}^{\infty} p^i v_i \right) \left( \sum_{i=0}^{\infty} p^i v_{i,x} \right) \right. \\ &\quad \left. - \left( \sum_{i=0}^{\infty} p^i u_{i,x} \right) \left( \sum_{i=0}^{\infty} p^i \bar{u}_i \right) - \left( \sum_{i=0}^{\infty} p^i u_i \right) \left( \sum_{i=0}^{\infty} p^i \bar{u}_{i,x} \right) \right] d\tau. \end{aligned} \quad (5.16)$$

Comparing the different coefficient of like power of  $p$ , we have

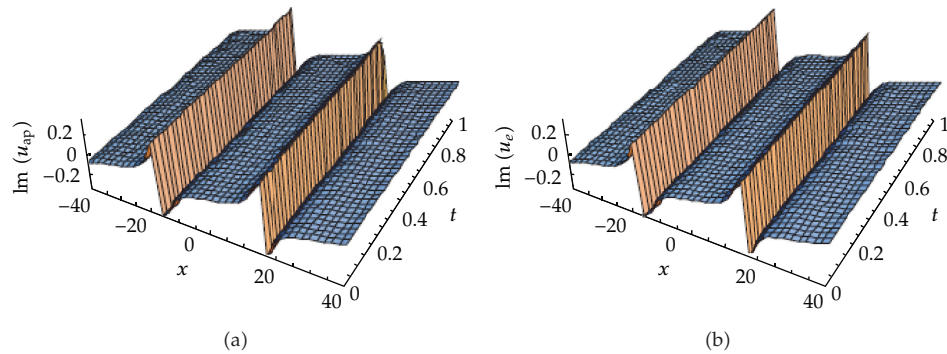
$$\begin{aligned}
 p^0 : u(x, 0) &= -c\alpha + 2ak \tanh(ikx), \\
 p^1 : iu_1(x, t) &= \int_0^t u_{0,xx}(x, \tau) d\tau + \int_0^t u_0(x, \tau) v_0(x, \tau) d\tau, \\
 p^2 : iu_2(x, t) &= \int_0^t u_{1,xx}(x, \tau) d\tau + \int_0^t u_0(x, \tau) v_1(x, \tau) d\tau + \int_0^t u_1(x, \tau) v_0(x, \tau) d\tau, \\
 p^3 : iu_3(x, t) &= \int_0^t u_{2,xx}(x, \tau) d\tau + \int_0^t u_0(x, \tau) v_2(x, \tau) d\tau + \int_0^t u_1(x, \tau) v_1(x, \tau) d\tau \\
 &\quad + \int_0^t u_2(x, \tau) v_0(x, \tau) d\tau, \\
 p^0 : v(x, 0) &= -2k^2 \operatorname{sech}^2(ikx), \\
 p^1 : v_1(x, t) &= - \int_0^t v_{0,xxx}(x, \tau) d\tau - 6 \int_0^t v_0(x, \tau) v_{0,x}(x, \tau) d\tau + \int_0^t u_0(x, \tau) \bar{u}_{0,x}(x, \tau) d\tau \\
 &\quad + \int_0^t u_{0,x}(x, \tau) \bar{u}_0(x, \tau) d\tau, \\
 p^2 : v_2(x, t) &= - \int_0^t v_{1,xxx}(x, \tau) d\tau - 6 \int_0^t v_1(x, \tau) v_{0,x}(x, \tau) d\tau - 6 \int_0^t v_{1,x}(x, \tau) v_0(x, \tau) d\tau \\
 &\quad + \int_0^t u_0(x, \tau) \bar{u}_{1,x}(x, \tau) d\tau + \int_0^t u_1(x, \tau) \bar{u}_{0,x}(x, \tau) d\tau + \int_0^t u_{0,x}(x, \tau) \bar{u}_1(x, \tau) d\tau \\
 &\quad + \int_0^t u_{1,x}(x, \tau) \bar{u}_0(x, \tau) d\tau,
 \end{aligned} \tag{5.17}$$

and so on. Consequently after some reduction with help of Maple or Mathematica, we get

$$\begin{aligned}
 u_0(x, t) &= -c\alpha + 2ak \tanh(ikx), \\
 u_1(x, t) &= -2itack^2 \operatorname{sech}^2(ikx), \\
 u_2(x, t) &= 2\alpha t^2 c^2 k^3 \operatorname{sech}^2(ikx) \tanh(ikx), \\
 u_3(x, t) &= -2it^3 \alpha c^3 k^4 \left[ 3 \operatorname{sech}^4(ikx) - 2 \operatorname{sech}^2(ikx) \right], \\
 v_0(x, t) &= -2k^2 \operatorname{sech}^2(ikx), \\
 v_1(x, t) &= -4itck^3 \operatorname{sech}^2(ikx) \tanh(ikx), \\
 v_2(x, t) &= -2c^2 t^2 k^4 \left[ 3 \operatorname{sech}^4(ikx) - 2 \operatorname{sech}^2(ikx) \right], \\
 v_3(x, t) &= -2ic^3 t^3 k^5 \left[ -4 \operatorname{sech}^4(ikx) \tanh(ikx) + \frac{4}{3} \operatorname{sech}^2(ikx) \tanh(ikx) \right].
 \end{aligned} \tag{5.18}$$

**Table 3:** The approximate solutions (5.19) and (5.20) in comparison with the exact solutions (5.14) and (5.15), respectively, if  $t = 0.5$ ,  $k = 0.1$ ,  $c = 0.2$ .

$x$	$ u_{\text{exact}} - u_{\text{app}} $	$ v_{\text{exact}} - v_{\text{app}} $
-2	$1.43476 \times 10^{-7}$	$1.21337 \times 10^{-8}$
-1.5	$1.34152 \times 10^{-7}$	$8.68819 \times 10^{-9}$
-1	$1.27786 \times 10^{-7}$	$5.6321 \times 10^{-9}$
-0.5	$1.24096 \times 10^{-7}$	$2.82205 \times 10^{-9}$
0	$1.22925 \times 10^{-7}$	$1.33341 \times 10^{-10}$
0.5	$1.2422 \times 10^{-7}$	$2.54966 \times 10^{-9}$
1	$1.28038 \times 10^{-7}$	$5.34219 \times 10^{-9}$
1.5	$1.34545 \times 10^{-7}$	$8.36762 \times 10^{-9}$
2	$1.44027 \times 10^{-7}$	$1.17669 \times 10^{-8}$



**Figure 3:** The approximate solution (5.19) shown in the figure (a) in comparison with the exact solution (5.14) shown in figure (b) if  $k = 0.1$ ,  $\alpha = 0.1$ , and  $c = 0.2$ .

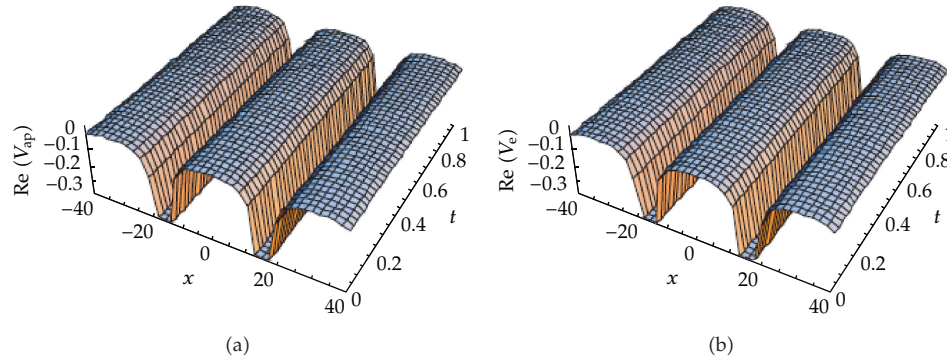
In this manner the other components can be obtained. Substituting (5.18) into (4.5), we obtain the approximate solutions of the initial value problem (1.3) and (5.13):

$$u(x, t) = -\alpha c + 2ak \tanh(ikx) - 2itack^2 \operatorname{sech}^2(ikx) + 2\alpha t^2 c^2 k^3 \operatorname{sech}^2(ikx) \tanh(ikx) - 2it^3 \alpha c^3 k^4 [3 \operatorname{sech}^4(ikx) - 2 \operatorname{sech}^2(ikx)] + \dots, \quad (5.19)$$

$$v(x, t) = -2k^2 \operatorname{sech}^2(ikx) - 4i tck^3 \operatorname{sech}^2(ikx) \tanh(ikx) - 2c^2 t^2 k^4 [3 \operatorname{sech}^4(ikx) - 2 \operatorname{sech}^2(ikx)] + \dots, \quad (5.20)$$

which are in the closed form of the exact solutions (5.14) and (5.15) using Taylor series expansion near  $t = 0$ .

The comparison between the exact solutions (5.14), (5.15) and the approximate solutions (5.19), (5.20) respectively, are shown in Table 3 and Figures 3 and 4. It seems that the errors are very small if  $t = 0.5$ ,  $k = 0.1$ ,  $c = 0.2$ .



**Figure 4:** The approximate solution (5.20) shown in (a) in comparison with the exact solution (5.15) shown in (b) if  $k = 0.1, c = 0.2, \alpha = 0.1$ .

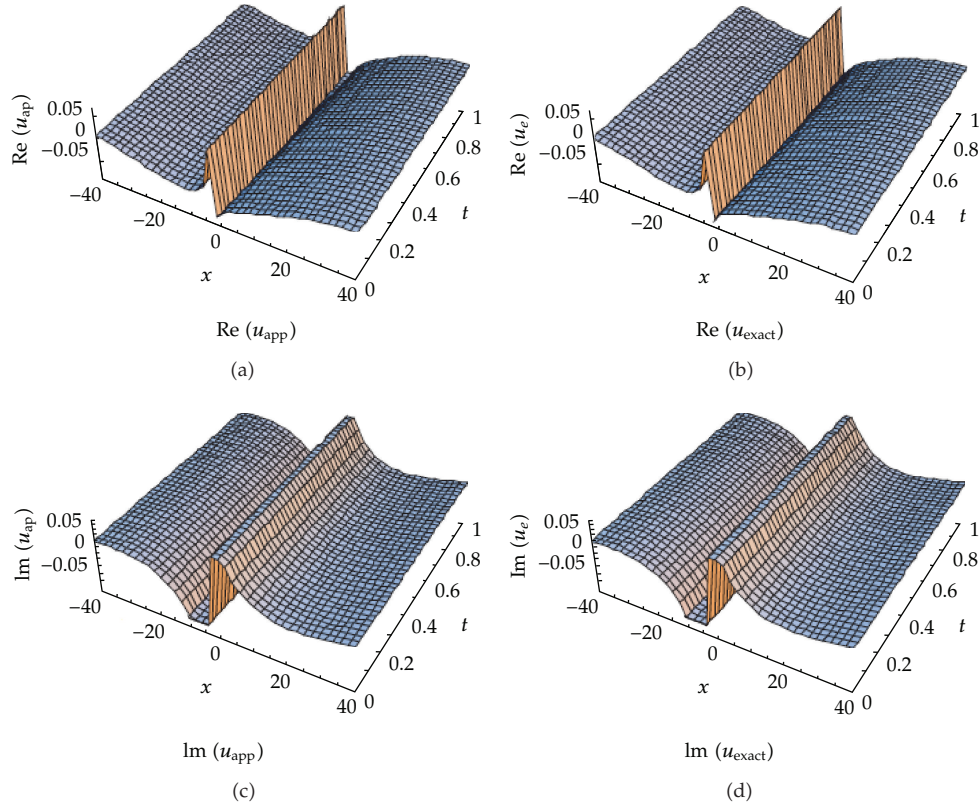
**Table 4:** The approximate solutions (5.27) and (5.28) in comparison with the exact solutions (5.22) and (5.23), respectively, if  $t = 0.5, k = 0.1, c = 0.2, \xi_0 = 2$ , and  $\xi_1 = 3$ .

$x$	$ u_{ex} - u_{app} $	$ v_{ex} - v_{app} $
-50	$8.13281 \times 10^{-5}$	$9.25371 \times 10^{-14}$
-40	$9.81849 \times 10^{-5}$	$3.88495 \times 10^{-13}$
-30	$1.22963 \times 10^{-4}$	$2.56975 \times 10^{-12}$
-20	$1.61595 \times 10^{-4}$	$4.11388 \times 10^{-11}$
-10	$4.68634 \times 10^{-10}$	$8.35654 \times 10^{-9}$
0	$6.24133 \times 10^{-3}$	$1.42883 \times 10^{-6}$
10	$5.81547 \times 10^{-4}$	$2.09106 \times 10^{-10}$
20	$2.70634 \times 10^{-4}$	$6.79057 \times 10^{-12}$
30	$1.73048 \times 10^{-4}$	$7.77128 \times 10^{-13}$
40	$1.26497 \times 10^{-5}$	$1.58651 \times 10^{-13}$
50	$9.94724 \times 10^{-5}$	$4.52416 \times 10^{-14}$

#### 5.4. Approximate Solution of the Nonlinear Long and Short Wave Resonance Equations with Initial Conditions Using MVIM

In this subsection, we find the solutions  $u(x, t)$  and  $v(x, t)$  satisfying the nonlinear long–short wave resonance equations (1.4) with the following initial conditions [42]:

$$\begin{aligned}
 v_0 = u(x, 0) &= 2\sqrt{\frac{k}{\beta}} \frac{1}{(x + \xi_1)} e^{i(kx + \xi_0)}, \\
 v_0 = v(x, 0) &= \frac{2}{(x + \xi_1)^2} - C,
 \end{aligned}
 \tag{5.21}$$



**Figure 5:** The approximate solution (5.27) in comparison with the exact solution (5.22) if  $k = 0.1, c = 0.2, \xi_0 = 2$ , and  $\xi_1 = 3$ .

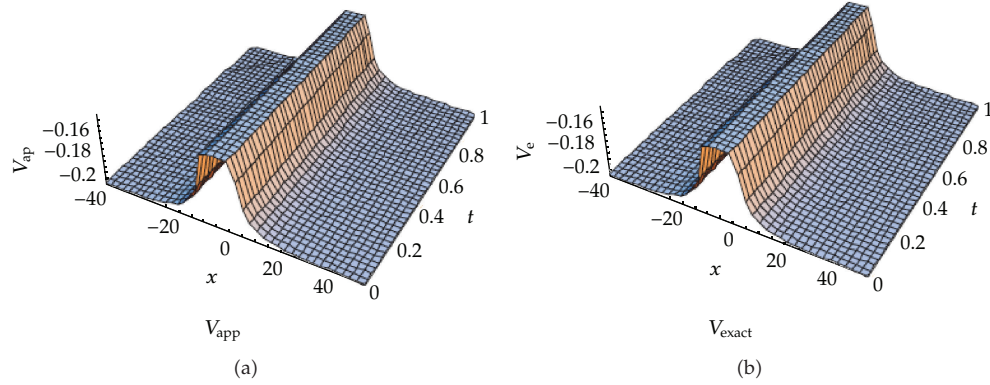
where  $\beta, k, C, \xi_0$  and  $\xi_1$  are arbitrary constants. These initial conditions follow by setting  $t = 0$  in the following exact solutions of (1.4):

$$u(x, t) = 2\sqrt{\frac{k}{\beta}} \frac{1}{(x - 2kt + \xi_1)} e^{i(kx + wt + \xi_0)}, \tag{5.22}$$

$$v(x, t) = \frac{2}{(x - 2kt + \xi_1)^2} - C, \tag{5.23}$$

where  $w = C - k^2$  is constant. These exact solutions have been derived by Shang [42] using the extended hyperbolic function method, which describes the resonance interaction between the long wave and the short wave. Let us now apply the MVIM to the initial value problem (1.4) and (5.21):

$$\begin{aligned} i \sum_{i=0}^{\infty} p^i u_i &= iu(x, 0) - p \int_0^t \left[ \sum_{i=0}^{\infty} p^i u_{i,xx} - \left( \sum_{i=0}^{\infty} p^i u_i \right) \left( \sum_{i=0}^{\infty} p^i v_i \right) \right] d\tau, \\ \sum_{i=0}^{\infty} p^i v_i &= v(x, 0) - p\beta \int_0^t \left[ \left( \sum_{i=0}^{\infty} p^i u_{i,x} \right) \left( \sum_{i=0}^{\infty} p^i \bar{u}_i \right) + \left( \sum_{i=0}^{\infty} p^i u_i \right) \left( \sum_{i=0}^{\infty} p^i \bar{u}_{i,x} \right) \right] d\tau. \end{aligned} \tag{5.24}$$



**Figure 6:** The approximate solution (5.28) in comparison with the exact solution (5.23) if  $k = 0.1, c = 0.2, \xi_0 = 2$ , and  $\xi_1 = 3$ .

Comparing the different coefficient of like power of  $p$ , we have

$$\begin{aligned}
 p^0 : u(x, 0) &= 2\sqrt{\frac{k}{\beta}} \frac{1}{(x + \xi_1)} e^{i(kx + \xi_0)}, \\
 p^1 : iu_1(x, t) &= -\int_0^t u_{0,xx}(x, \tau) d\tau + \int_0^t u_0(x, \tau) v_0(x, \tau) d\tau, \\
 p^2 : iu_2(x, t) &= -\int_0^t u_{1,xx}(x, \tau) d\tau + \int_0^t u_0(x, \tau) v_1(x, \tau) d\tau + \int_0^t u_1(x, \tau) v_0(x, \tau) d\tau, \\
 p^3 : iu_3(x, t) &= -\int_0^t u_{2,xx}(x, \tau) d\tau + \int_0^t u_0(x, \tau) v_2(x, \tau) d\tau + \int_0^t u_1(x, \tau) v_1(x, \tau) d\tau \\
 &\quad + \int_0^t u_2(x, \tau) v_0(x, \tau) d\tau, \\
 p^0 : v(x, 0) &= \frac{2}{(x + \xi_1)^2} - C, \\
 p^1 : v_1(x, t) &= -\beta \int_0^t u_0(x, \tau) \bar{u}_{0,x}(x, \tau) d\tau - \beta \int_0^t u_{0,x}(x, \tau) \bar{u}_0(x, \tau) d\tau, \\
 p^2 : v_2(x, t) &= -\beta \int_0^t u_0(x, \tau) \bar{u}_{1,x}(x, \tau) d\tau - \beta \int_0^t u_1(x, \tau) \bar{u}_{0,x}(x, \tau) d\tau \\
 &\quad - \beta \int_0^t u_{0,x}(x, \tau) \bar{u}_1(x, \tau) d\tau - \beta \int_0^t u_{1,x}(x, \tau) \bar{u}_0(x, \tau) d\tau, \\
 p^3 : v_3(x, t) &= -\beta \int_0^t u_{0,x}(x, \tau) \bar{u}_{2,x}(x, \tau) d\tau - \beta \int_0^t u_0(x, \tau) \bar{u}_{2,x}(x, \tau) d\tau - \beta \int_0^t u_{1,x}(x, \tau) \bar{u}_1(x, \tau) d\tau \\
 &\quad - \beta \int_0^t u_1(x, \tau) \bar{u}_{1,x}(x, \tau) d\tau - \beta \int_0^t u_{2,x}(x, \tau) \bar{u}_0(x, \tau) d\tau - \beta \int_0^t u_2(x, \tau) \bar{u}_{0,x}(x, \tau) d\tau,
 \end{aligned} \tag{5.25}$$

and so on. On substituting (5.21) into (5.25), we deduce that

$$\begin{aligned}
 u_0(x, t) &= 2\sqrt{\frac{k}{\beta}} \frac{1}{(x + \xi_1)} e^{i(kx + \xi_0)}, \\
 u_1(x, t) &= 2t\sqrt{\frac{k}{\beta}} \left[ \frac{i(C - k^2)}{(x + \xi_1)} + \frac{2k}{(x + \xi_1)^2} \right] e^{i(kx + \xi_0)}, \\
 u_2(x, t) &= 2t^2\sqrt{\frac{k}{\beta}} \left[ \frac{i(C - k^2)^2}{(x + \xi_1)} + \frac{4ik}{(x + \xi_1)^2} + \frac{4k}{(x + \xi_1)^3} \right] e^{i(kx + \xi_0)}, \\
 v_0(x, t) &= \frac{2}{(x + \xi_1)^2} - C, \\
 v_1(x, t) &= \frac{8kt}{(x + \xi_1)^3}, \\
 v_2(x, t) &= \frac{48kt^2}{(x + \xi_1)^4}, \\
 v_3(x, t) &= \frac{384kt^3}{(x + \xi_1)^5}.
 \end{aligned} \tag{5.26}$$

In this manner the other components can be obtained. Consequently, we obtain the following approximate solutions of the initial value problem (1.4) and (5.21):

$$\begin{aligned}
 u(x, t) &= 2\sqrt{\frac{k}{\beta}} \frac{1}{(x + \xi_1)} e^{i(kx + \xi_0)} + 2t\sqrt{\frac{k}{\beta}} \left[ \frac{i(C - k^2)}{(x + \xi_1)} + \frac{2k}{(x + \xi_1)^2} \right] e^{i(kx + \xi_0)} \\
 &\quad + 2t^2\sqrt{\frac{k}{\beta}} \left[ \frac{i(C - k^2)^2}{(x + \xi_1)} + \frac{4ik}{(x + \xi_1)^2} + \frac{4k}{(x + \xi_1)^3} \right] e^{i(kx + \xi_0)} + \dots,
 \end{aligned} \tag{5.27}$$

$$v(x, t) = \frac{2}{(x + \xi_1)^2} - C + \frac{8kt}{(x + \xi_1)^3} + \frac{48kt^2}{(x + \xi_1)^4} + \frac{384kt^3}{(x + \xi_1)^5} + \dots \tag{5.28}$$

which are in the closed forms of the exact solutions (5.22) and (5.23) using Taylor series expansion near  $t = 0$ .

The comparison between the exact solutions (5.22), (5.23) and the approximate solutions (5.27), (5.28) respectively is shown in Table 4 and Figures 5 and 6. It seems that the errors are very small if  $t = 0.5$ ,  $k = 0.1$ ,  $c = 0.2$ ,  $\xi_0 = 2$ , and  $\xi_1 = 3$ .

## 6. Conclusions

In this paper, the modified variational iteration method was applied for finding the approximate solutions for some nonlinear evolution equations in mathematical physics via the nonlinear Burgers-Fisher equation, nonlinear Kuramoto-Sivashinsky equation, nonlinear coupled Schrodinger KdV equations, and nonlinear long-short wave resonance equations with well-known initial conditions. It seems to us that the modified variational iteration method presents a rapid convergence solutions. It can be concluded that this method is



very powerful and efficient technique in finding approximate solutions for wide classes of nonlinear problems.

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