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Research Article

Numerical Solution of Weakly Singular Integrodifferential Equations on Closed Smooth Contour in Lebesgue Spaces

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The present paper deals with the justification of solvability conditions and properties of solutions for weakly singular integro-differential equations by collocation and mechanical quadrature methods. The equations are defined on an arbitrary smooth closed contour of the complex plane. Error estimates and convergence for the investigated methods are established in Lebesgue spaces.

1. Introduction

Singular integral equations (SIE) and singular integro-differential equations with Cauchy kernels (SIDE) and systems of such equations model many problems in elasticity theory, aerodynamics, mechanics, thermoelasticity and queuing analysis (see [1–6] and the literature cited therein). The general theory of SIE and SIDE has been widely investigated over the last decades [7–11]. It is known that the exact solution for SIDE can be found only in some particular cases. That is why there is a necessity to elaborate approximation methods for solving SIDE.

In the past, there was a lot of research in literature devoted to an approximate solution of SIE and SIDE by collocation and mechanical quadrature methods. The equations are defined on the unit circle centered at the origin or on the real axis, see for example [12–15]. However, the case when the contour of integration is an arbitrary smooth closed curve has not been studied enough.

It should be noted that conformal mapping from the arbitrary smooth closed contour to the unit circle does not solve the problem. Moreover, it makes it more difficult. In the present paper we consider the collocation and mechanical quadrature methods for the approximate solution of weakly SIDE. We use the Fejér points as collocation knots. In Section 2 we introduce the main definitions and notations. We present the numerical schemes of collocation and mechanical quadrature methods in Section 3. In Section 4 we formulate the auxiliary results. We use these results to prove the convergence theorems in Section 5.

We note that the convergence of the collocation method, reduction method and mechanical quadrature method for SIDE and systems of such equations in generalized Hölder spaces has been obtained in [16–18]. The equations are given on an arbitrary smooth closed contour (not weakly SIDE).

2. The Main Definitions and Notations

Let Γ be an arbitrary smooth closed contour bounding a simply connected region F^+ of the complex plane and let $t = 0 \in F^+$, $F^- = C \setminus \{F^+ \cup \Gamma\}$, where C is the complex plane. Let $z = \psi(w)$ be a function, mapping conformably the outside of unit circle $\Gamma_0 = \{|w| = 1\}$ on the domain F^- so that

$$\psi(\infty) = \infty, \qquad \psi^{(\prime)}(\infty) = 1. \tag{2.1}$$

We assume that the function $z = \psi(w)$ has the second derivative, satisfying on Γ_0 the Hölder condition with some parameter μ $(0 < \mu < 1)$; the class of such contours is denoted by $C(2;\mu)$ [19, 20].

Let $L_p(\Gamma)$ $(1 < p < \infty)$ be the space of complex functions with norm

$$\|g\|_p = \left(\frac{1}{l}\int_\Gamma |g|^p |d\tau|\right)^{1/p}, \tag{2.2}$$

where l is the length of Γ.

Let U_n be the Lagrange interpolating polynomial

$$(U_n g)(t) = \sum_{s=0}^{2n} g(t_s) \cdot l_s(t), \tag{2.3}$$

$$l_j(t) = \prod_{k=0,k\neq j}^{2n} \frac{t - t_k}{t_j - t_k}\left(\frac{t_j}{t}\right)^n \equiv \sum_{k=-n}^{n} \Lambda_k^{(j)} t^k, \quad t \in \Gamma,\ j = 0,\dots,2n. \tag{2.4}$$

3. Numerical Schemes of the Collocation Method and Mechanical Quadrature Method

In the complex space $L_p(\Gamma)$ $(1 < p < \infty)$ we consider the weakly singular integro-differential equation (SIDE):

$$(Mx \equiv)\sum_{r=0}^{\nu}\left[\widetilde{A}_r(t)x^{(r)}(t) + \widetilde{B}_r(t)\,\frac{1}{\pi i}\int_\Gamma \frac{x^{(r)}(\tau)}{\tau - t}\,d\tau \right.$$
$$\left. + \frac{1}{2\pi i}\int_\Gamma \frac{K_r(t,\tau)}{|t-\tau|^\gamma}\cdot x^{(r)}(\tau)d\tau\right] = f(t), \quad t \in \Gamma, \tag{3.1}$$

where $0 < \gamma < 1$, $\widetilde{A}_r(t)$, $\widetilde{B}_r(t)$, $K_r(t,\tau)(r = 0,\ldots,\nu)$ and $f(t)$ are known functions; $x^{(0)}(t) = x(t)$ is an unknown function; $x^{(r)}(t) = ((d^r x(t))/dt^r)$ $(r = 1,\ldots,\nu)$ (ν is a positive integer). Using the Riesz operators $P = 1/2(I + S)$, $Q = I - P$, (where I is the identity operator, and S is the singular operator (with Cauchy kernel)), we rewrite (3.1) in the following form convenient for consideration:

$$(Mx \equiv)\sum_{r=0}^{\nu}\left[A_r(t)\left(Px^{(r)}\right)(t) + B_r(t)\left(Qx^{(r)}\right)(t) \right.$$
$$\left. + \frac{1}{2\pi i}\int_\Gamma \frac{K_r(t,\tau)}{|t-\tau|^\gamma}\cdot x^{(r)}(\tau)d\tau\right] = f(t), \quad t \in \Gamma, \tag{3.2}$$

where $A_r(t) = \widetilde{A}_r(t) + \widetilde{B}_r(t)$, $B_r(t) = \widetilde{A}_r(t) - \widetilde{B}_r(t)$, $r = 0,\ldots,\nu$.

We search for a solution of (3.1) in the class of functions, satisfying the condition

$$\frac{1}{2\pi i}\int_\Gamma x(\tau)\tau^{-k-1}d\tau = 0, \quad k = 0,\ldots,\nu - 1. \tag{3.3}$$

In order to reduce the numerical schemes of collocation method we introduce a new integro-differential equation from the initial one. The weakly singular kernels are substituted by continuous ones. We obtain the new approximate equation

$$\left(M_\rho(x) \equiv\right)(M_0 x)(t) + \frac{1}{2\pi i}\sum_{r=0}^{\nu}\int_\Gamma K_{r,\rho}(t,\tau)x^{(r)}(\tau)d\tau = f(t), \quad t \in \Gamma, \tag{3.4}$$

where

$$K_{r,\rho}(t,\tau) = \begin{cases} \dfrac{K_r(t,\tau)}{|t-\tau|^\gamma}, & \text{when } |t-\tau| \geq \rho, \\ \dfrac{K_r(t,\tau)}{\rho^\gamma}, & \text{when } |t-\tau| < \rho. \end{cases} \tag{3.5}$$

ρ is an arbitrary positive number, M_0 is characteristic part of weakly SIDE. Equation (3.1) with the conditions (3.3) we denote as problem "(3.1)–(3.3)". We search for the approximate solution of problem (3.1)–(3.3) in polynomial form

$$x_{n,\rho}(t) = \sum_{k=0}^{n} \xi_{k,\rho}^{(n)} t^{k+\nu} + \sum_{k=-n}^{-1} \xi_{k,\rho}^{(n)} t^{k}, \quad t \in \Gamma, \tag{3.6}$$

where $\xi_{k,\rho}^{(n)} = \xi_{k,\rho}$ $(k = -n, \ldots, n)$ are unknown complex numbers. We note that the function $x_{n,\rho}(t)$, constructed by formula, obviously satisfies the condition (3.3). Let $R_n(t) = (M_\rho x_n)(t) - f(t)$ be residual of SIDE. The collocation method consists in setting it equal to zero at some chosen points t_j, $j = 0, \ldots, 2n$ on Γ and thus obtaining a linear algebraic system for unknowns $\xi_{k,\rho}$ which is determined by solving it:

$$R_n(t_j) = 0, \quad j = 0, \ldots, 2n. \tag{3.7}$$

Using the (3.7) we obtain a system of linear algebraic equations (SLAE) for collocation method:

$$\begin{aligned}
\sum_{r=0}^{\nu} & A_r(t_j) \sum_{k=0}^{n} \frac{(k+\nu)!}{(k+\nu-r)!} t_j^{k+\nu-r} \xi_{k,\rho} \\
&+ B_r(t_j) \sum_{k=1}^{n} (-1)^r \frac{(k+r-1)!}{(k-1)!} t_j^{-k-r} \times \xi_{-k,\rho} \\
&+ \frac{1}{2\pi i} \cdot \sum_{k=0}^{n} \frac{(k+\nu)!}{(k+\nu-r)!} \int_\Gamma K_{r,\rho}(t_j,\tau) \tau^{k+\nu-r} d\tau \cdot \xi_{k,\rho} \\
&+ \sum_{k=1}^{n} (-1)^r \frac{(k+r-1)!}{(k-1)!} \cdot \frac{1}{2\pi i} \int_\Gamma K_{r,\rho}(t_j,\tau) \tau^{-k-r} d\tau \cdot \xi_{-k,\rho} = f(t_j), \\
& \qquad j = 0, \ldots, 2n,
\end{aligned} \tag{3.8}$$

where t_j, $(j = 0, \ldots, 2n)$ are distinct points on Γ and $A_r(t) = \widetilde{A}_r(t) + \widetilde{B}_r(t)$, $B_r(t) = \widetilde{A}_r(t) - \widetilde{B}_r(t)$. We approximate the integrals in SLAE (3.8) by quadrature formula:

$$\frac{1}{2\pi i} \int_\Gamma g(\tau) \tau^{l+k} d\tau \cong \frac{1}{2\pi i} \int_\Gamma U_n\left(\tau^{l+1} \cdot g(\tau)\right) \tau^{k-1} d\tau, \tag{3.9}$$

where $k = 0, \ldots, n$, at $l = 0, 1, 2, \ldots$ and $k = -1, \ldots, -n$, for $l = -1, -2, \ldots$, and U_n is the Lagrange interpolation operator defined by formula (2.3).

Thus, we obtain the following SLAE from (3.8):

$$\begin{aligned}
&\sum_{r=0}^{\nu} A_r(t_j) \sum_{k=0}^{n} \frac{(k+\nu)!}{(k+\nu-r)!} t_j^{k+\nu-r} \xi_{k,\rho} \\
&\quad + B_r(t_j) \sum_{k=1}^{n} (-1)^r \frac{(k+r-1)!}{(k-1)!} t_j^{-k-r} \times \xi_{-k,\rho} \\
&\quad + \sum_{k=0}^{n} \frac{(k+\nu)!}{(k+\nu-r)!} \sum_{s=0}^{2n} K_{r,\rho}(t_j, t_s) t_s^{1+k-r} \Lambda_{-k}^{(s)} \xi_{k,\rho} \\
&\quad + \sum_{k=1}^{n} (-1)^r \frac{(k+r-1)!}{(k-1)!} \sum_{s=0}^{2n} K_{r,\rho}(t_j, t_s) t_s^{-k-r} \Lambda_{k}^{(s)} \xi_{-k,\rho} = f(t_j), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 0, \ldots, 2n.
\end{aligned} \tag{3.10}$$

4. Auxiliary Results

We formulate one result from [21], establishing the equivalence (in sense of solvability) of problem (3.1)–(3.3) and SIE. We use this result for proving Theorems 5.3 and 5.4. The functions $d^\nu(Px)(t)/dt^\nu$ and $d^\nu(Qx)(t)/dt^\nu$ can be represented by integrals of Cauchy type with the same density $v(t)$:

$$\begin{aligned}
\frac{d^\nu(Px)(t)}{dt^\nu} &= \frac{1}{2\pi i} \int_\Gamma \frac{v(\tau)}{\tau - t} d\tau, \quad t \in F^+, \\
\frac{d^\nu(Qx)(t)}{dt^\nu} &= \frac{t^{-\nu}}{2\pi i} \int_\Gamma \frac{v(\tau)}{\tau - t} d\tau, \quad t \in F^-.
\end{aligned} \tag{4.1}$$

Using the integral representation (4.1) we reduce the problem (3.1)–(3.3) to the equivalent (in sense of solvability) of SIE

$$\begin{aligned}
(\Upsilon v \equiv) C(t) v(t) &+ \frac{D(t)}{\pi i} \int_\Gamma \frac{v(\tau)}{\tau - t} d\tau \\
&+ \frac{1}{2\pi i} \int_\Gamma \frac{h(t,\tau)}{|\tau - t|^\gamma} v(\tau) d\tau = f(t), \quad t \in \Gamma,
\end{aligned} \tag{4.2}$$

for unknown $v(t)$ where

$$C(t) = \frac{1}{2}\left[A_\nu(t) + t^{-\nu}B_\nu(t)\right], \qquad D(t) = \frac{1}{2}\left[A_\nu(t) - t^{-\nu}B_\nu(t)\right], \tag{4.3}$$

$$\begin{aligned} h(t,\tau) = {} & \frac{1}{2}\left[K_\nu(t,\tau) + K_\nu(t,\tau)\tau^{-n}\right] - \frac{1}{2\pi i}\int_\Gamma \left[K_\nu(t,t_1) - K_\nu(t,t_1)t_1^{-n}\right]\frac{dt_1}{t_1-\tau} \\ & + \sum_{j=0}^{\nu-1}\left[A_j(t)\widetilde{M}_j(t,\tau) + \int_\Gamma K_j(t,t_1)\widetilde{M}_j(t_1,\tau)dt_1\right] \\ & - \sum_{j=0}^{\nu-1}\left[B_j(t)\widetilde{N}_j(t,\tau) + \int_\Gamma K_j(t,t_1)\widetilde{N}_j(t_1,\tau)dt_1\right], \end{aligned} \tag{4.4}$$

where $\widetilde{M}_j(t,\tau)$, $\widetilde{N}_j(t,\tau)$ $j = 0,\ldots,\nu$ are Hölder functions. An obvious form for these functions are given in [21]. By virtue of the properties of the functions $\widetilde{M}_j(t,\tau)$, $\widetilde{N}_j(t,\tau)$, $K_j(t,\tau)$, $A_j(t)$, $B_j(t)$, $j = 0,\ldots,\nu$ the function $h(t,\tau)$ is a continuous function in both variables.

Lemma 4.1. *The SIE* (4.2) *and problem* (3.1)–(3.3) *are equivalent in the sense of solvability. That is, for each solution $v(t)$ of SIE* (4.2) *there is a solution of problem* (3.1)–(3.3)*, determined by formulae*

$$(Px)(t) = \frac{(-1)^\nu}{2\pi i(\nu-1)!}\int_\Gamma v(\tau)\left[(\tau-t)^{\nu-1}\log\left(1-\frac{t}{\tau}\right) + \sum_{k=1}^{\nu-1}\widetilde{\alpha}_k\tau^{\nu-k-1}t^k\right]d\tau, \tag{4.5}$$

$$(Qx)(t) = \frac{(-1)^\nu}{2\pi i(\nu-1)!}\int_\Gamma v(\tau)\tau^{-\nu}\left[(\tau-t)^{\nu-1}\log\left(1-\frac{\tau}{t}\right) + \sum_{k=1}^{\nu-2}\widetilde{\beta}_k\tau^{\nu-k-1}t^k\right]d\tau, \tag{4.6}$$

where ($\widetilde{\alpha}_k = \sum_{j=0}^{k-1}((-1)^j C_{\nu-1}^j/(k-j))$, $k = 1,\ldots,\nu-1$, $\widetilde{\beta}_k = \sum_{j=k+1}^{\nu-1}((-1)^j C_{\nu-1}^j/(j-k))$, $k = 1,\ldots,\nu-2$ and $C_{\nu-1}^j$ are the binomial coefficients). On the other hand, for each solution $x(t)$ of the problem (3.1)–(3.3) *there is a solution $v(t)$*

$$v(t) = \frac{d^\nu(Px)(t)}{dt^\nu} + t^\nu\frac{d^\nu(Qx)(t)}{dt^\nu}, \tag{4.7}$$

to the SIE (4.2)*. Furthermore, for linearly independent solutions of* (4.2)*, there are corresponding linearly-independent solutions of the problem* (3.1)–(3.3) *from* (4.6) *and vice versa.*

In formulas (4.6) by $\log(1-t/\tau)$ we understand the branch which vanishes as $t = 0$ and by $\log(1-\tau/t)$ the branch which vanishes as $t = \infty$.

4.1. Estimates for Weakly Singular Integral Operators

Lemma 4.2. *Let $h(t,\tau) \in C(\Gamma \times \Gamma)$, and $\psi(t) \in L_p(\Gamma)$, $1 < p < \infty$. Then the function $H(t) = (1/2\pi i)\int_\Gamma (h(t,\tau)/|\tau - t|^\gamma)\psi(\tau)d\tau$, satisfies the inequality*

$$\|H\|_p \le d_1 \|\psi\|_p, \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad \|(\cdot)\|_p = \left|\frac{1}{l}\int_\Gamma |(\cdot)(\tau)|^p d\tau\right|^{1/p}. \tag{4.8}$$

By d_1, $d_2, \ldots,$ we denote the constants.

The proof can be found in [22].

Lemma 4.3. *Let the assumptions of Lemma 4.2 be satisfied; then $\|\chi_\rho\|_p \le d_2 \rho^{(1-\gamma)/q}\|\psi\|_p$, where $\chi_\rho = (1/2\pi i)\int_\Gamma [(h(t,\tau)/|\tau - t|^\gamma) - h_\rho(t,\tau)]\psi(\tau)d\tau$, $1/p + 1/q = 1$.*

The proof of this lemma can be found in [22].

5. Convergence Theorems

Define $\overset{\circ}{W}{}_p^{(\nu)}$ as

$$\overset{\circ}{W}{}_p^{(\nu)} = \left\{ g \in L_p(\Gamma) : g^{(\nu)} \in L_p(\Gamma), \frac{1}{2\pi i}\int_\Gamma g(\tau)\tau^{-k-1}d\tau = 0, k = 0, \ldots, \nu - 1 \right\}. \tag{5.1}$$

The norm in $\overset{\circ}{W}{}_p^{(\nu)}$ is determined by the equality

$$\|g\|_{p,\nu} = \left\|g^{(\nu)}\right\|_{L_p}. \tag{5.2}$$

We denote by $L_{p,\nu}$ the image of the space L_p with respect to the map $P + t^{-\nu}Q$ equipped with the norm of L_p. We formulate Lemmas 5.1 and 5.2 from [23]. We use these lemmas to prove the convergence theorems.

Lemma 5.1. *The differential operator $D^\nu : \overset{\circ}{W}{}_p^{(\nu)} \to L_{p,\nu}$, $(D^\nu g)(t) = g^{(\nu)}(t)$ is continuously invertible and its inverse operator $D^{-\nu} : L_{p,\nu} \to \overset{\circ}{W}{}_p^{(\nu)}$ is determined by the equality*

$$\begin{aligned}
&\left(D^{-\nu}g\right)(t) = \left(N^+g\right)(t) + \left(N^-g\right)(t), \\
&\left(N^+g\right)(t) = \frac{(-1)^\nu}{2\pi i(\nu-1)!}\int_\Gamma (Pg)(\tau)(\tau - t)^{\nu-1}\log\left(1 - \frac{t}{\tau}\right)d\tau, \\
&\left(N^-g\right)(t) = \frac{(-1)^{\nu-1}}{2\pi i(\nu-1)!}\int_\Gamma (Qg)(\tau)(\tau - t)^{\nu-1}\log\left(1 - \frac{\tau}{t}\right)d\tau.
\end{aligned} \tag{5.3}$$

From Lemma 5.1 Lemma 5.2 follows.

Lemma 5.2. *The operator* $B : \overset{\circ}{W}{}_p^{(\nu)} \to L_p, B = (P + t^{\nu} Q) D^{\nu}$ *is invertible and*

$$B^{-1} = D^{-\nu}\left(P + t^{-\nu} Q\right). \tag{5.4}$$

The proofs of Lemmas 5.1 and 5.2 can be found in [23].

The convergence of collocation method and mechanical quadrature method are given in the following theorems.

Theorem 5.3. *Let the following conditions be satisfied:*

(1) $\Gamma \in C(2, \mu)$, $0 < \mu < 1$;

(2) *the functions* $A_r(t)$ *and* $B_r(t)$ *belong to the space* $H_{\alpha}(\Gamma)$, $0 < \alpha < 1$;

(3) $A_{\nu}(t) B_{\nu}(t) \neq 0$, $t \in \Gamma$;

(4) *the index of the function* $t^{\nu} B_{\nu}^{-1}(t) A_{\nu}(t)$ *is equal to zero;*

(5) $K_r(t, \tau)(r = 0, \ldots, \nu) \in H_{\beta}(\Gamma \times \Gamma)$, $0 < \beta \leq 1$, *function* $f(t) \in C(\Gamma)$;

(6) *the operator* $M : \overset{\circ}{W}{}_p^{(\nu)} \to L_p(\Gamma)$ *is linear and invertible;*

(7) *the points* $t_j (j = 0, \ldots 2n)$ *form a system of Fejér knots on* Γ *[24, 25]:*

$$t_j = \varphi\left[\exp\left(\frac{2\pi i}{2n+1}(j - n)\right)\right], \quad j = 0, \ldots, 2n, \; i^2 = -1. \tag{5.5}$$

Then, the SLAE (3.8) *of collocation method has the unique solution* $\xi_k (k = -n, \ldots, n)$, *for numbers* $n \geq n_1$ *that are large enough and for numbers* ρ *small enough. The* ρ *satisfies the following inequality:*

$$\varepsilon_{\rho} = d_3 \rho^{(1-\gamma)/q} \left\| M^{-1} \right\|_p < q_8 < 1. \tag{5.6}$$

The approximate solutions $x_{n,\rho}(t)$, *constructed by formula* (3.6), *converge when* $n \to \infty$ *in the norm of space* $\overset{\circ}{W}{}_p^{(\nu)}$ *to the exact solution* $x(t)$ *of the problem* (3.1)–(3.3) *in sense of*

$$\lim_{\rho \to 0} \lim_{n \to \infty} \left\| x - x_{n,\rho} \right\|_{p,\nu} = 0, \tag{5.7}$$

and the following estimation for convergence holds:

$$\begin{gathered} \left\| x - x_{n,\rho} \right\|_{p,\nu} = O\left(\rho^{(1-\gamma)/q}\right) + O\left(\frac{1}{n^{\alpha}}\right) + O\left(\omega\left(f; \frac{1}{n}\right)\right) + O\left(\omega^t\left(h_{\rho}; \frac{1}{n}\right)\right) \overset{\text{def}}{=} \delta_n, \\ \left(\frac{1}{p} + \frac{1}{q} = 1\right). \end{gathered} \tag{5.8}$$

The $\omega(f;1/n)$ and $\omega^t(h;1/n)$ are modules of continuity, where

$$\omega\left(f;\frac{1}{n}\right) = \sup_{|t'-t''|\le 1/n} \left|f(t') - f(t'')\right|,$$
$$\omega^t\left(h;\frac{1}{n}\right) = \sup_{|t'-t''|\le 1/n} \left|h(t';\tau) - h(t'';\tau)\right|, \quad t',t'' \in \Gamma. \tag{5.9}$$

Proof. Using the conditions of Theorem 5.3 we have that the operator $M : \overset{o}{W}_{p,\nu} \to L_p(\Gamma)$ is invertible. We estimate the perturbation of M depending on ρ. Using Lemma 4.3 and the relation $M_\rho = M_0 + K_\rho$ we obtain

$$\|M - M_\rho\| = O\left(\rho^{(1-\gamma)/q}\right). \tag{5.10}$$

Let us show that the operator M_ρ is invertible for sufficiently small values ρ such that the inequality (5.6) is valid. Using the representation $M_\rho = M[I - M^{-1}(M - M_\rho)]$ and (5.10), we obtain from Banach theorem that the inverse operator $M_\rho^{-1} = [I - M^{-1}(M - M_\rho)]^{-1}M^{-1}$ exists. The following inequalities hold:

$$\left\|M_\rho^{-1}\right\| \le \frac{\|M^{-1}\|}{1-q}, \qquad \left\|M^{-1} - M_\rho^{-1}\right\| \le d_{11}\rho^{(1-\gamma)/q}\left\|M^{-1}\right\|. \tag{5.11}$$

The SLAE (3.8) of the collocation method for SIDE (3.1) for $\gamma \in (0;1)$ is equivalent to the operator equation

$$U_n M_\rho U_n x_{n,\rho} \equiv U_n M_0 U_n x_{n,\rho} + U_n \sum_{r=0}^{\nu} \left\{ \frac{1}{2\pi i} \int_\Gamma K_{r,\rho}(t,\tau) x_{n,\rho}^{(r)}(\tau) d\tau \right\} = U_n f, \tag{5.12}$$

where $K_{r,\rho}(t,\tau)$, $(r = 0,\dots,\nu)$ is defined by formula (3.5). Using the integral presentation (4.1), (5.12) is equivalent to the operator equation

$$U_n \Upsilon_\rho U_n v_{n,\rho} = U_n f, \tag{5.13}$$

where operator Υ_ρ is defined in (4.2), substituting Υ by Υ_ρ and $(h(t,\tau)/|\tau - t|^\gamma)$ by $h_\rho(t,\tau)$ (where $h_\rho(t,\tau)$ is calculated by formula (3.5)). Equation (5.13) represents the collocation method for SIE

$$\Upsilon_\rho v_\rho = f, \qquad v_\rho(t) \in L_p(\Gamma). \tag{5.14}$$

We should show that if $n(\geq n_1)$ is large enough and ρ satisfies the relation (5.6) the operator $U_n M_\rho U_n$ is invertible. The operator acts from the subspace $\overset{\circ}{X}_n = \{t^\nu \sum_{k=0}^{n} \xi_{k,\rho} t^k + \sum_{k=-n}^{-1} \xi_{k,\rho} t^k\}$ (the norm as in $\overset{\circ}{W}{}_p^{(\nu)}$) to the subspace

$$X_n = \sum_{k=-n}^{n} r_k t^k, \quad t \in \Gamma. \tag{5.15}$$

(the norm as in $L_p(\Gamma)$.)

Using formulas (4.1) the $d^\nu(Px_{n,\rho})(t)/dt^\nu$ and $d^\nu(Qx_{n,\rho})(t)/dt^\nu$ can be represented by Cauchy-type integrals with the same density $v_{n,\rho}(t)$:

$$\begin{aligned} \frac{d^\nu(Px_{n,\rho})(t)}{dt^\nu} &= \frac{1}{2\pi i}\int_\Gamma \frac{v_{n,\rho}(\tau)}{\tau - t}d\tau, \quad t \in F^+, \\ \frac{d^\nu(Qx_{n,\rho})(t)}{dt^\nu} &= \frac{t^{-\nu}}{2\pi i}\int_\Gamma \frac{v_{n,\rho}(\tau)}{\tau - t}d\tau, \quad t \in F^-. \end{aligned} \tag{5.16}$$

Using the formulas

$$(Px)^{(r)}(t) = P\left(x^{(r)}\right)(t), \qquad (Qx)^{(r)}(t) = Q\left(x^{(r)}\right)(t), \tag{5.17}$$

and relations (4.1) we obtain from (5.16)

$$v_{n,\rho}(t) = \sum_{k=0}^{n} \frac{(k+\nu)!}{k!} t^k \xi_{k,\rho} + (-1)^\nu \sum_{k=1}^{n} \frac{(k+\nu-1)!}{(k-1)!} t^{-k} \xi_{-k,\rho}. \tag{5.18}$$

We obtain from previous relation that $v_{n,\rho}(t) \in X_n, t \in \Gamma$.

The collocation method for SIE was considered in [19, 20, 26], where sufficient conditions for solvability and convergence of this method were obtained. From (5.16), Lemma 4.1, and $v_{n,\rho}(t) \in X_n$ we conclude that if function $v_{n,\rho}(t)$ is the solution of (5.13) then the function $x_{n,\rho}(t)$ is the discrete solution for the system $U_n M U_n x_{n,\rho} = U_n f$ and vice versa. We can determine the function $v_{n,\rho}(t)$ from relations (4.6):

$$\begin{aligned} (Px_{n,\rho})(t) &= \frac{(-1)^\nu}{2\pi i(\nu-1)!}\int_\Gamma v_{n,\rho}(\tau)\left[(\tau-t)^{\nu-1}\log\left(1-\frac{t}{\tau}\right) + \sum_{k=1}^{\nu-1}\tilde{\alpha}_k \tau^{\nu-k-1}t^k\right]d\tau, \\ (Qx_{n,\rho})(t) &= \frac{(-1)^\nu}{2\pi i(\nu-1)!}\int_\Gamma v_{n,\rho}(\tau)\tau^{-\nu}\left[(\tau-t)^{\nu-1}\log\left(1-\frac{\tau}{t}\right) + \sum_{k=1}^{\nu-1}\tilde{\beta}_k \tau^{\nu-k-1}t^k\right]d\tau. \end{aligned} \tag{5.19}$$

From the conditions (3), (4), and (6) of Theorem 5.3 and Lemmas 5.1 and 5.2, the invertibility of operator $\Upsilon : L_p(\Gamma) \to L_p(\Gamma)$ follows. From Banach theorem and Lemma 4.3 for small numbers ρ (ρ satisfies the relation (5.6)) we have that the operator $\Upsilon_\rho : L_p(\Gamma) \to L_p(\Gamma)$ is invertible. We should show that for (5.13) all conditions of the Theorem 1 are satisfied from [19, 20]. Theorem 1 [20] gives the convergence of the collocation method for SIE in spaces $L_p(\Gamma)$. From condition 3 of Theorem 1 [20] and from (4.3) we obtain the condition 3 of Theorem 5.3. From the equality

$$[C(t) - D(t)]^{-1}[C(t) + D(t)] = t^{\nu} B_{\nu}^{-1} A_q(t), \tag{5.20}$$

we conclude that the index of the function $[C(t) - D(t)]^{-1}[C(t) + D(t)]$ is equal to zero, which coincides with condition (4) of Theorem 5.3. Other conditions of Theorem 5.3 coincide with conditions of Theorem 1 [20]. Conditions (1)–(6) in Theorem 5.3 provide the validity of all conditions of Theorem 1 [20]. Therefore, beginning with numbers $n \geq n_1$ (5.13) is uniquely solvable for numbers ρ small enough where ρ satisfies the relation (5.6). The approximate solutions $v_{n,\rho}(t)$ of (5.13) converge to the exact solution of (4.2) in the norm of the space $L_p(\Gamma)$ as $n \to \infty$. Therefore (5.12) and the SLAE (3.10) have the unique solutions for $(n \geq n_1)$. From Theorem 1 [20] the following estimation holds:

$$\left\| v_\rho - v_{n,\rho} \right\|_p \leq O\left(\frac{1}{n^\alpha}\right) + O\left(\omega\left(f; \frac{1}{n}\right)\right) + O\left(\omega^t\left(h; \frac{1}{n}\right)\right), \tag{5.21}$$

where $O(\omega^t(h; 1/n)$ and $O(\omega(f; 1/n)$ are modulus of continuity. From (4.1) and (5.19) we obtain

$$\left(Px_\rho\right)^{(\nu)}(t) = \left(Pv_\rho\right)(t), \qquad \left(Qx_\rho\right)^{(\nu)}(t) = t^{-\nu}\left(Qv_\rho\right)(t). \tag{5.22}$$

Therefore we have

$$\left(Px_{n,\rho}\right)^{(\nu)}(t) = \left(Pv_{n,\rho}\right)(t), \qquad \left(Qx_{n,\rho}\right)^{(\nu)}(t) = t^{-\nu}\left(Qv_{n,\rho}\right)(t). \tag{5.23}$$

We proceed to get an error estimate

$$\begin{aligned} \left\| x_\rho - x_{n,\rho} \right\|_{p,\nu} &= \left\| x_\rho^{(\nu)} - x_{n,\rho}^{(\nu)} \right\|_{[L_p]} \\ &\leq \left\| P(v_\rho - v_{n,\rho}) \right\|_{[L_p]} + \left\| t^{-\nu} Q(v_\rho - v_{n,\rho}) \right\|_{[L_p]} \\ &\leq \|P\| \cdot \left\| v_\rho - v_{n,\rho} \right\|_{[L_p]} + \left\| t^{-\nu} \right\| \|Q\| \cdot \left\| v_\rho - v_{n,\rho} \right\|_{[L_p]} \\ &\leq \left(\|P\| + \left\| t^{-\nu} \right\| \|Q\|\right) \left\| v_\rho - v_{n,\rho} \right\|_{[L_p]}. \end{aligned} \tag{5.24}$$

Using the inequality

$$\begin{aligned} \left\| t^{-\nu} \right\|_{L_p} &= \left(\frac{1}{l} \int_\Gamma \left| t^{-\nu} \right|^p dt\right)^{1/p} = \left(\frac{1}{l} \int_\Gamma \left| t^{-\nu p} \right| dt\right)^{1/p} \\ &\leq \left(\frac{1}{l} \frac{1}{\min\limits_{t \in \Gamma} |t|^{p\nu}} l\right)^{1/p} = \left(\frac{1}{\min\limits_{t \in \Gamma} |t|^{p\nu}}\right)^{1/p} = c_1. \end{aligned} \tag{5.25}$$

From (5.21), (5.24), and (5.11), and from the inequality

$$\left\|x - x_{n,\rho}\right\|_{p,\nu} \leq \left\|M^{-1}f - M_{\rho}^{-1}\right\|_{p,\nu} + \left\|x_{\rho} - x_{n,\rho}\right\|_{p,\nu} \tag{5.26}$$

we obtain the relation (5.8). Thus Theorem 5.3 is proved. □

Theorem 5.4. *Let all conditions of Theorem 5.3 be satisfied. Then the SLAE* (3.10) *has a unique solution* $\xi_{k,\rho}$, $k = -n, \ldots, n$ *for numbers* $n \geq n_2 (\geq n_1)$ *large enough and for numbers* ρ *small enough* (ρ *satisfies the relation* (5.6)). *The approximate solutions* $x_{n,\rho}(t)$ *converge when* $n \to \infty$ *and* $\rho \to 0$ *in the norm* $\overset{\circ}{W}{}_p^{(\nu)}$ *to the exact solution* $x(t)$ *of the problem* (3.1)–(3.3) *and the following estimation for the convergence is true:*

$$\left\|x - x_{n,\rho}\right\|_{p,\nu} = \delta_n + O\left(\omega^{\tau}\left(h; \frac{1}{n}\right)\right). \tag{5.27}$$

Proof. It is easy to verify that SLAE (3.10) is equivalent to the operational equation

$$\begin{aligned} U_n \Bigg\{ \sum_{r=0}^{\nu} \Bigg[& A_r(t)\left(Px_{n,\rho}^{(r)}\right)(t) + B_r(t)\left(Qx_{n,\rho}^{(r)}\right)(t) \\ & + \frac{1}{2\pi i}\int_{\Gamma} \frac{1}{\tau} U_n^{(\tau)}\left[\tau^{\nu+1-r}K_{\rho}(t,\tau)\right]\left(Px_{n,\rho}^{(r)}\right)(\tau)d\tau \\ & + \frac{1}{2\pi i}\int_{\Gamma} \frac{1}{\tau} U_n^{(\tau)}\left[\tau^{-r-1}K_{\rho}(t,\tau)\right]\left(Qx_{n,\rho}^{(r)}\right)(\tau)d\tau \Bigg] \Bigg\} = U_n f, \end{aligned} \tag{5.28}$$

which after the application of integral representation (5.19) is equivalent (in the same sense of solvability) to the operator equation

$$U_n\left\{C(t)v_{n,\rho}(t) + D(t)\left(Sv_{n,\rho}\right)(t) + \frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\tau}U_n^{(\tau)}\left[\tau h_{\rho}(t,\tau)\right]\cdot v_{n,\rho}(\tau)d\tau\right\} = U_n f, \tag{5.29}$$

where the functions $C(t)$, $D(t)$, and $h_{\rho}(t,\tau)$ are determined above. The equation (5.28) represents an equation of the mechanical quadrature method for (5.14). It is easy to verify (as in the proof of Theorem 5.3), that the conditions of Theorem 5.4 provide the validity of all conditions of Theorem 2 from [19, 26] (for the mechanical quadrature method). It follows that (5.29) is uniquely solvable for $n \geq n_2$ and ρ small enough. Moreover, the approximate solutions $v_{n,\rho}(t) \in X_n$ of this equation converge to the exact solution $v_{\rho}(t)$ of SIE (4.2) in the norm $L_p(\Gamma)$ as $n \to \infty$ and the following estimation is true:

$$\left\|v_{\rho} - v_{n,\rho}\right\|_p = O\left(\frac{1}{n^{\alpha}}\right) + O\left(\omega\left(f; \frac{1}{n}\right)\right) + O\left(\omega^{\tau}\left(h; \frac{1}{n}\right)\right) + O\left(\omega^{t}\left(h; \frac{1}{n}\right)\right). \tag{5.30}$$

The function $x_{n,\rho}(t)$ can be expressed via the function $v_{n,\rho}(t)$ by formula (5.19). Using the definition of the norm in the space $L_p(\Gamma)$, and the relations (4.6), (5.30), and equality (5.26) we obtain (5.27). Theorem 5.4 is proved. □

6. Conclusion

In this paper, we have proposed the numerical schemes of the collocation method and mechanical quadrature method for solving of weakly SIDE. The equations are defined on an arbitrary smooth closed contour. The convergence of these methods was proved in Lebesgue spaces.

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