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Research Article

Periodic Wave Solutions and Their Limits for the Generalized KP-BBM Equation

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We use the bifurcation method of dynamical systems to study the periodic wave solutions and their limits for the generalized KP-BBM equation. A number of explicit periodic wave solutions are obtained. These solutions contain smooth periodic wave solutions and periodic blow-up solutions. Their limits contain periodic wave solutions, kink wave solutions, unbounded wave solutions, blow-up wave solutions, and solitary wave solutions.

1. Introduction

The Benjamin-Bona-Mahony (BBM) equation [1],

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1.1}$$

has been proposed as a model for propagation of long waves where nonlinear dispersion is incorporated.

The Kadomtsov-Petviashvili (KP) equation [2] is given by

$$u_t + a(u_x + u_{xxx})_x + u_{yy} = 0, \tag{1.2}$$

which is a weekly two-dimensional generalization of the KdV equation in the sense that it accounts for slowly varying transverse perturbations of unidirectional KdV solitons moving along the x-direction.

Wazwaz [3] presented the Kadomtsov-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation

$$\left(u_t + u_x + a(u^2)_x - bu_{xxt}\right)_x + ru_{yy} = 0, \tag{1.3}$$

and the generalized KP-BBM equation

$$\left(u_t + u_x + a(u^3)_x - bu_{xxt}\right)_x + ru_{yy} = 0. \tag{1.4}$$

Wazwaz [3, 4] obtained some solitons solution and periodic solutions of (1.3) by using the sine-cosine method and the extended tanh method. Abdou [5] used the extended mapping method with symbolic computation to obtain some periodic solutions of (1.3), solitary wave solution, and triangular wave solution. Song et al. [6] obtained exact solitary wave solutions of (1.3) by using bifurcation method of dynamical systems.

The aim of this paper is to study the traveling wave solutions and their phase portraits for (1.4) by using the bifurcation method and qualitative theory of dynamical systems [6–15]. Through some special phase orbits, we obtain a number of smooth periodic wave solutions and periodic blow-up solutions. Their limits contain periodic wave solutions, kink wave solutions, unbounded solutions, blow-up wave solutions, and solitary wave solutions.

The remainder of this paper is organized as follows. In Section 2, by using the bifurcation theory of planar dynamical systems, two phase portraits for the corresponding traveling wave system of (1.4) are given under different parameter conditions. In Section 3, we present our main results and their proofs. A short conclusion will be given in Section 4.

2. Phase Portraits

To derive our results, we give some preliminaries in this section. For given positive constant wave speed c, substituting $u = \varphi(\xi)$ with $\xi = x + y - ct$ into the generalized KP-BBM equation (1.4), it follows that

$$\left(-c\varphi' + \varphi' - a(\varphi^3)' + bc\varphi'''\right)' + r\varphi'' = 0. \tag{2.1}$$

Integrating (2.1) twice and letting the first integral constant be zero, we have

$$(r - c + 1)\varphi - a\varphi^3 + bc\varphi'' = g_1, \tag{2.2}$$

where g_1 is the second constant of integration.

Letting $\phi = \varphi'$, we get the following planar system:

$$\begin{aligned}\frac{d\varphi}{d\xi} &= \phi,\\ \frac{d\phi}{d\xi} &= \alpha\varphi^3 - \beta\varphi + g,\end{aligned} \tag{2.3}$$

where $\alpha = a/bc$, $\beta = (r + 1 - c)/bc$ and $g = g_1/bc$.

Obviously, the system (2.3) is a Hamiltonian system with Hamiltonian function

$$H(\varphi, \phi) = \phi^2 - \frac{\alpha}{2}\varphi^4 + \beta\varphi^2 - 2g\varphi = h, \tag{2.4}$$

where h is Hamiltonian.

Now we consider the phase portraits of system (2.3). Set

$$\begin{aligned} f_0(\varphi) &= \alpha\varphi^3 - \beta\varphi, \\ f(\varphi) &= \alpha\varphi^3 - \beta\varphi + g. \end{aligned} \tag{2.5}$$

Obviously, $f_0(\varphi)$ has three zero points, φ_-, φ_0 and φ_+, which are given as follows:

$$\varphi_- = -\sqrt{\frac{\beta}{\alpha}}, \qquad \varphi_0 = 0, \qquad \varphi_+ = \sqrt{\frac{\beta}{\alpha}}. \tag{2.6}$$

It is easy to obtain two extreme points of $f_0(\varphi)$ as follows:

$$\varphi_-^* = -\sqrt{\frac{\beta}{3\alpha}}, \qquad \varphi_+^* = \sqrt{\frac{\beta}{3\alpha}}. \tag{2.7}$$

Letting

$$g_0 = \left| f_0(\varphi_-^*) \right| = \left| f_0(\varphi_+^*) \right| = \frac{2|\beta|}{3}\sqrt{\frac{\beta}{3\alpha}}, \tag{2.8}$$

then it is easily seen that g_0 is the extreme values of $f_0(\varphi)$.

Let $(\varphi_i, 0)$ be one of the singular points of system (2.3), then the characteristic values of the linearized system of system (2.3) at the singular points $(\varphi_i, 0)$ are

$$\lambda_\pm = \pm\sqrt{f'(\varphi_i)}. \tag{2.9}$$

From the qualitative theory of dynamical systems, we therefore know that

(i) if $f'(\varphi_i) > 0$, $(\varphi_i, 0)$ is a saddle point.

(ii) if $f'(\varphi_i) < 0$, $(\varphi_i, 0)$ is a center point.

(iii) if $f'(\varphi_i) = 0$, $(\varphi_i, 0)$ is a degenerate saddle point.

Therefore, we obtain the phase portraits of system (2.3) in Figures 1 and 2.

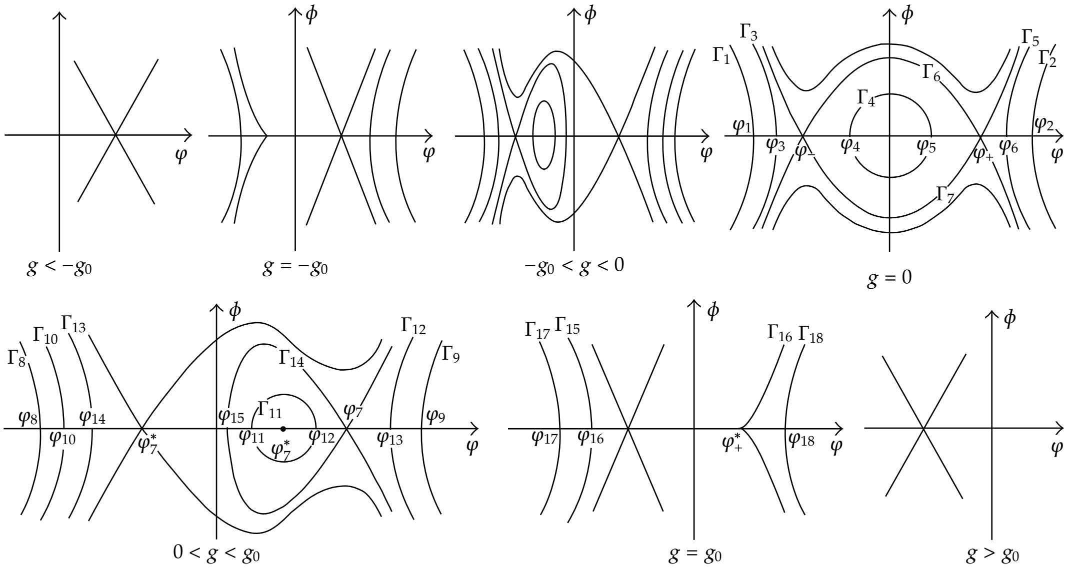

Figure 1: The phase portraits of system (2.3) when $\alpha > 0$, $\beta > 0$.

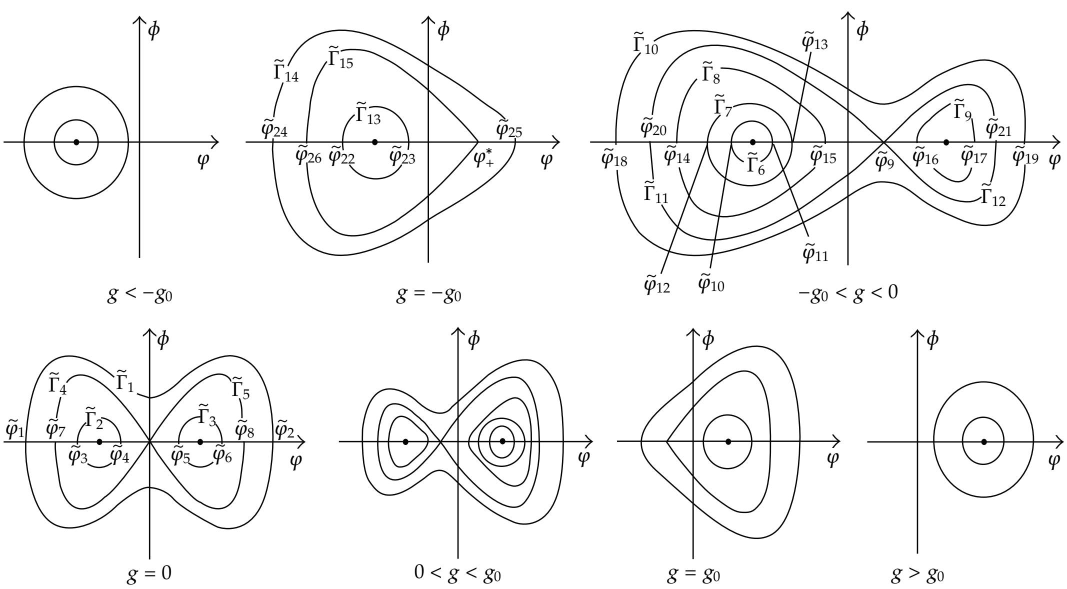

Figure 2: The phase portraits of system (2.3) when $\alpha < 0$, $\beta < 0$.

3. Main Results and Their Proofs

In this section, we state our main results.

Proposition 3.1. *For given positive constants* c *and* g_0, (1.4) *has the following periodic wave solutions when* $\alpha > 0$ *and* $\beta > 0$.

(1) *If* $g = 0$, *we get eight periodic blow-up wave solutions*

$$u_{1\pm}(x,y,t) = \pm\sqrt{\frac{2\beta}{\alpha}}\csc\sqrt{\beta}\xi, \tag{3.1}$$

$$u_{2\pm}(x,y,t) = \pm\sqrt{\frac{2\beta}{\alpha}}\sec\sqrt{\beta}\xi, \tag{3.2}$$

$$u_{3\pm}(x,y,t) = \pm\frac{\varphi_6}{\operatorname{sn}\left(\varphi_6\sqrt{\alpha/2}\xi, \varphi_5/\varphi_6\right)}, \tag{3.3}$$

$$u_{4\pm}(x,y,t) = \pm\sqrt{\frac{\varphi_6^2 - \varphi_5^2\left(\operatorname{sn}\left(\varphi_6\sqrt{\alpha/2}\xi, \varphi_5/\varphi_6\right)\right)^2}{1 - \left(\operatorname{sn}\left(\varphi_6\sqrt{\alpha/2}\xi, \varphi_5/\varphi_6\right)\right)^2}}, \tag{3.4}$$

two periodic wave solutions

$$u_{5\pm}(x,y,t) = \pm\varphi_5\operatorname{sn}\left(\varphi_6\sqrt{\frac{\alpha}{2}}\xi, \frac{\varphi_5}{\varphi_6}\right), \tag{3.5}$$

two kink wave solutions

$$u_{6\pm}(x,y,t) = \pm\sqrt{\frac{\beta}{\alpha}}\tanh\sqrt{\frac{\beta}{2}}\xi, \tag{3.6}$$

and two unbounded wave solutions

$$u_{7\pm}(x,y,t) = \pm\sqrt{\frac{\beta}{\alpha}}\coth\sqrt{\frac{\beta}{2}}\xi. \tag{3.7}$$

(2) *If* $0 < g < g_0$, *we get four periodic blow-up wave solutions*

$$u_{8\pm}(x,y,t) = \varphi_7^* - \frac{2\gamma_1}{\delta_1 \pm \sqrt{\eta_1}\cos\sqrt{\alpha\gamma_1/2}\xi}, \tag{3.8}$$

$$\begin{aligned}
u_9(x,y,t) &= \frac{\varphi_{10}(\varphi_{11}-\varphi_{13}) + \varphi_{11}(\varphi_{13}-\varphi_{10})(\operatorname{sn}(\omega_1\xi,k_1))^2}{\varphi_{11}-\varphi_{13} + (\varphi_{13}-\varphi_{10})(\operatorname{sn}(\omega_1\xi,k_1))^2}, \\
u_{10}(x,y,t) &= \frac{\varphi_{13}(\varphi_{12}-\varphi_{10}) - \varphi_{12}(\varphi_{13}-\varphi_{10})(\operatorname{sn}(\omega_1\xi,k_1))^2}{\varphi_{12}-\varphi_{10} - (\varphi_{13}-\varphi_{10})(\operatorname{sn}(\omega_1\xi,k_1))^2},
\end{aligned} \tag{3.9}$$

where

$$\varphi_7^* = \frac{1}{2}\left(-\varphi_7 + \sqrt{\frac{4\beta}{\alpha} - 3\varphi_7^2}\right), \qquad \gamma_1 = \frac{-4\beta + 3\varphi_7\left(\alpha\varphi_7 + \sqrt{4\alpha\beta - 3\alpha^2\varphi_7^2}\right)}{\alpha},$$
$$\delta_1 = -2\varphi_7 + \frac{2\sqrt{4\alpha\beta - 3\alpha^2\varphi_7^2}}{\alpha}, \qquad \eta_1 = \frac{4\varphi_7\left(\alpha\varphi_7 + \sqrt{4\alpha\beta - 3\alpha^2\varphi_7^2}\right)}{\alpha}, \tag{3.10}$$
$$\omega_1 = \frac{\sqrt{\alpha(\varphi_{13} - \varphi_{11})(\varphi_{12} - \varphi_{10})}}{2\sqrt{2}}, \qquad k_1 = \sqrt{\frac{(\varphi_{12} - \varphi_{11})(\varphi_{13} - \varphi_{10})}{(\varphi_{13} - \varphi_{11})(\varphi_{12} - \varphi_{10})}},$$

a periodic wave solution

$$u_{11}(x, y, t) = \frac{\varphi_{11}(\varphi_{10} - \varphi_{12}) + \varphi_{10}(\varphi_{12} - \varphi_{11})(\mathrm{sn}(\omega_1\xi, k_1))^2}{\varphi_{10} - \varphi_{12} + (\varphi_{12} - \varphi_{11})(\mathrm{sn}(\omega_1\xi, k_1))^2}, \tag{3.11}$$

a blow-up wave solution

$$u_{12}(x, y, t) = \varphi_7 + \frac{2(\beta - 3\alpha\varphi_7^2)}{2\alpha\varphi_7 - \sqrt{2\alpha(\beta - \alpha\varphi_7^2)}\cosh\sqrt{3\alpha\varphi_7^2 - \beta}\,\xi}, \tag{3.12}$$

a solitary wave solution

$$u_{13}(x, y, t) = \varphi_7 + \frac{2(\beta - 3\alpha\varphi_7^2)}{2\alpha\varphi_7 + \sqrt{2\alpha(\beta - \alpha\varphi_7^2)}\cosh\sqrt{3\alpha\varphi_7^2 - \beta}\,\xi}, \tag{3.13}$$

and an unbounded wave solution

$$u_{14}(x, y, t) = \varphi_7 - \frac{2(\beta - 3\alpha\varphi_7^2)\,\mathrm{csch}\sqrt{3\alpha\varphi_7^2 - \beta}\,\xi}{\sqrt{-2\alpha\beta + 6\alpha^2\varphi_7^2} + 2\alpha\varphi_7\tanh\left(\sqrt{3\alpha\varphi_7^2 - \beta}\,\xi/2\right)}. \tag{3.14}$$

(3) *If $g = g_0$, we get three blow-up wave solutions*

$$u_{15}(x, y, t) = \frac{18\sqrt{2} - \beta\sqrt{3\beta}\xi^3}{\sqrt{\alpha}(18\xi - 3\beta\xi^3)},$$
$$u_{16}(x, y, t) = \frac{18\sqrt{2} + \beta\sqrt{3\beta}\xi^3}{\sqrt{\alpha}(-18\xi + 3\beta\xi^3)}, \tag{3.15}$$
$$u_{17}(x, y, t) = \frac{\sqrt{3\beta}(9 + 2\beta\xi^2)}{\sqrt{\alpha}(-9 + 6\beta\xi^2)},$$

and a periodic blow-up wave solution

$$u_{18}(x,y,t) = \frac{-A_1\varphi_{17} + B_1\varphi_{18} + (A_1\varphi_{17} + B_1\varphi_{18})\,\mathrm{cn}\left(\sqrt{\alpha A_1 B_1/2}\xi, k_2\right)}{-A_1 + B_1 + (A_1 + B_1)\,\mathrm{cn}\left(\sqrt{\alpha A_1 B_1/2}\xi, k_2\right)}, \tag{3.16}$$

where

$$A_1 = \sqrt{\left(\varphi_{18} - \frac{c_1 + \overline{c}_1}{2}\right)^2 - \frac{(c_1 - \overline{c}_1)^2}{4}}, \qquad B_1 = \sqrt{\left(\varphi_{17} - \frac{c_1 + \overline{c}_1}{2}\right)^2 - \frac{(c_1 - \overline{c}_1)^2}{4}},$$
$$k_2 = \sqrt{\frac{(A_1 + B_1)^2 - (\varphi_{18} - \varphi_{17})^2}{4A_1B_1}}, \tag{3.17}$$

c_1 and $\overline{c}_1$ are conjugate complex numbers.

Proof. (1) If $g = 0$, we will consider three kinds of orbits.

(i) From the phase portrait, we note that there are two special orbits Γ_1 and Γ_2, which have the same Hamiltonian with that of the center point $(0,0)$. In (φ,ϕ) plane, the expressions of these two orbits are given as

$$\phi = \pm\sqrt{\frac{\alpha}{2}\varphi^2(\varphi - \varphi_1)(\varphi - \varphi_2)}, \tag{3.18}$$

where $\varphi_1 = -\sqrt{2\beta/\alpha}$ and $\varphi_2 = \sqrt{2\beta/\alpha}$.

Substituting (3.18) into $d\varphi/d\xi = \phi$ and integrating them along the two orbits Γ_1 and Γ_2, it follows that

$$\pm\int_{\varphi}^{\infty} \frac{1}{s\sqrt{(s-\varphi_1)(s-\varphi_2)}}ds = \sqrt{\frac{\alpha}{2}}\int_0^{\xi} ds,$$
$$\pm\int_{\varphi_2}^{\varphi} \frac{1}{s\sqrt{(s-\varphi_1)(s-\varphi_2)}}ds = \sqrt{\frac{\alpha}{2}}\int_0^{\xi} ds. \tag{3.19}$$

From (3.19) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get four periodic blow-up solutions $u_{1\pm}(x,y,t)$ and $u_{2\pm}(x,y,t)$ as (3.1) and (3.2).

(ii) From the phase portrait, we note that there are three special orbits Γ_3, Γ_4, and Γ_5 passing the points $(\varphi_3,0)$, $(\varphi_4,0)$, $(\varphi_5,0)$, and $(\varphi_6,0)$. In (φ,ϕ) plane, the expressions of the orbit are given as

$$\phi = \pm\sqrt{\frac{\alpha}{2}(\varphi - \varphi_3)(\varphi - \varphi_4)(\varphi - \varphi_5)(\varphi - \varphi_6)}, \tag{3.20}$$

where $\varphi_3 = -\varphi_6$, $\varphi_4 = -\sqrt{2\beta/\alpha - \varphi_6^2}$, $\varphi_5 = \sqrt{2\beta/\alpha - \varphi_6^2}$ and $\sqrt{\beta/\alpha} < \varphi_6 < \sqrt{2\beta/\alpha}$.

Substituting (3.20) into $d\varphi/d\xi = \phi$ and integrating them along Γ_3, Γ_4, and Γ_5, we have

$$\begin{aligned}
&\pm\int_{\varphi}^{\infty} \frac{1}{\sqrt{(s-\varphi_3)(s-\varphi_4)(s-\varphi_5)(s-\varphi_6)}}ds = \sqrt{\frac{\alpha}{2}}\int_0^{\xi} ds,\\
&\pm\int_{\varphi_6}^{\varphi} \frac{1}{\sqrt{(s-\varphi_3)(s-\varphi_4)(s-\varphi_5)(s-\varphi_6)}}ds = \sqrt{\frac{\alpha}{2}}\int_0^{\xi} ds,\\
&\pm\int_{0}^{\varphi} \frac{1}{\sqrt{(s-\varphi_3)(s-\varphi_4)(s-\varphi_5)(s-\varphi_6)}}ds = \sqrt{\frac{\alpha}{2}}\int_0^{\xi} ds.
\end{aligned} \tag{3.21}$$

From (3.21) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get four periodic blow-up wave solutions $u_{3\pm}(x, y, t)$, $u_{4\pm}(x, y, t)$ as (3.3), (3.4) and two periodic solutions $u_{5\pm}(x, y, t)$ as (3.5).

(iii) From the phase portrait, we see that there are two heteroclinic orbits Γ_6 and Γ_7 connected at saddle points $(\varphi_-, 0)$ and $(\varphi_+, 0)$. In (φ, ϕ) plane, the expressions of the heteroclinic orbits are given as

$$\phi = \pm\sqrt{\frac{\alpha}{2}(\varphi-\varphi_-)^2(\varphi-\varphi_+)^2}. \tag{3.22}$$

Substituting (3.22) into $d\varphi/d\xi = \phi$ and integrating them along the heteroclinic orbits Γ_6 and Γ_7, it follows that

$$\begin{aligned}
&\pm\int_0^{\varphi} \frac{1}{(s-\varphi_-)(\varphi_+-s)}ds = \sqrt{\frac{\alpha}{2}}\int_0^{\xi} ds,\\
&\pm\int_{\varphi}^{\infty} \frac{1}{(s-\varphi_-)(\varphi_+-s)}ds = \sqrt{\frac{\alpha}{2}}\int_0^{\xi} ds.
\end{aligned} \tag{3.23}$$

From (3.23) and noting that $u = \varphi(\xi)$ and $\xi = x+y-ct$, we get two kink wave solutions $u_{6\pm}(x, y, t)$ as (3.6) and two unbounded solutions $u_{7\pm}(x, y, t)$ as (3.7).

(2) If $0 < g < g_0$, we set the largest solution of $f(\varphi) = 0$ be $\varphi_7(\sqrt{\beta/3\alpha} < \varphi_7 < \sqrt{\beta/\alpha})$, then we can get another two solutions of $f(\varphi) = 0$ as follows:

$$\begin{aligned}
\varphi_7^{*} &= \frac{1}{2}\left(-\varphi_7 + \sqrt{\frac{4\beta}{\alpha} - 3\varphi_7^2}\right),\\
\varphi_7^{\star} &= \frac{1}{2}\left(-\varphi_7 - \sqrt{\frac{4\beta}{\alpha} - 3\varphi_7^2}\right).
\end{aligned} \tag{3.24}$$

(i) From the phase portrait, we see that there are two special orbits Γ_8 and Γ_9, which have the same Hamiltonian with that of the center point $(\varphi_7^*, 0)$. In (φ, ϕ)-plane, the expressions of the orbits are given as

$$\phi = \pm\sqrt{\frac{\alpha}{2}(\varphi - \varphi_7^*)^2(\varphi - \varphi_8)(\varphi - \varphi_9)}, \tag{3.25}$$

where

$$\begin{aligned}
\varphi_8 &= \frac{1}{2}\left(\varphi_7 - \sqrt{\frac{4\beta}{\alpha} - 3\varphi_7^2} - 2\sqrt{\varphi_7^2 + \varphi_7\sqrt{\frac{4\beta}{\alpha} - 3\varphi_7^2}}\right), \\
\varphi_9 &= \frac{1}{2}\left(\varphi_7 - \sqrt{\frac{4\beta}{\alpha} - 3\varphi_7^2} + 2\sqrt{\varphi_7^2 + \varphi_7\sqrt{\frac{4\beta}{\alpha} - 3\varphi_7^2}}\right).
\end{aligned} \tag{3.26}$$

Substituting (3.25) into $d\varphi/d\xi = \phi$ and integrating them along the orbits, it follows that

$$\begin{aligned}
\pm\int_{\varphi_8}^{\varphi} \frac{1}{\sqrt{(s - \varphi_7^*)^2(s - \varphi_8)(s - \varphi_9)}}\,ds &= \sqrt{\frac{\alpha}{2}}\int_0^{\xi} ds, \\
\pm\int_{\varphi_9}^{\varphi} \frac{1}{\sqrt{(s - \varphi_7^*)^2(s - \varphi_8)(s - \varphi_9)}}\,ds &= \sqrt{\frac{\alpha}{2}}\int_0^{\xi} ds.
\end{aligned} \tag{3.27}$$

From (3.27) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get two periodic blow-up wave solutions $u_{8\pm}(x, y, t)$ as (3.8).

(ii) From the phase portrait, we note that there are three special orbits Γ_{10}, Γ_{11}, and Γ_{12} passing the points $(\varphi_{10}, 0)$, $(\varphi_{11}, 0)$, $(\varphi_{12}, 0)$, and $(\varphi_{13}, 0)$. In (φ, ϕ) plane, the expressions of the orbit are given as

$$\phi = \pm\sqrt{\frac{\alpha}{2}(\varphi - \varphi_{10})(\varphi - \varphi_{11})(\varphi - \varphi_{12})(\varphi - \varphi_{13})}, \tag{3.28}$$

where $\varphi_8 < \varphi_{10} < \varphi_{14} < \varphi_7^* < \varphi_{15} < \varphi_{11} < \varphi_{12} < \varphi_7 < \varphi_{13} < \varphi_9$.

Substituting (3.28) into $d\varphi/d\xi = \phi$ and integrating them along Γ_{10}, Γ_{11}, and Γ_{12}, we have

$$\int_{\varphi}^{\varphi_{10}} \frac{1}{\sqrt{(\varphi_{13}-s)(\varphi_{12}-s)(\varphi_{11}-s)(\varphi_{10}-s)}} ds = \sqrt{\frac{\alpha}{2}} \int_0^{\xi} ds,$$
$$\int_{\varphi_{13}}^{\varphi} \frac{1}{\sqrt{(s-\varphi_{13})(s-\varphi_{12})(s-\varphi_{11})(s-\varphi_{10})}} ds = \sqrt{\frac{\alpha}{2}} \int_0^{\xi} ds, \tag{3.29}$$
$$\int_{\varphi_{11}}^{\varphi} \frac{1}{\sqrt{(\varphi_{13}-s)(\varphi_{12}-s)(s-\varphi_{11})(s-\varphi_{10})}} ds = \sqrt{\frac{\alpha}{2}} \int_0^{\xi} ds.$$

From (3.29) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get two periodic blow-up wave solutions $u_9(x, y, t)$, $u_{10}(x, y, t)$ as (3.9) and a periodic wave solution $u_{11}(x, y, t)$ as (3.11).

(iii) From the phase portrait, we see that there are a spacial orbit Γ_{13}, which passes the point $(\varphi_{14}, 0)$, and a homoclinic orbit Γ_{14} passing the saddle point $(\varphi_7, 0)$. In (φ, ϕ) plane, the expressions of the orbits are given as

$$\phi = \pm\sqrt{\frac{\alpha}{2}(\varphi-\varphi_7)^2(\varphi-\varphi_{14})(\varphi-\varphi_{15})}, \tag{3.30}$$

where

$$\varphi_{14} = \frac{-\alpha\varphi_7 - \sqrt{2\alpha\beta - 2\alpha^2\varphi_7^2}}{\alpha},$$
$$\varphi_{15} = \frac{-\alpha\varphi_7 + \sqrt{2\alpha\beta - 2\alpha^2\varphi_7^2}}{\alpha}. \tag{3.31}$$

Substituting (3.30) into $d\varphi/d\xi = \phi$ and integrating them along the orbits, it follows that

$$\pm\int_{\varphi_{14}}^{\varphi} \frac{1}{\sqrt{(s-\varphi_7)^2(s-\varphi_{14})(s-\varphi_{15})}} ds = \sqrt{\frac{\alpha}{2}} \int_0^{\xi} ds,$$
$$\pm\int_{\varphi_{15}}^{\varphi} \frac{1}{\sqrt{(s-\varphi_7)^2(s-\varphi_{14})(s-\varphi_{15})}} ds = \sqrt{\frac{\alpha}{2}} \int_0^{\xi} ds, \tag{3.32}$$
$$\pm\int_{\varphi}^{\infty} \frac{1}{\sqrt{(s-\varphi_7)^2(s-\varphi_{15})(s-\varphi_{15})}} ds = \sqrt{\frac{\alpha}{2}} \int_0^{\xi} ds.$$

From (3.32) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get a blow-up solution $u_{12}(x, y, t)$ as (3.12), a solitary wave solution $u_{13}(x, y, t)$ as (3.13), and an unbounded wave solution $u_{14}(x, y, t)$ as (3.14).

(3) If $g = g_0$, we will consider two kinds of orbits.

(i) From the phase portrait, we see that there are two orbits Γ_{15} and Γ_{16}, which have the same Hamiltonian with the degenerate saddle point $(\varphi_+^*, 0)$. In (φ, ϕ) plane, the expressions of these two orbits are given as

$$\phi = \pm\sqrt{\frac{\alpha}{2}(\varphi - \varphi_+^*)^3(\varphi - \varphi_{16})}, \tag{3.33}$$

where

$$\varphi_{16} = -\sqrt{\frac{3\beta}{\alpha}}. \tag{3.34}$$

Substituting (3.33) into $d\varphi/d\xi = \phi$ and integrating them along these two orbits Γ_{15} and Γ_{16}, it follows that

$$\begin{aligned} \pm\int_{\varphi}^{+\infty} \frac{1}{\sqrt{(s - \varphi_+^*)^3(s - \varphi_{16})}} ds &= \sqrt{\frac{\alpha}{2}} \int_0^{\xi} ds, \\ \pm\int_{\varphi_{16}}^{\varphi} \frac{1}{\sqrt{(s - \varphi_+^*)^3(s - \varphi_{16})}} ds &= \sqrt{\frac{\alpha}{2}} \int_0^{\xi} ds. \end{aligned} \tag{3.35}$$

From (3.35) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get three blow-up solutions $u_{15}(x, y, t)$, $u_{16}(x, y, t)$, and $u_{17}(x, y, t)$ as (3.15).

(ii) From the phase portrait, we see that there are two special orbits Γ_{17} and Γ_{18} passing the points $(\varphi_{17}, 0)$ and $(\varphi_{18}, 0)$. In (φ, ϕ) plane, the expressions of the orbits are given as

$$\phi = \pm\sqrt{\frac{\alpha}{2}(\varphi - \varphi_{18})(\varphi - \varphi_{17})(\varphi - c_1)(\varphi - \overline{c}_1)}, \tag{3.36}$$

where $\varphi_{17} < \varphi_{16} < \varphi_+^* < \varphi_{18}$, c_1 and $\overline{c}_1$ are conjugate complex numbers.

Substituting (3.36) into $d\varphi/d\xi = \phi$ and integrating them along Γ_{17} and Γ_{18}, we have

$$\pm\int_{\varphi_{18}}^{\varphi} \frac{1}{\sqrt{(s - \varphi_{18})(s - \varphi_{17})(s - c_1)(s - \overline{c}_1)}} ds = \sqrt{\frac{\alpha}{2}} \int_0^{\xi} ds. \tag{3.37}$$

From (3.37) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get a periodic blow-up wave solutions $u_{18}(x, y, t)$ as (3.16).

Thus, we obtain the results given in Proposition 3.1. □

Proposition 3.2. *For given positive constants c and g_0,* (1.4) *has the following periodic wave solution when $\alpha < 0$ and $\beta < 0$.*

(1) *If $g = 0$, we get four periodic wave solutions*

$$u_{19}(x,y,t) = \widetilde{\varphi}_1 \operatorname{cn}\left(\sqrt{\beta - \alpha\widetilde{\varphi}_1^2}\xi, -\widetilde{\varphi}_1\sqrt{\frac{\alpha}{-2\beta + 2\alpha\varphi_1^2}}\right), \quad (3.38)$$

$$u_{20}(x,y,t) = \widetilde{\varphi}_2 \operatorname{cn}\left(\sqrt{\beta - \alpha\widetilde{\varphi}_2^2}\xi, \widetilde{\varphi}_2\sqrt{\frac{\alpha}{-2\beta + 2\alpha\varphi_2^2}}\right), \quad (3.39)$$

$$u_{21}(x,y,t) = \frac{-\kappa_1\widetilde{\varphi}_6 + \lambda_1\widetilde{\varphi}_6 \operatorname{sn}^2(\omega_2\xi, k_3)}{\kappa_1 + \lambda_1 \operatorname{sn}^2(\omega_2\xi, k_3)}, \quad (3.40)$$

$$u_{22}(x,y,t) = \sqrt{\widetilde{\varphi}_6^2 - \left(2\widetilde{\varphi}_6^2 - \frac{2\beta}{\alpha}\right)\operatorname{sn}^2\left(\widetilde{\varphi}_6\sqrt{-\frac{\alpha}{2}}\xi, k_4\right)}, \quad (3.41)$$

where

$$\kappa_1 = \widetilde{\varphi}_6 + \sqrt{\frac{2\beta}{\alpha} - \widetilde{\varphi}_6^2}, \qquad \lambda_1 = \widetilde{\varphi}_6 - \sqrt{\frac{2\beta}{\alpha} - \widetilde{\varphi}_6^2},$$
$$\omega_2 = \frac{\sqrt{-\alpha}\widetilde{\varphi}_6 + \sqrt{-2\beta + \alpha\widetilde{\varphi}_6^2}}{2\sqrt{2}}, \qquad k_3 = \frac{\lambda_1}{\kappa_1}, \qquad k_4 = \frac{\sqrt{2\widetilde{\varphi}_6^2 - 2\beta/\alpha}}{\widetilde{\varphi}_6}, \quad (3.42)$$

and two solitary wave solutions

$$u_{23\pm}(x,y,t) = \pm\sqrt{\frac{2\beta}{\alpha}}\operatorname{sech}\sqrt{-\beta}\xi. \quad (3.43)$$

(2) *If $-g_0 < g < 0$, we get six periodic wave solutions*

$$u_{24}(x,y,t) = \frac{A_2\widetilde{\varphi}_{10} + \widetilde{\varphi}_{11}B_2 + (A_2\widetilde{\varphi}_{10} - \widetilde{\varphi}_{11}B_2)\operatorname{cn}\left(\sqrt{-\alpha A_2B_2/2}\xi, k_5\right)}{A_2 + B_2 + (A_2 - B_2)\operatorname{cn}\left(\sqrt{-\alpha A_2B_2/2}\xi, k_5\right)}, \quad (3.44)$$

$$u_{25}(x,y,t) = \widetilde{\varphi}_9^* + \frac{2\theta}{-\mu + \sqrt{q}\cos\sqrt{(\alpha\theta/2)}\xi}, \quad (3.45)$$

$$u_{26}(x,y,t) = \frac{\widetilde{\varphi}_{14}(\widetilde{\varphi}_{17} - \widetilde{\varphi}_{15}) + \widetilde{\varphi}_{17}(\widetilde{\varphi}_{15} - \widetilde{\varphi}_{14})\operatorname{sn}^2(\omega_3\xi, k_6)}{\widetilde{\varphi}_{17} - \widetilde{\varphi}_{15} + (\widetilde{\varphi}_{15} - \widetilde{\varphi}_{14})\operatorname{sn}^2(\omega_3\xi, k_6)}, \quad (3.46)$$

$$u_{27}(x,y,t) = \frac{\widetilde{\varphi}_{17}(-\widetilde{\varphi}_{16} + \widetilde{\varphi}_{14}) - \widetilde{\varphi}_{14}(\widetilde{\varphi}_{17} - \widetilde{\varphi}_{16})\operatorname{sn}^2(\omega_3\xi, k_6)}{-\widetilde{\varphi}_{16} + \widetilde{\varphi}_{14} - (\widetilde{\varphi}_{17} - \widetilde{\varphi}_{16})\operatorname{sn}^2(\omega_3\xi, k_6)}, \quad (3.47)$$

$$u_{28}(x,y,t) = \frac{A_3\tilde{\varphi}_{18} + \tilde{\varphi}_{19}B_3 + (A_3\tilde{\varphi}_{18} - \tilde{\varphi}_{19}B_3)\,\mathrm{cn}\left(\sqrt{-\alpha A_3 B_3/2}\xi, k_7\right)}{A_3 + B_3 + (A_3 - B_3)\,\mathrm{cn}\left(\sqrt{-\alpha A_3 B_3/2}\xi, k\right)}, \tag{3.48}$$

$$u_{29}(x,y,t) = \frac{A_3\tilde{\varphi}_{18} + \tilde{\varphi}_{19}B_3 - (A_3\tilde{\varphi}_{18} - \tilde{\varphi}_{19}B_3)\,\mathrm{cn}\left(\sqrt{-\alpha A_3 B_3/2}\xi, k_7\right)}{A_3 + B_3 - (A_3 - B_3)\,\mathrm{cn}\left(\sqrt{-\alpha A_3 B_3/2}\xi, k_7\right)}, \tag{3.49}$$

where

$$\begin{gathered}
A_2 = \sqrt{\left(\tilde{\varphi}_{11} - \frac{c_3 + \overline{c}_3}{2}\right)^2 - \frac{(c_3 - \overline{c}_3)^2}{4}}, \qquad B_2 = \sqrt{\left(\tilde{\varphi}_{10} - \frac{c_3 + \overline{c}_3}{2}\right)^2 - \frac{(c_3 - \overline{c}_3)^2}{4}}, \\
A_3 = \sqrt{\left(\tilde{\varphi}_{19} - \frac{c_4 + \overline{c}_4}{2}\right)^2 - \frac{(c_4 - \overline{c}_4)^2}{4}}, \qquad B_3 = \sqrt{\left(\tilde{\varphi}_{18} - \frac{c_4 + \overline{c}_4}{2}\right)^2 - \frac{(c_4 - \overline{c}_4)^2}{4}}, \\
k_5 = \sqrt{\frac{(\tilde{\varphi}_{11} - \tilde{\varphi}_{10})^2 - (A_2 - B_2)^2}{4A_2B_2}}, \qquad k_6 = \sqrt{\frac{(\tilde{\varphi}_{17} - \tilde{\varphi}_{16})(\tilde{\varphi}_{15} - \tilde{\varphi}_{14})}{(\tilde{\varphi}_{17} - \tilde{\varphi}_{15})(\tilde{\varphi}_{16} - \tilde{\varphi}_{14})}}, \\
\omega_3 = \frac{\sqrt{-\alpha(\tilde{\varphi}_{17} - \tilde{\varphi}_{15})(\tilde{\varphi}_{16} - \tilde{\varphi}_{14})}}{2\sqrt{2}}, \qquad k_7 = \sqrt{\frac{(\tilde{\varphi}_{19} - \tilde{\varphi}_{18})^2 - (A_3 - B_3)^2}{4A_3B_3}}, \\
\tilde{\varphi}_9^* = \frac{-\alpha\tilde{\varphi}_9 - \sqrt{4\alpha\beta - 3\alpha^2\tilde{\varphi}_9^2}}{2\alpha}, \qquad \theta = -\frac{4\beta}{\alpha} + 3\tilde{\varphi}_9^2 + 3\tilde{\varphi}_9\sqrt{\frac{4\beta}{\alpha} - 3\tilde{\varphi}_9^2}, \\
\mu = 2\tilde{\varphi}_9 - 2\sqrt{\frac{4\beta}{\alpha} - 3\tilde{\varphi}_9^2}, \qquad q = 4\tilde{\varphi}_9^2 + 4\tilde{\varphi}_9\sqrt{\frac{4\beta}{\alpha} - 3\tilde{\varphi}_9^2},
\end{gathered} \tag{3.50}$$

c_3, $\overline{c}_3$, c_4 *and* $\overline{c}_4$ *are complex numbers.*

And two solitary wave solutions

$$u_{30}(x,y,t) = \tilde{\varphi}_9 + \frac{2\beta - 6\alpha\tilde{\varphi}_9^2}{2\alpha\tilde{\varphi}_9 + \sqrt{2\alpha\beta - 2\alpha^2\tilde{\varphi}_9^2}\cosh\sqrt{-\beta + 3\alpha\tilde{\varphi}_9^2}\xi}, \tag{3.51}$$

$$u_{31}(x,y,t) = \tilde{\varphi}_9 + \frac{2\beta - 6\alpha\tilde{\varphi}_9^2}{2\alpha\tilde{\varphi}_9 - \sqrt{2\alpha\beta - 2\alpha^2\tilde{\varphi}_9^2}\cosh\sqrt{-\beta + 3\alpha\tilde{\varphi}_9^2}\xi}. \tag{3.52}$$

(3) *If* $g = -g_0$, *we get two periodic wave solutions as follows:*

$$u_{32}(x,y,t) = \frac{A_4\tilde{\varphi}_{22} + \tilde{\varphi}_{23}B_4 + (A_4\tilde{\varphi}_{22} - \tilde{\varphi}_{23}B_4)\,\mathrm{cn}\left(\sqrt{-\alpha A_4 B_4/2}\xi, k_8\right)}{A_4 + B_4 + (A_4 - B_4)\,\mathrm{cn}\left(\sqrt{-\alpha A_4 B_4/2}\xi, k_8\right)}, \tag{3.53}$$

$$u_{33}(x,y,t) = \frac{A_5\widetilde{\varphi}_{24} + \widetilde{\varphi}_{25}B_5 + (A_5\widetilde{\varphi}_{24} - \widetilde{\varphi}_{25}B_5)\operatorname{cn}\left(\sqrt{-\alpha A_5 B_5/2}\xi, k_9\right)}{A_5 + B_5 + (A_5 - B_5)\operatorname{cn}\left(\sqrt{-\alpha A_5 B_5/2}\xi, k_9\right)}, \tag{3.54}$$

where

$$\begin{gathered}
A_4 = \sqrt{\left(\widetilde{\varphi}_{23} - \frac{c_5 + \overline{c}_5}{2}\right)^2 - \frac{(c_5 - \overline{c}_5)^2}{4}}, \qquad B_4 = \sqrt{\left(\widetilde{\varphi}_{22} - \frac{c_5 + \overline{c}_5}{2}\right)^2 - \frac{(c_5 - \overline{c}_5)^2}{4}}, \\
A_5 = \sqrt{\left(\widetilde{\varphi}_{25} - \frac{c_6 + \overline{c}_6}{2}\right)^2 - \frac{(c_6 - \overline{c}_6)^2}{4}}, \qquad B_6 = \sqrt{\left(\widetilde{\varphi}_{24} - \frac{c_6 + \overline{c}_6}{2}\right)^2 - \frac{(c_6 - \overline{c}_6)^2}{4}}, \\
k_8 = \sqrt{\frac{(\widetilde{\varphi}_{23} - \widetilde{\varphi}_{22})^2 - (A_4 - B_4)^2}{4A_4B_4}}, \qquad k_9 = \sqrt{\frac{(\widetilde{\varphi}_{25} - \widetilde{\varphi}_{24})^2 - (A_5 - B_5)^2}{4A_5B_5}},
\end{gathered} \tag{3.55}$$

c_5, $\overline{c}_5$, c_6 *and* $\overline{c}_6$ *are complex numbers.*

And a solitary wave solution

$$u_{34}(x,y,t) = \sqrt{\frac{3\beta}{\alpha}\frac{9 + 2\beta\xi^2}{-9 + 6\beta\xi^2}}. \tag{3.56}$$

Proof. (1) If $g = 0$, we set

$$\widetilde{\varphi}_2 > \sqrt{\frac{2\beta}{\alpha}}, \qquad \sqrt{\frac{\beta}{\alpha}} < \widetilde{\varphi}_6 < \sqrt{\frac{2\beta}{\alpha}}. \tag{3.57}$$

(i) From the phase portrait, we see that there are a closed orbit $\widetilde{\Gamma}_1$ passing the points $(\widetilde{\varphi}_1, 0)$ and $(\widetilde{\varphi}_2, 0)$. In (φ, ϕ) plane, the expressions of the closed orbits are given as

$$\phi = \pm\sqrt{-\frac{\alpha}{2}(\widetilde{\varphi}_2 - \varphi)(\varphi - \widetilde{\varphi}_1)(\varphi - c_2)(\varphi - \overline{c}_2)}, \tag{3.58}$$

where $\widetilde{\varphi}_1 = -\widetilde{\varphi}_2$, $c_2 = i\sqrt{\widetilde{\varphi}_2^2 - 2\beta/\alpha}$ and $\overline{c}_2 = -i\sqrt{\widetilde{\varphi}_2^2 - 2\beta/\alpha}$.

Substituting (3.58) into $d\varphi/d\xi = \phi$ and integrating them along the orbit $\widetilde{\Gamma}_1$, we have

$$\begin{gathered}
\pm\int_{\widetilde{\varphi}_1}^{\varphi} \frac{1}{\sqrt{(\widetilde{\varphi}_2 - s)(s - \widetilde{\varphi}_1)(s - c_2)(s - \overline{c}_2)}} ds = \sqrt{-\frac{\alpha}{2}}\int_0^{\xi} ds, \\
\pm\int_{\widetilde{\varphi}_2}^{\varphi} \frac{1}{\sqrt{(\widetilde{\varphi}_2 - s)(s - \widetilde{\varphi}_1)(s - c_2)(s - \overline{c}_2)}} ds = \sqrt{-\frac{\alpha}{2}}\int_0^{\xi} ds.
\end{gathered} \tag{3.59}$$

From (3.59) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we obtain the periodic wave solutions $u_{19}(x,y,t)$ as (3.38) and $u_{20}(x,y,t)$ as (3.39).

(ii) From the phase portrait, we see that there are two closed orbits $\widetilde{\Gamma}_2$ and $\widetilde{\Gamma}_3$ passing the points $(\widetilde{\varphi}_3, 0)$, $(\widetilde{\varphi}_4, 0)$, $(\widetilde{\varphi}_5, 0)$, and $(\widetilde{\varphi}_6, 0)$. In (φ, ϕ) plane, the expressions of the closed orbits are given as

$$\phi = \pm\sqrt{-\frac{\alpha}{2}(\varphi - \widetilde{\varphi}_3)(\varphi - \widetilde{\varphi}_4)(\varphi - \widetilde{\varphi}_5)(\widetilde{\varphi}_6 - \varphi)}, \tag{3.60}$$

where $\widetilde{\varphi}_3 = -\widetilde{\varphi}_6$, $\widetilde{\varphi}_4 = -\sqrt{2\beta/\alpha - \widetilde{\varphi}_6^2}$ and $\widetilde{\varphi}_5 = \sqrt{2\beta/\alpha - \widetilde{\varphi}_6^2}$.

Substituting (3.60) into $d\varphi/d\xi = \phi$ and integrating them along $\widetilde{\Gamma}_2$ and $\widetilde{\Gamma}_3$, we have

$$\begin{gathered}
\pm\int_{\widetilde{\varphi}_3}^{\varphi} \frac{1}{\sqrt{(\widetilde{\varphi}_6 - s)(\widetilde{\varphi}_5 - s)(\widetilde{\varphi}_4 - s)(s - \widetilde{\varphi}_3)}} ds = \sqrt{-\frac{\alpha}{2}} \int_0^{\xi} ds, \\
\pm\int_{\widetilde{\varphi}_6}^{\varphi} \frac{1}{\sqrt{(\widetilde{\varphi}_6 - s)(s - \widetilde{\varphi}_5)(s - \widetilde{\varphi}_4)(s - \widetilde{\varphi}_3)}} ds = \sqrt{-\frac{\alpha}{2}} \int_0^{\xi} ds.
\end{gathered} \tag{3.61}$$

From (3.61) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we obtain the periodic wave solutions $u_{21}(x, y, t)$ as (3.40) and $u_{22}(x, y, t)$ as (3.41).

(iii) From the phase portrait, we see that there are two symmetric homoclinic orbits $\widetilde{\Gamma}_4$ and $\widetilde{\Gamma}_5$ connected at the saddle point $(0, 0)$. In (φ, ϕ) plane, the expressions of the homoclinic orbits are given as

$$\phi = \pm\varphi\sqrt{-\frac{\alpha}{2}(\varphi - \widetilde{\varphi}_7)(\widetilde{\varphi}_8 - \varphi)}, \tag{3.62}$$

where $\widetilde{\varphi}_7 = -\sqrt{2\beta/\alpha}$ and $\widetilde{\varphi}_8 = \sqrt{2\beta/\alpha}$.

Substituting (3.62) into $d\varphi/d\xi = \phi$ and integrating them along the orbits $\widetilde{\Gamma}_4$ and $\widetilde{\Gamma}_5$, we have

$$\begin{gathered}
\pm\int_{\widetilde{\varphi}_7}^{\varphi} \frac{1}{s\sqrt{(s - \widetilde{\varphi}_7)(\widetilde{\varphi}_8 - s)}} ds = \sqrt{-\frac{\alpha}{2}} \int_0^{\xi} ds, \\
\pm\int_{\widetilde{\varphi}_8}^{\varphi} \frac{1}{s\sqrt{(s - \widetilde{\varphi}_7)(\widetilde{\varphi}_8 - s)}} ds = \sqrt{-\frac{\alpha}{2}} \int_0^{\xi} ds.
\end{gathered} \tag{3.63}$$

From (3.63) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we obtain the solitary wave solutions $u_{23\pm}(x, y, t)$ as (3.43).

(2) If $-g_0 < g < 0$, we set the middle solution of $f(\varphi) = 0$ be $\widetilde{\varphi}_9 (0 < \widetilde{\varphi}_9 < \sqrt{\beta/3\alpha})$, then we can get another two solutions of $f(\varphi) = 0$ as follows:

$$\widetilde{\varphi}_9^{*} = \frac{-\alpha\widetilde{\varphi}_9 - \sqrt{4\alpha\beta - 3\alpha^2\widetilde{\varphi}_9^2}}{2\alpha},$$
$$\widetilde{\varphi}_9^{\star} = \frac{-\alpha\widetilde{\varphi}_9 + \sqrt{4\alpha\beta - 3\alpha^2\widetilde{\varphi}_9^2}}{2\alpha}. \quad (3.64)$$

(i) From the phase portrait, we see that there are a closed orbit $\widetilde{\Gamma}_6$ passing the points $(\widetilde{\varphi}_{10}, 0)$ and $(\widetilde{\varphi}_{11}, 0)$. In (φ, ϕ) plane, the expressions of the closed orbits are given as

$$\phi = \pm\sqrt{-\frac{\alpha}{2}(\widetilde{\varphi}_{11} - \varphi)(\varphi - \widetilde{\varphi}_{10})(\varphi - c_3)(\varphi - \overline{c}_3)}, \quad (3.65)$$

where $\widetilde{\varphi}_{12} < \widetilde{\varphi}_{10} < \widetilde{\varphi}_9^{\star} < \widetilde{\varphi}_{11} < \widetilde{\varphi}_{13}$, c_3 and $\overline{c}_3$ are conjugate complex numbers.

Substituting (3.37) into $d\varphi/d\xi = \phi$ and integrating them along $\widetilde{\Gamma}_6$, we have

$$\pm\int_{\widetilde{\varphi}_{10}}^{\varphi} \frac{1}{\sqrt{(\widetilde{\varphi}_{11} - s)(s - \widetilde{\varphi}_{10})(s - c_3)(s - \overline{c}_3)}} ds = \sqrt{-\frac{\alpha}{2}} \int_0^{\xi} ds. \quad (3.66)$$

From (3.66) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get a periodic wave solution $u_{24}(x, y, t)$ as (3.44).

(ii) From the phase portrait, we note that there is a special orbit $\widetilde{\Gamma}_7$, which has the same Hamiltonian with that of $(\varphi_9^{*}, 0)$. In (φ, ϕ) plane, the expressions of the orbits are given as

$$\phi = \pm\sqrt{-\frac{\alpha}{2}(\varphi - \widetilde{\varphi}_9^{*})^2(\varphi - \widetilde{\varphi}_{12})(\widetilde{\varphi}_{13} - \varphi)}, \quad (3.67)$$

where

$$\widetilde{\varphi}_{12} = \frac{\alpha\widetilde{\varphi}_9 + \sqrt{4\alpha\beta - 3\alpha^2\widetilde{\varphi}_9^2} + 2\sqrt{\alpha\widetilde{\varphi}_9\left(\alpha\widetilde{\varphi}_9 - \sqrt{4\alpha\beta - 3\alpha^2\widetilde{\varphi}_9^2}\right)}}{2\alpha},$$
$$\widetilde{\varphi}_{13} = \frac{\alpha\widetilde{\varphi}_9 + \sqrt{4\alpha\beta - 3\alpha^2\widetilde{\varphi}_9^2} - 2\sqrt{\alpha\widetilde{\varphi}_9\left(\alpha\widetilde{\varphi}_9 - \sqrt{4\alpha\beta - 3\alpha^2\widetilde{\varphi}_9^2}\right)}}{2\alpha}. \quad (3.68)$$

Substituting (3.67) into $d\varphi/d\xi = \phi$ and integrating them along $\widetilde{\Gamma}_7$, it follows that

$$\pm\int_{\widetilde{\varphi}_{12}}^{\varphi} \frac{1}{\sqrt{(\widetilde{\varphi}_{13} - s)(s - \widetilde{\varphi}_9^{*})^2(s - \widetilde{\varphi}_{12})}} ds = \sqrt{-\frac{\alpha}{2}} \int_0^{\xi} ds. \quad (3.69)$$

From (3.69) and noting that $u = \varphi(\xi)$ and $\xi = x+y-ct$, we get a periodic wave solution $u_{25}(x,y,t)$ as (3.45).

(iii) From the phase portrait, we note that there are two closed orbits $\widetilde{\Gamma}_8$ and $\widetilde{\Gamma}_9$ passing the points $(\widetilde{\varphi}_{14},0)$, $(\widetilde{\varphi}_{15},0)$, $(\widetilde{\varphi}_{16},0)$, and $(\widetilde{\varphi}_{17},0)$. In (φ,ϕ) plane, the expressions of the orbits are given as

$$\phi = \pm\sqrt{-\frac{\alpha}{2}(\varphi-\widetilde{\varphi}_{14})(\varphi-\widetilde{\varphi}_{15})(\varphi-\widetilde{\varphi}_{16})(\widetilde{\varphi}_{17}-\varphi)}, \tag{3.70}$$

where $\widetilde{\varphi}_{20} < \widetilde{\varphi}_{14} < \widetilde{\varphi}_{12} < \widetilde{\varphi}_{10} < \widetilde{\varphi}_9^{\star} < \widetilde{\varphi}_{11} < \widetilde{\varphi}_{13} < \widetilde{\varphi}_{15} < \widetilde{\varphi}_9 < \widetilde{\varphi}_{16} < \widetilde{\varphi}_9^{*} < \widetilde{\varphi}_{17} < \widetilde{\varphi}_{21}$.

Substituting (3.70) into $d\varphi/d\xi = \phi$ and integrating them along $\widetilde{\Gamma}_8$ and $\widetilde{\Gamma}_9$, we have

$$\begin{aligned}
&\pm\int_{\widetilde{\varphi}_{14}}^{\varphi}\frac{1}{\sqrt{(\widetilde{\varphi}_{17}-s)(\widetilde{\varphi}_{16}-s)(\widetilde{\varphi}_{15}-s)(s-\widetilde{\varphi}_{14})}}ds = \sqrt{-\frac{\alpha}{2}}\int_0^{\xi}ds,\\
&\pm\int_{\widetilde{\varphi}_{17}}^{\varphi}\frac{1}{\sqrt{(\widetilde{\varphi}_{17}-s)(s-\widetilde{\varphi}_{16})(s-\widetilde{\varphi}_{15})(s-\widetilde{\varphi}_{14})}}ds = \sqrt{-\frac{\alpha}{2}}\int_0^{\xi}ds.
\end{aligned} \tag{3.71}$$

From (3.71) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get two periodic wave solutions $u_{26}(x,y,t)$ as (3.46) and $u_{27}(x,y,t)$ as (3.47).

(iv) From the phase portrait, we note that there is a special orbit $\widetilde{\Gamma}_{10}$ passing the points $(\widetilde{\varphi}_{18},0)$ and $(\widetilde{\varphi}_{19},0)$. In (φ,ϕ) plane, the expressions of the orbit are given as

$$\phi = \pm\sqrt{-\frac{\alpha}{2}(\widetilde{\varphi}_{19}-\varphi)(\varphi-\widetilde{\varphi}_{18})(\varphi-c_4)(\varphi-\overline{c}_4)}, \tag{3.72}$$

where $\widetilde{\varphi}_{18} < \widetilde{\varphi}_{20} < \widetilde{\varphi}_{21} < \widetilde{\varphi}_{19}$, c_4 and $\overline{c}_4$ are conjugate complex numbers.

Substituting (3.72) into $d\varphi/d\xi = \phi$ and integrating it along $\widetilde{\Gamma}_{10}$, we have

$$\pm\int_{\widetilde{\varphi}_{18}}^{\varphi}\frac{1}{\sqrt{(\widetilde{\varphi}_{19}-s)(s-\widetilde{\varphi}_{18})(s-c_4)(s-\overline{c}_4)}}ds = \sqrt{-\frac{\alpha}{2}}\int_0^{\xi}ds. \tag{3.73}$$

From (3.73) and noting that $u = \varphi(\xi)$ and $\xi = x+y-ct$, we get a periodic wave solution $u_{28}(x,y,t)$ as (3.48).

If $\varphi(\xi)$ is a traveling wave solution, then $\varphi(\xi + q)$ is a traveling wave solution too. Taking $q = 2K$ and noting that $\mathrm{cn}(u+2K) = -\mathrm{cn}\,u$, we get a periodic wave solution $u_{29}(x,y,t)$ as (3.49).

(v) From the phase portrait, we note that there are two homoclinic orbits $\widetilde{\Gamma}_{11}$ and $\widetilde{\Gamma}_{12}$ connected at the saddle point $(\widetilde{\varphi}_9,0)$. In (φ,ϕ) plane, the expressions of the orbits are given as

$$\phi = \pm\sqrt{-\frac{\alpha}{2}(\varphi-\widetilde{\varphi}_9)^2(\varphi-\widetilde{\varphi}_{20})(\widetilde{\varphi}_{21}-\varphi)}, \tag{3.74}$$

where

$$\widetilde{\varphi}_{20}=\frac{-\alpha\widetilde{\varphi}_9+\sqrt{2\alpha\beta-2\alpha^2\widetilde{\varphi}_9^2}}{\alpha},\qquad \widetilde{\varphi}_{21}=\frac{-\alpha\widetilde{\varphi}_9-\sqrt{2\alpha\beta-2\alpha^2\widetilde{\varphi}_9^2}}{\alpha}. \tag{3.75}$$

Substituting (3.74) into $d\varphi/d\xi=\phi$ and integrating them along $\widetilde{\Gamma}_{11}$ and $\widetilde{\Gamma}_{12}$, it follows that

$$\pm\int_{\widetilde{\varphi}_{20}}^{\varphi}\frac{1}{\sqrt{(s-\widetilde{\varphi}_9)^2(s-\widetilde{\varphi}_{20})(\widetilde{\varphi}_{21}-s)}}ds=\sqrt{-\frac{\alpha}{2}}\int_0^{\xi}ds,$$
$$\pm\int_{\widetilde{\varphi}_{21}}^{\varphi}\frac{1}{\sqrt{(s-\widetilde{\varphi}_9)^2(s-\widetilde{\varphi}_{20})(\widetilde{\varphi}_{21}-s)}}ds=\sqrt{-\frac{\alpha}{2}}\int_0^{\xi}ds. \tag{3.76}$$

From (3.76) and noting that $u=\varphi(\xi)$ and $\xi=x+y-ct$, we get two solitary wave solutions $u_{30}(x,y,t)$ as (3.51) and $u_{31}(x,y,t)$ as (3.52).

(3) If $g=-g_0$, we will consider two kinds of orbits.

(i) From the phase portrait, we note that there is a closed orbit $\widetilde{\Gamma}_{13}$ passing the points $(\widetilde{\varphi}_{22},0)$ and $(\varphi_{23},0)$. In (φ,ϕ) plane, the expressions of the orbit are given as

$$\phi=\pm\sqrt{-\frac{\alpha}{2}(\widetilde{\varphi}_{23}-\varphi)(\varphi-\widetilde{\varphi}_{22})(\varphi-c_5)(\varphi-\overline{c}_5)}, \tag{3.77}$$

where $-\sqrt{3\beta/\alpha}<\widetilde{\varphi}_{22}<-2\sqrt{\beta/3\alpha}<\widetilde{\varphi}_{23}<\sqrt{\beta/3\alpha}$, c_5 and $\overline{c}_5$ are conjugate complex numbers.

Substituting (3.77) into $d\varphi/d\xi=\phi$ and integrating it along $\widetilde{\Gamma}_{13}$, we have

$$\pm\int_{\widetilde{\varphi}_{22}}^{\varphi}\frac{1}{\sqrt{(\widetilde{\varphi}_{23}-s)(s-\widetilde{\varphi}_{22})(s-c_5)(s-\overline{c}_5)}}ds=\sqrt{-\frac{\alpha}{2}}\int_0^{\xi}ds. \tag{3.78}$$

From (3.78) and noting that $u=\varphi(\xi)$ and $\xi=x+y-ct$, we get a periodic wave solutions $u_{32}(x,y,t)$ as (3.53).

(ii) From the phase portrait, we note that there is a closed orbit $\widetilde{\Gamma}_{14}$ passing the points $(\widetilde{\varphi}_{24},0)$ and $(\widetilde{\varphi}_{25},0)$. In (φ,ϕ) plane, the expressions of the orbit are given as

$$\phi=\pm\sqrt{-\frac{\alpha}{2}(\widetilde{\varphi}_{25}-\varphi)(\varphi-\widetilde{\varphi}_{24})(\varphi-c_6)(\varphi-\overline{c}_6)}, \tag{3.79}$$

where $\widetilde{\varphi}_{24}<-\sqrt{3\beta/\alpha}<\widetilde{\varphi}_{22}<-2\sqrt{\beta/3\alpha}<\widetilde{\varphi}_{23}<\sqrt{\beta/3\alpha}<\widetilde{\varphi}_{25}$, c_6 and $\overline{c}_6$ are conjugate complex numbers.

Substituting (3.79) into $d\varphi/d\xi = \phi$ and integrating them along Γ_{14}, we have

$$\pm\int_{\tilde{\varphi}_{24}}^{\varphi} \frac{1}{\sqrt{(\tilde{\varphi}_{25} - s)(s - \tilde{\varphi}_{24})(s - c_6)(s - \overline{c}_6)}} ds = \sqrt{-\frac{\alpha}{2}} \int_0^{\xi} ds. \tag{3.80}$$

From (3.80) and noting that $u = \varphi(\xi)$ and $\xi = x+y-ct$, we get a periodic wave solutions $u_{33}(x, y, t)$ as (3.54).

(iii) From the phase portrait, we see that there is a homoclinic orbit $\widetilde{\Gamma}_{15}$, which passes the degenerate saddle point $(\varphi_+^*, 0)$. In (φ, ϕ) plane, the expressions of the homoclinic orbit are given as

$$\phi = \pm\sqrt{-\frac{\alpha}{2}(\varphi_+^* - \varphi)^3 (\varphi - \tilde{\varphi}_{26})}, \tag{3.81}$$

where

$$\tilde{\varphi}_{26} = -\sqrt{\frac{3\beta}{\alpha}}. \tag{3.82}$$

Substituting (3.81) into $d\varphi/d\xi = \phi$ and integrating them along Γ_{15}, it follows that

$$\pm\int_{\tilde{\varphi}_{26}}^{\varphi} \frac{1}{(s - \varphi_+^*)\sqrt{(s - \varphi_+^*)(\tilde{\varphi}_{26} - s)}} ds = \sqrt{-\frac{\alpha}{2}} \int_0^{\xi} ds. \tag{3.83}$$

From (3.83) and noting that $u = \varphi(\xi)$ and $\xi = x + y - ct$, we get a solitary wave solution $u_{34}(x, y, t)$ as (3.56).

Thus, the derivation of Proposition 3.2 has been finished. □

Proposition 3.3. *For these solutions, the following are their relations.*

(1) *When φ_6 tends to $\sqrt{2\beta/\alpha}$, the periodic blow-up wave solutions $u_{3\pm}$ and $u_{4\pm}$ tend to periodic blow-up wave solutions $u_{1\pm}$ and $u_{2\pm}$, that is,*

$$\lim_{\varphi_6 \to \sqrt{2\beta/\alpha}} u_{3\pm}(x, y, t) = u_{1\pm}(x, y, t), \qquad \lim_{\varphi_6 \to \sqrt{2\beta/\alpha}} u_{4\pm}(x, y, t) = u_{2\pm}(x, y, t). \tag{3.84}$$

(2) *When φ_6 tends to $\sqrt{\beta/\alpha}$, the periodic wave solutions $u_{5\pm}$ tend to kink wave solutions $u_{6\pm}$, that is,*

$$\lim_{\varphi_6 \to \sqrt{\beta/\alpha}} u_{5\pm}(x, y, t) = u_{6\pm}(x, y, t). \tag{3.85}$$

(3) *When φ_6 tends to $\sqrt{\beta/\alpha}$, the periodic blow-up wave solutions $u_{3\pm}$ tend to unbounded wave solutions $u_{7\pm}$, that is,*

$$\lim_{\varphi_6 \to \sqrt{\beta/\alpha}} u_{3\pm}(x, y, t) = u_{7\pm}(x, y, t). \tag{3.86}$$

(4) *When φ_{13} tends to φ_9, the periodic blow-up wave solution u_9 tends to periodic blow-up wave solution u_{8-}, that is,*

$$\lim_{\varphi_{13} \to \varphi_9} u_9(x, y, t) = u_{8-}(x, y, t). \tag{3.87}$$

(5) *When φ_{13} tends to φ_9, the periodic blow-up wave solution u_{10} tends to periodic blow-up wave solution u_{8+}, that is,*

$$\lim_{\varphi_{13} \to \varphi_9} u_{10}(x, y, t) = u_{8+}(x, y, t). \tag{3.88}$$

(6) *When φ_{13} tends to φ_7, the periodic wave solution u_{11} tends to solitary wave solution u_{13}, that is,*

$$\lim_{\varphi_{13} \to \varphi_7} u_{11}(x, y, t) = u_{13}(x, y, t). \tag{3.89}$$

(7) *When φ_{13} tends to φ_7, the periodic blow-up wave solution u_{10} tends to unbounded wave solution u_{14}, that is,*

$$\lim_{\varphi_{13} \to \varphi_7} u_{10}(x, y, t) = u_{14}(x, y, t). \tag{3.90}$$

(8) *When φ_{18} tends to $\sqrt{\beta/3\alpha}$, the periodic blow-up wave solution u_{17} tends to blow-up wave solution u_{16}, that is,*

$$\lim_{\varphi_{18} \to \sqrt{\beta/3\alpha}} u_{17}(x, y, t) = u_{16}(x, y, t). \tag{3.91}$$

(9) *When $\tilde{\varphi}_2$ tends to $\sqrt{2\beta/\alpha}$, the periodic wave solutions u_{19} and u_{20} tend to solitary wave solutions $u_{23\pm}$, that is,*

$$\lim_{\tilde{\varphi}_2 \to \sqrt{2\beta/\alpha}} u_{19}(x, y, t) = u_{23-}(x, y, t), \qquad \lim_{\tilde{\varphi}_2 \to \sqrt{2\beta/\alpha}} u_{20}(x, y, t) = u_{23+}(x, y, t). \tag{3.92}$$

(10) *When $\tilde{\varphi}_6$ tend to $\sqrt{2\beta/\alpha}$, the periodic wave solutions u_{21} and u_{22} tends to solitary wave solutions $u_{23\pm}$, that is,*

$$\lim_{\tilde{\varphi}_6 \to \sqrt{2\beta/\alpha}} u_{21}(x, y, t) = u_{23-}(x, y, t), \qquad \lim_{\tilde{\varphi}_6 \to \sqrt{2\beta/\alpha}} u_{22}(x, y, t) = u_{23+}(x, y, t). \tag{3.93}$$

(11) *When $\tilde{\varphi}_{11}$ tends to $\tilde{\varphi}_{13}$, the periodic wave solution u_{24} tends to periodic wave solution u_{25}, that is,*

$$\lim_{\tilde{\varphi}_{11}\to\tilde{\varphi}_{13}} u_{24}(x,y,t) = u_{25}(x,y,t). \tag{3.94}$$

(12) *When $\tilde{\varphi}_{17}$ tends to $\tilde{\varphi}_9^*$, the periodic wave solution u_{26} tends to periodic wave solution u_{25}, that is,*

$$\lim_{\tilde{\varphi}_{17}\to\tilde{\varphi}_9^*} u_{26}(x,y,t) = u_{25}(x,y,t). \tag{3.95}$$

(13) *When $\tilde{\varphi}_{17}$ tends to $\tilde{\varphi}_{21}$, the periodic wave solutions u_{26} and u_{27} tend to solitary wave solutions u_{30} and u_{31}, that is,*

$$\lim_{\tilde{\varphi}_{17}\to\tilde{\varphi}_{21}} u_{26}(x,y,t) = u_{30}(x,y,t), \qquad \lim_{\tilde{\varphi}_{17}\to\tilde{\varphi}_{21}} u_{27}(x,y,t) = u_{31}(x,y,t). \tag{3.96}$$

(14) *When $\tilde{\varphi}_{19}$ tends to $\tilde{\varphi}_{21}$, the periodic wave solutions u_{28} and u_{29} tend to solitary wave solutions u_{30} and u_{31}, that is,*

$$\lim_{\tilde{\varphi}_{19}\to\tilde{\varphi}_{21}} u_{28}(x,y,t) = u_{30}(x,y,t), \qquad \lim_{\tilde{\varphi}_{19}\to\tilde{\varphi}_{21}} u_{29}(x,y,t) = u_{31}(x,y,t). \tag{3.97}$$

(15) *When $\tilde{\varphi}_{22}$ and $\tilde{\varphi}_{24}$ tend to $\sqrt{3\beta/\alpha}$, the periodic wave solutions u_{32} and u_{33} tend to solitary wave solution u_{34}, that is,*

$$\lim_{\tilde{\varphi}_{22}\to\sqrt{3\beta/\alpha}} u_{32}(x,y,t) = \lim_{\tilde{\varphi}_{24}\to\sqrt{3\beta/\alpha}} u_{33}(x,y,t) = u_{34}(x,y,t). \tag{3.98}$$

Proof. (1) Letting $\varphi_6 \to \sqrt{2\beta/\alpha}$, it follows that $\varphi_5 \to 0$ and $\operatorname{sn}(\varphi_6\sqrt{\alpha/2}\xi, \varphi_5/\varphi_6) \to \operatorname{sn}(\sqrt{\beta}\xi, 0) = \sin\sqrt{\beta}\xi$, and we have

$$\begin{aligned}
\lim_{\varphi_6\to\sqrt{2\beta/\alpha}} u_{3\pm}(x,y,t) &= \lim_{\varphi_6\to\sqrt{2\beta/\alpha}} \pm\frac{\varphi_6}{\operatorname{sn}\left(\varphi_6\sqrt{\alpha/2}\xi, \varphi_5/\varphi_6\right)} = \pm\frac{\sqrt{2\beta/\alpha}}{\sin\sqrt{\beta}\xi} = u_{1\pm}(x,y,t), \\
\lim_{\varphi_6\to\sqrt{2\beta/\alpha}} u_{4\pm}(x,y,t) &= \lim_{\varphi_6\to\sqrt{2\beta/\alpha}} \pm\sqrt{\frac{\varphi_6^2-\varphi_5^2\left(\operatorname{sn}\left(\varphi_6\sqrt{\alpha/2}\xi, \varphi_5/\varphi_6\right)\right)^2}{1-\left(\operatorname{sn}\left(\varphi_6\sqrt{\alpha/2}\xi, \varphi_5/\varphi_6\right)\right)^2}} \\
&= \pm\frac{\sqrt{2\beta/\alpha}}{\cos\sqrt{\beta}\xi} = u_{2\pm}(x,y,t).
\end{aligned} \tag{3.99}$$

Therefore, the periodic blow-up solutions $u_{1\pm}(x,y,t)$ are the limit of the elliptic function periodic blow-up solutions $u_{3\pm}(x,y,t)$ and the periodic blow-up solutions $u_{2\pm}(x,y,t)$ are the limit of the elliptic function periodic blow-up solutions $u_{4\pm}(x,y,t)$.

(2) Letting $\varphi_6 \to \sqrt{\beta/\alpha}$, it follows that $\varphi_5 \to \sqrt{\beta/\alpha}$ and $\operatorname{sn}(\varphi_5\sqrt{\alpha/2}\xi, \varphi_5/\varphi_6) \to \operatorname{sn}(\sqrt{\beta/2}\xi, 1) = \tanh\sqrt{\beta/2}\xi$, and we have

$$\begin{aligned}\lim_{\varphi_6 \to \sqrt{\beta/\alpha}} u_{5\pm}(x,y,t) &= \lim_{\varphi_6 \to \sqrt{\beta/\alpha}} \pm\varphi_5 \operatorname{sn}\left(\varphi_6\sqrt{\frac{\alpha}{2}}\xi, \frac{\varphi_5}{\varphi_6}\right) \\ &= \pm\sqrt{\frac{\beta}{\alpha}}\tanh\sqrt{\frac{\beta}{2}}\xi = u_{6\pm}(x,y,t).\end{aligned} \tag{3.100}$$

Therefore, the kink wave solutions $u_{6\pm}(x,y,t)$ are the limit of the elliptic function periodic solutions $u_{5\pm}(x,y,t)$.

(4) Letting $\varphi_{13} \to \varphi_9$, it follows that $\varphi_{12} \to \varphi_7^*$, $\varphi_{11} \to \varphi_7^*$, $\varphi_{10} \to \varphi_8$ and $\operatorname{sn}(\omega_1\xi, k_1) \to \operatorname{sn}(\omega_1\xi, 0) = \sin\omega_1\xi$, and we have

$$\begin{aligned}\lim_{\varphi_{13} \to \varphi_9} u_9(x,y,t) &= \lim_{\varphi_{13} \to \varphi_9} \frac{\varphi_{10}(\varphi_{11}-\varphi_{13}) + \varphi_{11}(\varphi_{13}-\varphi_{10})(\operatorname{sn}(\omega_1\xi,k_1))^2}{\varphi_{11}-\varphi_{13}+(\varphi_{13}-\varphi_{10})(\operatorname{sn}(\omega_1\xi,k_1))^2} \\ &= \varphi_7^* - \frac{2\gamma_1}{\delta_1 - \sqrt{\eta_1}\cos\sqrt{\alpha\gamma_1/2}\xi} = u_{8-}(x,y,t).\end{aligned} \tag{3.101}$$

Therefore, the trigonometric function periodic blow-up wave solution $u_{8-}(x,y,t)$ is the limit of the elliptic function periodic blow-up solution $u_9(x,y,t)$.

(9) Letting $\tilde{\varphi}_2 \to \sqrt{2\beta/\alpha}$, it follows that $\tilde{\varphi}_1 \to -\sqrt{2\beta/\alpha}$, $\sqrt{\beta - \alpha\tilde{\varphi}_1^2} \to \sqrt{-\beta}$ and $-\tilde{\varphi}_1\sqrt{\alpha/(-2\beta + 2\alpha\varphi_1^2)} \to 1$, and we have

$$\begin{aligned}\lim_{\tilde{\varphi}_2 \to \sqrt{2\beta/\alpha}} u_{19}(x,y,t) &= \lim_{\tilde{\varphi}_2 \to \sqrt{2\beta/\alpha}} \tilde{\varphi}_1 \operatorname{cn}\left(\sqrt{\beta - \alpha\tilde{\varphi}_1^2}\xi, -\tilde{\varphi}_1\sqrt{\frac{\alpha}{-2\beta + 2\alpha\varphi_1^2}}\right) \\ &= -\sqrt{\frac{2\beta}{\alpha}}\operatorname{sech}\sqrt{-\beta}\xi = u_{23-}(x,y,t).\end{aligned} \tag{3.102}$$

Therefore, the solitary wave solution $u_{23-}(x,y,t)$ is the limit of the elliptic function periodic wave solution $u_{19}(x,y,t)$.

(15) Letting $\tilde{\varphi}_{22} \to -\sqrt{3\beta/\alpha}$, it follows that $\tilde{\varphi}_{23} \to \sqrt{\beta/3\alpha}$, $c_5 \to \sqrt{\beta/3\alpha}$, $\overline{c}_5 \to \sqrt{\beta/3\alpha}$, $A_4 \to 0$, $B_4 \to 4\sqrt{\beta/3\alpha}$ and $\text{cn}(\sqrt{-\alpha A_4 B_4/2}, k_8) \to \text{cn}(0, k_8) = 1$, and we have

$$\begin{aligned}
\lim_{\tilde{\varphi}_{22} \to -\sqrt{3\beta/\alpha}} u_{32}(x,y,t) &= \lim_{\varphi_{22} \to -\sqrt{3\beta/\alpha}} \frac{A_4\tilde{\varphi}_{22} + \tilde{\varphi}_{23}B_4 + (A_4\tilde{\varphi}_{22} - \tilde{\varphi}_{23}B_4)\,\text{cn}\left(\sqrt{-\alpha A_4 B_4/2}\xi, k_8\right)}{A_4 + B_4 + (A_4 - B_4)\,\text{cn}\left(\sqrt{-\alpha A_4 B_4/2}\xi, k_8\right)} \\
&= \lim_{A_4 \to 0} \frac{A_4\tilde{\varphi}_{22} + \tilde{\varphi}_{23}B_4 + (A_4\tilde{\varphi}_{22} - \tilde{\varphi}_{23}B_4)\,\text{cn}\left(\sqrt{-\alpha A_4 B_4/2}\xi, k_8\right)}{A_4 + B_4 + (A_4 - B_4)\,\text{cn}\left(\sqrt{-\alpha A_4 B_4/2}\xi, k_8\right)} \\
&= \lim_{A_4 \to 0} \frac{2\sqrt{-2\alpha A_4 B_4}(\tilde{\varphi}_{22} + \tilde{\varphi}_{22}\chi_1) + B_4(A_4\tilde{\varphi}_{22} - B_4\tilde{\varphi}_{23})\alpha\xi\chi_2\chi_3}{2\sqrt{-2\alpha A_4 B_4}(1 + \chi_1) + B_4(A_4 - B_4)\alpha\xi\chi_2\chi_3} \\
&= \sqrt{\frac{3\beta}{\alpha}} \frac{9 + 2\beta\xi^2}{-9 + 6\beta\xi^2} = u_{34}(x,y,t),
\end{aligned} \tag{3.103}$$

where $\chi_1 = \text{cn}(\sqrt{-\alpha A_4 B_4/2}, k_8)$, $\chi_2 = \text{dn}(\sqrt{-\alpha A_4 B_4/2}, k_8)$, $\chi_3 = \text{sn}(\sqrt{-\alpha A_4 B_4/2}, k_8)$.

Therefore, the fractional function solitary wave solution $u_{34}(x,y,t)$ is the limit of the elliptic function periodic wave solution $u_{32}(x,y,t)$.

Similarly, we can derive the others case. This has proved Proposition 3.3. □

Finally, we will show that the periodic wave solution $u_{5+}(x,y,t)$ evolutes into the kink wave solution $u_{6+}(x,y,t)$ when $\varphi_6 \to \sqrt{\beta/\alpha}$. We take some suitable choices of the parameters, such as

$$a = 1, \qquad b = 1, \qquad r = 5 \qquad c = 2, \tag{3.104}$$

as an illustrative sample and draw their plots (see Figure 3).

Remark 3.4. One may find that we only consider the case when $g \geq 0$ in Proposition 3.1 (when $g \leq 0$ in Proposition 3.2). In fact, we may get exactly the same solutions in the opposite case.

4. Conclusion

In this paper, we have obtained many traveling wave solutions for the generalized KP-BBM equation (1.4) by employing the bifurcation method and qualitative theory of dynamical systems. The traveling wave solutions have been given in Propositions 3.1 and 3.2. On the other hand, in Proposition 3.3, we prove that the periodic wave solutions, kink wave solutions, blow-up wave solutions, unbounded solutions, and solitary wave solutions can be obtained from the limits of the smooth periodic wave solutions or periodic blow-up solutions.

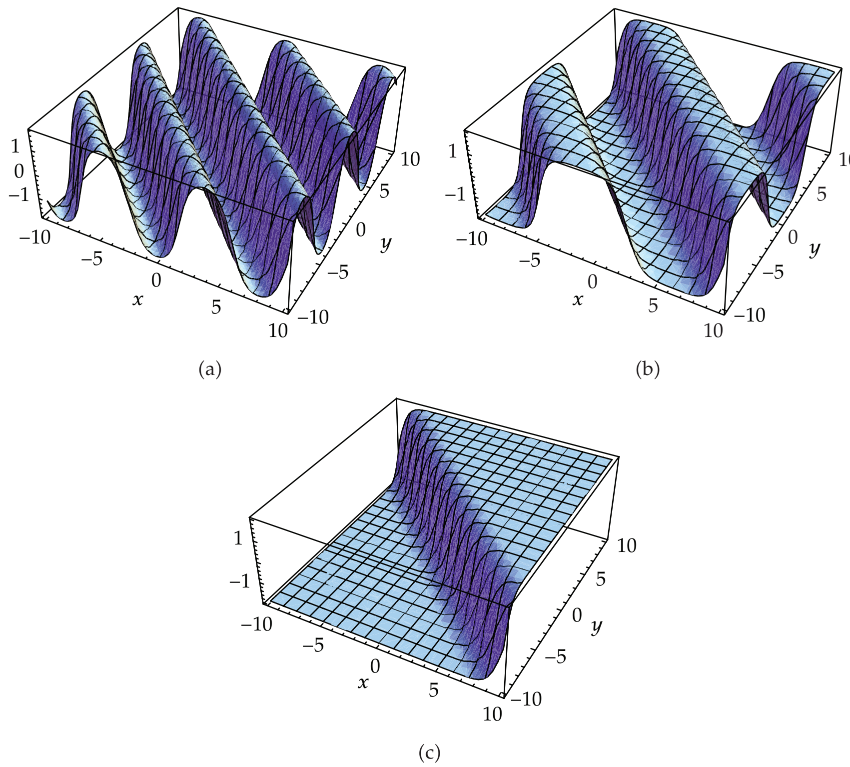

Figure 3: The periodic wave solution $u_{5+}(x, y, t)$ evolute into the kink wave solution $u_{6+}(x, y, t)$ with the conditions (3.104). (a) $\varphi_6 = 2.2$; (b) $\varphi_6 = 2.01$; (c) $\varphi_6 = 2$.

The method can be applied to many other nonlinear evolution equations and we believe that many new results wait for further discovery by this method.

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