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Research Article

On Six Solutions for *m*-Point Differential Equations System with Two Coupled Parallel Sub-Super Solutions

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Under the assumption of two coupled parallel subsuper solutions, the existence of at least six solutions for a kind of second-order m-point differential equations system is obtained using the fixed point index theory. As an application, an example to demonstrate our result is given.

1. Introduction

In this paper, we consider the following second-order m-point boundary value problems of nonlinear equations system

$$\begin{gathered}
-\varphi''(t) = f_1(\varphi(t)) + f_2(\psi(t)), \quad t \in [0,1], \\
-\psi''(t) = g_1(\psi(t)) + g_2(\varphi(t)), \quad t \in [0,1], \\
\varphi'(0) = 0, \qquad \varphi(1) = \sum_{i=1}^{m-2} \alpha_i \varphi(\xi_i), \\
\psi'(0) = 0, \qquad \psi(1) = \sum_{i=1}^{m-2} \alpha_i \psi(\xi_i),
\end{gathered} \tag{1.1}$$

where $f_i, g_i : \mathbb{R}^1 \to \mathbb{R}^1 (i = 1, 2)$ are continuous and α_i, ξ_i satisfying

(H_0) $\sum_{i=1}^{m-2} \alpha_i \in (0,1)$ with $\alpha_i \in (0,+\infty)$ for $i = 1,2,\ldots,m-2$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$.

Multipoint boundary value problems arise in many applied sciences for example, the vibrations of a guy wire composed of N parts with a uniform cross-section throughout, but different densities in different parts can be set up as a multipoint boundary value problems (see [1]). Many problems in the theory of elastic stability can be modelled by multipoint boundary value problems (see [2]). The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [3]. Subsequently, Gupta [4] studied certain three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, the solvability of more general nonlinear multipoint boundary value problems has been discussed by several authors using various methods. We refer the readers to [5–12] and the references therein.

In the recent years, many authors have studied existence and multiplicity results for solutions of multipoint boundary value problems via the well-ordered upper and lower solutions method, see [8, 13, 14] and the references therein. However, only in very recent years, some authors considered the multiplicity of solutions under conditions of non-well-ordered upper and lower solutions. For some abstract results concerning conditions of non-well-ordered upper and lower solutions, the readers are referred to recent works [15–18].

In [19], Xu et al. considered the following second-order three-point boundary value problem

$$\begin{aligned} &y''(t) + f(t,y) = 0, \quad t \in [0,1], \\ &y(0) = 0, \qquad y(1) - \alpha y(\eta) = 0, \end{aligned} \tag{1.2}$$

where $0 < \eta < 1,\ 0 < \alpha < 1,\ f \in C([0,1] \times \mathbb{R}^1, \mathbb{R}^1)$. He obtained the following result. First, let us give the following condition $(H_0)'$ to be used later.
$(H_0)'$ There exists $M > 0$ such that

$$f(t,x_2) - f(t,x_1) \geq -M(x_2 - x_1), \quad t \in [0,1],\ x_2 \geq x_1. \tag{1.3}$$

Let the function e be $e = e(t) = t$ for $t \in [0,1]$.

Theorem 1.1. *Suppose that $(H_0)'$ holds, u_1 and u_2 are two strict lower solutions of (1.2), v_1 and v_2 are two strict upper solutions of (1.2), and $u_1 < v_1,\ u_2 < v_2,\ u_2 \nleq v_1,\ u_1 \nleq v_2$. Moreover, assume*

$$\begin{aligned} -\xi_0 e \leq u_2 - u_1 \leq \xi_0 e, \\ -\xi_0 e \leq v_2 - v_1 \leq \xi_0 e, \end{aligned} \tag{1.4}$$

for some $\xi_0 > 0$. Then, the three-point boundary value problem (1.2) has at least six solutions.

We would also like to mention the result of Yang [20], in [20]. Yang studied the following integral boundary value problem

$$-(au')' + bu = g(t)f(t,u),$$
$$(\cos\gamma_0)u(0) - (\sin\gamma_0)u'(0) = \int_0^1 u(\tau)d\alpha(\tau), \tag{1.5}$$
$$(\cos\gamma_1)u(1) + (\sin\gamma_1)u'(1) = \int_0^1 u(\tau)d\beta(\tau),$$

where $\gamma_0 \in [0,\pi/2]$ and $\gamma_1 \in [0,\pi/2]$, $\int_0^1 u(\tau)d\alpha(\tau)$ and $\int_0^1 u(\tau)d\beta(\tau)$ denote the Riemann-Stieltjes integrals of u with respect to α and β, respectively. Some sufficient conditions for the existence of either none, or one, or more positive solutions of the problem (1.5) were established. The main tool used in the proofs of existence results is a fixed point theorem in a cone, due to Krasnoselskii and Zabreiko.

At the same time, we note that Webb and Lan [21] have considered the first eigenvalue of the following linear problem

$$u''(t) + \lambda u(t) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2}\alpha_i u(\eta_i), \tag{1.6}$$

they also investigated the existence and multiplicity of positive solutions of several related nonlinear multipoint boundary value problems. Furthermore, Ma and O'Regan [22] studied the spectrum structure of the problem (1.6), and the authors obtained the concrete computational method and the corresponding properties of real eigenvalue of (1.6) by constructing an auxiliary function. Their work is very fundamental to further study for multipoint boundary value problems. By extending and improving the work in [22], Rynne [23] showed that the associated Sturm-Liouville problem consisting of (1.6) has a strictly increasing sequence of simple eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ with eigenfunctions $\phi_n(t) = \sin(\sqrt{\lambda_n}t)$.

Very recently, Kong et al. [24] were concerned with the general boundary value problem with a variable w

$$u''(t) + w(t)f(u) = 0, \quad t \in (a,b),$$
$$\cos\alpha u(a) - \sin\alpha u'(a) = 0, \quad \alpha \in [0,\pi), \quad u(b) = \sum_{i=1}^{m-2}k_i u(\eta_i) \tag{1.7}$$

By relating (1.7) to the eigenvalues of a linear Sturm-Liouville problem with a two-point separated boundary condition, the existence and nonexistence of nodal solutions of (1.7) were obtained. We also point out that Webb [25] made the excellent remark on some existence results of symmetric positive solutions obtained in some recent papers and the author also corrected the values of the principle eigenvalue previously given in some examples.

In this paper, by means of two coupled parallel subsuper solutions, we obtain some sufficient conditions for the existence of six solutions for (1.1) and our main tool is based on the fixed point index theory. At the end of this paper, we will give an example which illustrates that our work is true. Our method stems from the paper [18].

2. Preliminaries and a Lemma

In the section, we shall give some preliminaries and a lemma which are fundamental to prove our main result.

Let E be an ordered Banach space in which the partial ordering $\leq$ is induced by a cone $P \subset E$. A cone P is said to be normal if there exists a constant $N > 0$, such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, the smallest N is called the normal constant of P. P is called solid, if $\mathrm{int}P \neq \emptyset$, that is, P has nonempty interior. Every cone P in E defines a partial ordering in E given by $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, we write $x < y$; if cone P is solid and $y - x \in \mathrm{int}P$, we write $x \ll y$. P is called total if $E = \overline{P - P}$. Let $B : E \to E$ be a bounded linear operator. B is said to be positive if $B(P) \subset P$. An operator A is strongly increasing, that is, $x < y$ implies $Ax \ll Ay$. If A is a linear operator, A is strongly increasing implying A is strongly positive.

Let E be an ordered Banach space, P a total cone in E, the partial ordering $\leq$ induced by P. $B : E \to E$ is a positive completely continuous linear operator. Let $r(B) > 0$ the spectral radius of B, B^* the conjugated operator of B, and P^* the conjugated cone of P. Since $P \subset E$ is a total cone (i.e., $E = \overline{P - P}$), according to the famous Krein-Rutman theorem (see [26]), we infer that if $r(B) \neq 0$, then there exist $\overline{\varphi} \in P \setminus \{\theta\}$ and $g^* \in P^* \setminus \{\theta\}$, such that

$$\begin{aligned} B\overline{\varphi} &= r(B)\overline{\varphi}, \\ B^* g^* &= r(B) g^*. \end{aligned} \tag{2.1}$$

Fixed $\overline{\varphi} \in P \setminus \{\theta\}$, $g^* \in P^* \setminus \{\theta\}$ such that (2.1) holds. For $\delta > 0$, let

$$P(g^*, \delta) = \{x \in P,\ g^*(x) \geq \delta\|x\|\}, \tag{2.2}$$

then $P(g^*, \delta)$ is also a cone in E. One can refer [26–28] for definition and properties about the cones.

Definition 2.1 (see [29]). Let B be a positive linear operator. The operator B is said to satisfy condition **H**, if there exist $\overline{\varphi} \in P \setminus \{\theta\}$, $g^* \in P^* \setminus \{\theta\}$, and $\delta > 0$ such that (2.1) holds, and B maps P into $P(g^*, \delta)$.

Lemma 2.2 (see [10]). *Suppose that $d = 1 - \sum_{i=1}^{m-2} \alpha_i \neq 0$. Then, the BVP*

$$\begin{gathered} -u''(t) = 0, \quad t \in (0,1), \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \end{gathered} \tag{2.3}$$

has Green's function

$$G(t,s) = \widetilde{G}(t,s) + \frac{\sum_{i=1}^{m-2} \alpha_i \widetilde{G}(\xi_i, s)}{1 - \sum_{i=1}^{m-2} \alpha_i}, \tag{2.4}$$

where

$$\widetilde{G}(t,s) = \begin{cases} 1-t, & 0 \le s \le t \le 1, \\ 1-s, & 0 \le t \le s \le 1. \end{cases} \tag{2.5}$$

For convenience, we list the following hypotheses which will be used in our main result.

(H_1) $f_i,\ g_i (i=1,2)$ *are strictly increasing;*

(H_2) *there exist constants* $k>0,\ l>0$ *and* $D>0$ *such that*

(i) $|f_1(\pm k) \pm l| < N^{-1}k$,

(ii) $|g_1(\pm k) \pm D| < N^{-1}k$,

(iii) $|f_2(\pm k)| \le l$, *and*

(iv) $|g_2(\pm k)| \le D$, *where* $N = \max_{t\in[0,1]} \int_0^1 G(t,s)ds$;

(H_3) there exist constants $0 < c_1 < k,\ -k < c_2 < 0,\ 0 < c_3 < k,\ -k < c_4 < 0$, such that, for all $t \in [0,1]$, we have

(i) $c_1 < \int_0^1 G(t,s) f_1(c_1)ds - Nl$,

(ii) $c_2 > \int_0^1 G(t,s) g_1(c_2)ds + ND$,

(iii) $c_3 < \int_0^1 G(t,s) g_1(c_3)ds - ND$, and

(iv) $c_4 > \int_0^1 G(t,s) f_1(c_4)ds + Nl$;

(H_4)

(i) $\lim_{|\varphi| \to +\infty} (f_1(\varphi) + f_2(\varphi))/\varphi \ge 2\lambda_1$ uniformly for $\varphi \in \mathbb{R}$,

(ii) $\lim_{|\Psi| \to +\infty} (g_1(\varphi) + g_2(\varphi))/\varphi \ge 2\lambda_1$ *uniformly for* $\varphi \in \mathbb{R}$, *where* λ_1 *is the first eigenvalue of the following boundary value problem:*

$$\begin{gathered} -u''(t) = \lambda u, \quad t \in (0,1), \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i). \end{gathered} \tag{2.6}$$

It is well known that $\lambda_1 = r^{-1}(H)$, where linear operator $H : C[0,1] \to C[0,1]$ is defined as $\mathrm{Hu}(t) = \int_0^1 G(t,s)u(s)ds$.

3. Main Results

Theorem 3.1. *Assume* (H_0), (H_1)–(H_4) *hold, then BVP* (1.1) *has at least six distinct continuous solutions.*

Proof. It is easy to check that BVP (1.1) is equivalent to the following integral equation systems:

$$\begin{aligned}\varphi(t) &= \int_0^1 G(t,s)\left[f_1(\varphi(s)) + f_2(\psi(s))\right]ds,\\ \psi(t) &= \int_0^1 G(t,s)\left[g_1(\psi(s)) + g_2(\varphi(s))\right]ds,\end{aligned} \tag{3.1}$$

where $G(t,s)$ is defined as in Lemma 2.2. By (H_0), we know that $G(t,s) \geq 0$, for all $t,s \in [0,1]$.

Let $E = C[0,1] \times C[0,1]$, define the norm in E as $\|(\varphi,\psi)\| = \|\varphi\| + \|\psi\|$. Then, E is a Banach space with this norm. Let $P = \{(\varphi,\psi) \in E \mid \varphi(t) \geq 0, \psi(t) \geq 0, \text{for all } t \in [0,1]\}$, $Q = \{\varphi \in C[0,1] \mid \varphi(t) \geq 0, \text{ for all } t \in [0,1]\}$. Then, $P = Q \times Q$ is a normal and solid cone. Set $T : E \to E$, such that

$$T(\varphi,\psi) = \left(\int_0^1 G(t,s)\left[f_1(\varphi(s)) + f_2(\psi(s))\right]ds, \int_0^1 G(t,s)\left[g_1(\psi(s)) + g_2(\varphi(s))\right]ds\right), \tag{3.2}$$

it is clear that the solutions of (1.1) are equivalent to the fixed points of T.

Set $0 < \xi_1 \leq \lambda_1$, let

$$\begin{aligned}f(\varphi,\psi) &= \frac{(f_1(\varphi) + f_2(\psi), g_1(\psi) + g_2(\varphi))}{\lambda_1 + \xi_1},\\ K(\varphi,\psi) &= ((\lambda_1 + \xi_1)H\varphi, (\lambda_1 + \xi_1)H\psi) = (H_1\varphi, H_1\psi),\end{aligned} \tag{3.3}$$

where $H_1 = (\lambda_1 + \xi_1)H$, then $T = Kf$. It is easy to see that H is a strongly positive completely continuous operator, and it follows from $G(t,s) \geq 0$ and the continuity of $G(t,s)$ that K is a strongly positive completely continuous operator. Since f_i, $g_i : R^1 \to R^1 (i = 1,2)$ are strictly increasing continuous functions, we know that f is a strictly increasing continuous bounded operator. By $T = Kf$, we can prove that T is completely continuous. We infer from the increasing properties of K and f that T is increasing.

Let $\varphi_1 \equiv c_1$, $\psi_1 \equiv -k$, $\varphi_2 \equiv k$, $\psi_2 \equiv c_2$, $\varphi_3 \equiv -k$, $\psi_3 \equiv c_3$, $\varphi_4 \equiv c_4$, $\psi_4 \equiv k$, then $(\varphi_i,\psi_i) (i = 1,2,3,4)$ satisfy

$$(\varphi_1,\psi_1) < (\varphi_2,\psi_2), \qquad (\varphi_3,\psi_3) < (\varphi_4,\psi_4), \qquad (\varphi_1,\psi_1) \nleq (\varphi_4,\psi_4), \qquad (\varphi_3,\psi_3) \nleq (\varphi_2,\psi_2). \tag{3.4}$$

By (H_2)(iii) and (H_3)(i), we have

$$\int_0^1 G(t,s)\left[f_1(\varphi_1)+f_2(\psi_1)\right]ds \geq \int_0^1 G(t,s)f_1(c_1)ds - Nl > c_1 = \varphi_1. \tag{3.5}$$

It follows from (H_2)(ii),(iv), and the increasing property of g_2 that

$$\int_0^1 G(t,s)\left[g_1(\psi_1)+g_2(\varphi_1)\right]ds > (-k)N^{-1}\int_0^1 G(t,s)ds \geq (-k)N^{-1}N = -k = \psi_1. \tag{3.6}$$

Equations (3.2), (3.5), and (3.6) imply that

$$(\varphi_1,\psi_1) < T(\varphi_1,\psi_1). \tag{3.7}$$

Similarly, by (H_1)–(H_3), we obtain

$$T(\varphi_2,\psi_2) < (\varphi_2,\psi_2), \qquad (\varphi_3,\psi_3) < T(\varphi_3,\psi_3), \qquad T(\varphi_4,\psi_4) < (\varphi_4,\psi_4). \tag{3.8}$$

By [20, Lemma 3], we get that H_1 satisfies condition **H**. Therefore, there exist $j_0^* \in Q^* \setminus \{\theta\}$, $\delta > 0$, such that

$$H_1^* j_0^* = r(H_1) j_0^*, \tag{3.9}$$

$$j_0^*(H_1\varphi) \geq \delta\|H_1\varphi\|, \quad \forall \varphi \in Q. \tag{3.10}$$

By the definition of spectral radius of completely continuous operator, we have $r(K) = r(H_1)$, and combining (3.9), we infer that

$$H_1^* j_0^* = r(K) j_0^*. \tag{3.11}$$

Let $j^*((\varphi,\psi)) = j_0^*(\varphi) + j_0^*(\psi)$, for all $(\varphi,\psi) \in E$, then $j^* \in P^* \setminus \{\theta\}$. According to the proof in [18], we can get that K satisfies condition **H**.

By condition (H_4), we obtain that there exists $C > 0$, such that

$$f_1(\varphi) + f_2(\psi) \geq (\lambda_1 + \xi_1)\varphi, \quad \varphi \geq C, \ \psi \in \mathbb{R}, \tag{3.12}$$

$$f_1(\varphi) + f_2(\psi) \leq (\lambda_1 + \xi_1)\varphi, \quad \varphi \leq -C, \ \psi \in \mathbb{R}, \tag{3.13}$$

$$g_1(\psi) + g_2(\varphi) \geq (\lambda_1 + \xi_1)\psi, \quad \psi \geq C, \ \varphi \in \mathbb{R}, \tag{3.14}$$

$$g_1(\psi) + g_2(\varphi) \leq (\lambda_1 + \xi_1)\psi, \quad \psi \leq -C, \ \varphi \in \mathbb{R}. \tag{3.15}$$

Equations (3.12)–(3.15) imply

$$\left(f_1(\varphi) + f_2(\psi), g_1(\psi) + g_2(\varphi)\right) \geq (\lambda_1 + \xi_1)(\varphi,\psi), \quad \forall\ \varphi,\psi \geq C, \tag{3.16}$$

$$\left(f_1(\varphi) + f_2(\psi), g_1(\psi) + g_2(\varphi)\right) \leq (\lambda_1 + \xi_1)(\varphi,\psi), \quad \forall \varphi,\psi \leq -C. \tag{3.17}$$

Since $f_1(\varphi), g_1(\psi)$ are continuous in $[-(c_1+k), C]$, so they are bounded, then there exists $h > 0$ such that

$$f_1(\varphi) \geq -h, \qquad g_1(\psi) \geq -h, \qquad -\|(\varphi_1, \psi_1)\| = -(c_1+k) \leq \varphi, \quad \psi \leq C. \tag{3.18}$$

By virtue of (3.18) and the increasing properties of f_2 and g_2, one shows

$$\begin{aligned}
&(f_1(\varphi) + f_2(\psi), g_1(\psi) + g_2(\varphi)) \\
&\quad \geq \left(-h - \left|f_2(-\|(\varphi_1, \psi_1)\|)\right|, -h - \left|g_2(-\|(\varphi_1, \psi_1)\|)\right|\right), \quad -\|(\varphi_1, \psi_1)\| \leq \varphi, \quad \psi \leq C.
\end{aligned} \tag{3.19}$$

In addition, if φ, ψ satisfy $\varphi \geq C$, $-\|(\varphi_1, \psi_1)\| \leq \psi \leq C$, then it follows from (3.12), (3.18), and the increasing property of g_2 that

$$\begin{aligned}
&(f_1(\varphi) + f_2(\psi), g_1(\psi) + g_2(\varphi)) \\
&\quad \geq \left((\lambda_1 + \xi_1)\varphi, -h + g_2(-\|(\varphi_1, \psi_1)\|)\right) \\
&\quad \geq (\lambda_1 + \xi_1)(\varphi, \psi) - \left((\lambda_1 + \xi_1)C + h + \left|g_2(-\|(\varphi_1, \psi_1)\|)\right|, (\lambda_1 + \xi_1)C + h + \left|g_2(-\|(\varphi_1, \psi_1)\|)\right|\right) \\
&\quad = (\lambda_1 + \xi_1)(\varphi, \psi) - (d_1, d_1),
\end{aligned} \tag{3.20}$$

where

$$d_1 = (\lambda_1 + \xi_1)C + h + \left|g_2(-\|(\varphi_1, \psi_1)\|)\right|. \tag{3.21}$$

Similarly, if φ, ψ satisfy $\psi \geq C$, $-\|(\varphi_1, \psi_1)\| \leq \varphi \leq C$, then combining the increasing property of f_2 with (3.14) and (3.18), we know that

$$(f_1(\varphi) + f_2(\psi), g_1(\psi) + g_2(\varphi)) \geq (\lambda_1 + \xi_1)(\varphi, \psi) - (d_2, d_2), \tag{3.22}$$

where $d_2 = (\lambda_1 + \xi_1)C + h + |f_2(-\|(\varphi_1, \psi_1)\|)|$. Let $d = \max\{d_1, d_2\}$. By (3.21) and (3.22), we get that if $\varphi \geq C$, $-\|(\varphi_1, \psi_1)\| \leq \psi \leq C$ or $\psi \geq C$, $-\|(\varphi_1, \psi_1)\| \leq \varphi \leq C$, it is obvious that

$$(f_1(\varphi) + f_2(\psi), g_1(\psi) + g_2(\varphi)) \geq (\lambda_1 + \xi_1)(\varphi, \psi) - (d, d). \tag{3.23}$$

It follows from (3.16), (3.19), and (3.23) that

$$(f_1(\varphi) + f_2(\psi), g_1(\psi) + g_2(\varphi)) \geq (\lambda_1 + \xi_1)(\varphi, \psi) - (d, d), \quad \varphi, \psi \geq -\|(\varphi_1, \psi_1)\|. \tag{3.24}$$

In a similar way, from (3.12) and (3.14), we can show that there exists $e > 0$ such that

$$(f_1(\varphi) + f_2(\psi), g_1(\psi) + g_2(\varphi)) \geq (\lambda_1 + \xi_1)(\varphi, \psi) - (e, e), \quad \varphi, \psi \geq -\|(\varphi_3, \psi_3)\|. \tag{3.25}$$

Let $a = \max\{d, e\}$. It follows from (3.24) and (3.25) that if $\varphi, \psi \geq -\|(\varphi_1, \psi_1)\|$ or $\varphi, \psi \geq -\|(\varphi_3, \psi_3)\|$, then

$$\left(f_1(\varphi) + f_2(\psi), g_1(\psi) + g_2(\varphi)\right) \geq (\lambda_1 + \xi_1)(\varphi, \psi) - (a, a). \tag{3.26}$$

In a similar way, from (3.13) and (3.15), we can prove that there exists constant $\widetilde{a}$ such that if $\varphi, \psi \leq \|(\varphi_2, \psi_2)\|$ or $\varphi, \psi \leq \|(\varphi_4, \psi_4)\|$, then

$$\left(f_1(\varphi) + f_2(\psi), g_1(\psi) + g_2(\varphi)\right) \leq (\lambda_1 + \xi_1)(\varphi, \psi) - (\widetilde{a}, \widetilde{a}). \tag{3.27}$$

Let $P((\varphi_1, \psi_1)) = \{(u, v) \in E \mid (u, v) \geq (\varphi_1, \psi_1)\}$, $P((\varphi_3, \psi_3)) = \{(u, v) \in E \mid (u, v) \geq (\varphi_3, \psi_3)\}$, $P((\varphi_2, \psi_2)) = \{(u, v) \in E \mid (u, v) \leq (\varphi_2, \psi_2)\}$, $P((\varphi_4, \psi_4)) = \{(u, v) \in E \mid (u, v) \leq (\varphi_4, \psi_4)\}$, for all $(\varphi, \psi) \in P((\varphi_1, \psi_1)) \cup P((\varphi_3, \psi_3))$, then $\varphi(t), \psi(t) \geq -\|(\varphi_1, \psi_1)\|$ or $\varphi(t), \psi(t) \geq -\|(\varphi_3, \psi_3)\|$, for all $t \in [0, 1]$; therefore, in virtue of expression of B, f, and (3.26), we have

$$\begin{aligned}
&Kf(\varphi(t), \psi(t)) \\
&\quad \geq K(\varphi(t), \psi(t)) - \frac{(\lambda_1 + \xi_1)a}{(\lambda_1 + \xi_1)} \left(\int_0^1 G(t, s)ds, \int_0^1 G(t, s)ds\right) \\
&\quad = \left(r^{-1}(K) + \epsilon\right) K(\varphi(t), \psi(t)) - (u_1(t), v_1(t)),
\end{aligned} \tag{3.28}$$

where $\epsilon = 1 - r^{-1}(K)$, $(u_1(t), v_1(t)) = a(\int_0^1 G(t, s)ds, \int_0^1 G(t, s)ds)$. Since $0 < \xi_1 \leq \lambda_1$, one can show

$$0 < \epsilon = 1 - \frac{\lambda_1}{\lambda_1 + \xi_1} = \frac{\xi_1}{\lambda_1 + \xi_1} \leq \frac{\lambda_1}{\lambda_1 + \xi_1} = r^{-1}(K). \tag{3.29}$$

This implies that there exist $(u_1, v_1) \in E$ and $0 < \xi_2 \leq r^{-1}(K)$ such that

$$Kf(\varphi, \psi) \geq \left(r^{-1}(K) + \xi_2\right) K(\varphi, \psi) - (u_1, v_1), \quad \forall(\varphi, \psi) \in P\big((\varphi_1, \psi_1)\big) \cup P\big((\varphi_3, \psi_3)\big). \tag{3.30}$$

Similarly, we get by (3.27) that there exist $(u_2, v_2) \in E$ and $0 < \xi_3 \leq r^{-1}(K)$ such that

$$Kf(\varphi, \psi) \leq \left(r^{-1}(K) + \xi_3\right) K(\varphi, \psi) - (u_2, v_2), \quad \forall(\varphi, \psi) \in P\big((\varphi_2, \psi_2)\big) \cup P\big((\varphi_4, \psi_4)\big). \tag{3.31}$$

We get by (3.7) that

$$T : P\big((\varphi_1, \psi_1)\big) \longrightarrow P\big((\varphi_1, \psi_1)\big). \tag{3.32}$$

Let $E_1 = \{(u,v) \in E \mid (\varphi_1, \psi_1) \leq (u,v) \leq (\varphi_2, \psi_2)\}$. Since P is normal, then E_1 is bounded (see [28]). Choose $M_1 > 0$ such that

$$M_1 > \max\left\{ \frac{1}{\xi_2\delta}\Big(\xi_2\delta\|(\varphi_1,\psi_1)\| + \xi_2\delta\|T(\varphi_1,\psi_1)\| + r^{-1}(K)g^*((u_1,v_1)) - \xi_2 g^*(T(\varphi_1,\psi_1))\Big), \right.$$
$$\left. \sup_{(u,v)\in E_1} \|(u,v)\| + \|(\varphi_1,\psi_1)\| \right\}. \tag{3.33}$$

Let $\Omega_1 = \{(u,v) \in P((\varphi_1,\psi_1)) \mid \|(u,v) - (\varphi_1,\psi_1)\| < M_1, (u,v) \ngeq (\varphi_3,\psi_3)\}$, then $E_1 \subset \Omega_1$ and Ω_1 is a bounded open set. By the proof of Theorem 2.1 in [18], we can show that

$$(u,v) - T(u,v) \neq \lambda K(\overline{u},\overline{v}), \quad \forall \lambda \geq 0,\ (u,v) \in \partial\Omega_1, \tag{3.34}$$

where $(\overline{u},\overline{v})$ satisfies $K(\overline{u},\overline{v}) = r(K)(\overline{u},\overline{v})$.

Equation (3.34) implies that T has no fixed point on $\partial\Omega_1$. It is easy to prove that $P((\varphi_1,\psi_1))$ is a retract of E, which together with (3.32) implies that the fixed point index $i(T,\Omega_1,P((\varphi_1,\psi_1)))$ over Ω_1 with respect to $P((\varphi_1,\psi_1))$ is well defined, and a standard proof yields

$$i(T + sK(\overline{u},\overline{v}),\Omega_1,P((\varphi_1,\psi_1))) = 0. \tag{3.35}$$

Set $\overline{H}(t,(u,v)) = (1-t)T(u,v) + t(T(u,v) + sK(\overline{u},\overline{v}))$, $(t,(u,v)) \in [0,1] \times \overline{\Omega_1}$, then for any $(u,v) \in \overline{\Omega_1}$, $t \in [0,1]$, we have $\overline{H}(t,(u,v)) \in P((\varphi_1,\psi_1))$. It follows from (3.34) that $H(t,(u,v)) \neq (u,v)$, for all $(t,(u,v)) \in [0,1] \times \partial\Omega_1$, and by (3.35) and the homotopy invariance of the fixed point index, we get

$$i(T,\Omega_1,P((\varphi_1,\psi_1))) = i(T + sK(\overline{u},\overline{v}),\Omega_1,P((\varphi_1,\psi_1))) = 0. \tag{3.36}$$

Let $W_1 = \{(u,v) \in P((\varphi_1,\psi_1)) \mid (u,v) \ll (\varphi_2,\psi_2)\}$. By means of usual method (see [30]), we get that

$$i(T,W_1,P((\varphi_1,\psi_1))) = 1. \tag{3.37}$$

It is evident that A has no fixed point on ∂W_1, by (3.36), (3.37), and the additivity of the fixed point index, we have

$$i\Big(T,\Omega_1 \setminus \overline{W_1},P((\varphi_1,\psi_1))\Big) = i(T,\Omega_1,P((\varphi_1,\psi_1))) - i(T,W_1,P((\varphi_1,\psi_1))) = 0 - 1 = -1. \tag{3.38}$$

Set $E_2 = \{(u,v) \in E \mid (\varphi_3, \psi_3) \leq (u,v) \leq (\varphi_4, \psi_4)\}$, and choose $M_2 > 0$ such that

$$M_2 > \max\left\{ \frac{1}{\xi_2 \delta}\Big(\xi_2 \delta \|(\varphi_3, \psi_3)\| + \xi_2 \delta \|T(\varphi_3, \psi_3)\| + r^{-1}(K) g^*((u_1, v_1)) - \xi_2 g^*(T(\varphi_3, \psi_3))\Big), \right.$$
$$\left. \sup_{(u,v)\in E_2} \|(u,v)\| + \|(\varphi_3, \psi_3)\| \right\}. \tag{3.39}$$

Let

$$\begin{gathered} \Omega_2 = \left\{(u,v) \in P((\varphi_3, \psi_3)) \mid \|(u,v) - (\varphi_3, \psi_3)\| < M_2, (u,v) \not\geq (\varphi_1, \psi_1)\right\}, \\ W_2 = \left\{(u,v) \in P((\varphi_3, \psi_3)) \mid (u,v) \ll (\varphi_4, \psi_4)\right\}. \end{gathered} \tag{3.40}$$

Similarly to the proof of (3.37) and (3.38), we get that

$$\begin{gathered} i(T, W_2, P((\varphi_3, \psi_3))) = 1, \\ i\Big(T, \Omega_2 \setminus \overline{W_2}, P((\varphi_3, \psi_3))\Big) = -1. \end{gathered} \tag{3.41}$$

Choose $M_3, M_4 > 0$ such that

$$M_3 > \max\left\{ \frac{1}{\xi_3 \delta}\Big(\xi_3 \delta \|(\varphi_2, \psi_2)\| + \xi_3 \delta \|T(\varphi_2, \psi_2)\| + r^{-1}(K) g^*((u_2, v_2)) - \xi_3 g^*(T(\varphi_2, \psi_2))\Big) \right.$$
$$\left. \sup_{(u,v)\in E_1} \|(u,v)\| + \|(\varphi_2, \psi_2)\| \right\},$$
$$M_4 > \max\left\{ \frac{1}{\xi_3 \delta}\Big(\xi_3 \delta \|(\varphi_4, \psi_4)\| + \xi_3 \delta \|T(\varphi_4, \psi_4)\| + r^{-1}(K) g^*((u_2, v_2)) - \xi_3 g^*(T(\varphi_4, \psi_4))\Big), \right.$$
$$\left. \sup_{(u,v)\in E_2} \|(u,v)\| + \|(\varphi_4, \psi_4)\| \right\}. \tag{3.42}$$

Set

$$\begin{gathered} \Omega_3 = \left\{(u,v) \in P((\varphi_2, \psi_2)) \mid \|(u,v) - (\varphi_2, \psi_2)\| < M_3, (u,v) \not\leq (\varphi_4, \psi_4)\right\}, \\ \Omega_4 = \left\{(u,v) \in P((\varphi_4, \psi_4)) \mid \|(u,v) - (\varphi_4, \psi_4)\| < M_4, (u,v) \not\leq (\varphi_2, \psi_2)\right\}, \\ W_3 = \left\{(u,v) \in P((\varphi_2, \psi_2)) \mid (\varphi_1, \psi_1) \ll (u,v)\right\}, \\ W_4 = \left\{(u,v) \in P((\varphi_4, \psi_4)) \mid (\varphi_2, \psi_2) \ll (u,v)\right\}. \end{gathered} \tag{3.43}$$

By virtue of (3.31) and the same method as that for (3.34), we have

$$(u,v) - T(u,v) \neq -\mu K(\overline{u},\overline{v}), \quad \forall \mu \geq 0,\ (u,v) \in \partial\Omega_3 \cup \partial\Omega_4. \tag{3.44}$$

By (3.44), similarly to the proof of (3.38), we can prove that

$$i\Big(T, \Omega_3 \setminus \overline{W_3}, P((\varphi_2,\psi_2))\Big) = i\Big(T, \Omega_4 \setminus \overline{W_4}, P((\varphi_4,\psi_4))\Big) = -1. \tag{3.45}$$

Equations (3.37)–(3.41), (3.45) imply that T has at least six distinct fixed points, that is, the system of differential equations (1.1) has at least six solution in $C[0,1] \times C[0,1]$. □

4. An Example

In this section, we present a simple example to explain our results.

Consider the following second-order three-point BVP for nonlinear equations system:

$$\begin{gathered}
-\varphi''(t) = f_1(\varphi(t)) + f_2(\psi(t)), \quad t \in [0,1], \\
-\psi''(t) = g_1(\psi(t)) + g_2(\varphi(t)), \quad t \in [0,1], \\
\varphi'(0) = 0, \qquad \varphi(1) = \frac{1}{2}\varphi\left(\frac{1}{4}\right), \\
\psi'(0) = 0, \qquad \psi(1) = \frac{1}{2}\psi\left(\frac{1}{4}\right),
\end{gathered} \tag{4.1}$$

where $f_1(\varphi) = (\varphi^3/30000) + 5\varphi^{1/3}$, $g_1(\psi) = (\psi^3/30000) + 4\psi^{1/5}$, $f_2(\psi) = (1/6) \cdot \sqrt[5]{100} + 1/12 \arctan 6\psi$, $g_2(\varphi) = (1/16) \cdot \sqrt[3]{100} + (1/30) \arctan 7\varphi$, $\alpha_1 = 1/2$, $\xi_1 = 1/4$, $N = \max_{t\in[0,1]} \int_0^1 G(t,s)ds = 31/32$, $f_i, g_i : \mathbb{R}^1 \to \mathbb{R}^1$ are strictly increasing continuous functions, and condition (H_1) is satisfied. Choose $k = 100$, $l = 5/12 + \pi/24, D = 5/16 + \pi/60$. Some direct calculations show

$$\begin{aligned}
\left|f_1(\pm 100) \pm l\right| &\leq \frac{10^6}{3\cdot 10^4} + 5\cdot\sqrt[3]{100} + \frac{5}{12} + \frac{\pi}{24} \\
&< \frac{32}{31}\cdot 100 = N^{-1}k, \\
\left|g_1(\pm 100) \pm D\right| &\leq \frac{100}{3} + 4\cdot\sqrt[5]{100} + \frac{5}{16} + \frac{\pi}{60} \\
&< 34 + 4\cdot\frac{5}{2} < \frac{32}{31}\cdot 100 = N^{-1}k, \\
\left|f_2(\pm 100)\right| &\leq \frac{1}{6}\cdot\sqrt[5]{100} + \frac{1}{12}\arctan 6\cdot 100
\end{aligned}$$

$$< \frac{1}{6} \cdot \frac{5}{2} + \frac{1}{12} \cdot \frac{\pi}{2} = \frac{5}{12} + \frac{\pi}{24} = l,$$

$$\begin{aligned}
\left|g_2(\pm 100)\right| &\le \frac{1}{16} \cdot \sqrt[3]{100} + \frac{1}{30} \arctan 7 \cdot 100 \\
&< \frac{1}{16} \cdot 5 + \frac{1}{30} \cdot \frac{\pi}{2} = \frac{5}{16} + \frac{\pi}{60} = D,
\end{aligned} \tag{4.2}$$

Therefore, condition (H_2) is satisfied.

Choosing $c_1 = 1/8,\ c_2 = -1,\ c_3 = 1/4,\ c_4 = -1$, it is easy to check that

$$\begin{aligned}
\int_0^1 G(t,s) f_1(c_1) ds - Nl &\ge \frac{15}{32}\left(\frac{1}{8^3 \cdot 3 \cdot 10^4} + \frac{5}{2}\right) - \frac{31}{32}\left(\frac{5}{12} + \frac{\pi}{24}\right) \\
&\ge \frac{15}{32} \cdot \frac{5}{2} - \left(\frac{5}{12} + \frac{1}{6}\right) = \frac{75}{64} - \frac{7}{12} > \frac{1}{8} = c_1, \\
\int_0^1 G(t,s)\ g_1(c_2) ds + ND &\le \frac{15}{32}\left(\frac{-1}{3 \cdot 10^4} - 4\right) + \frac{31}{32}\left(\frac{5}{16} + \frac{\pi}{60}\right) \\
&\le \frac{15}{32} \cdot (-4) + \frac{6}{16} = -\frac{3}{2} < -1 = c_2, \\
\int_0^1 G(t,s) g_1(c_3) ds - ND &\ge \frac{15}{32}\left[\frac{1}{4^3 \cdot 3 \cdot 10^4} + 4\left(\frac{1}{4}\right)^{1/5}\right] - \frac{31}{32}\left(\frac{5}{16} + \frac{\pi}{60}\right) \\
&\ge \frac{15}{32} \cdot 4\sqrt[5]{\frac{1}{4}} - \frac{3}{8} \ge \frac{15}{32} \cdot 4 \cdot \frac{3}{4} - \frac{3}{8} \\
&= \frac{23}{32} > \frac{1}{4} = c_3, \\
\int_0^1 G(t,s) f_1(c_4) ds + Nl &\le \frac{15}{32}\left(\frac{-1}{3 \cdot 10^4} - 5\right) + \frac{31}{32}\left(\frac{5}{12} + \frac{\pi}{24}\right) \\
&\le \frac{15}{32}(-5) + \frac{7}{12} = \frac{-75}{32} + \frac{7}{12} < -1 = c_4.
\end{aligned} \tag{4.3}$$

Therefore, condition (H_3) is satisfied. At last, we will check condition (H_4), by the method of [9, 31], and we consider the linear eigenvalue problem

$$\begin{aligned}
u'' + \lambda u = 0, &\quad 0 < t < 1, \\
u'(0) = 0, &\quad u(1) = \frac{1}{2} u\left(\frac{1}{4}\right).
\end{aligned} \tag{4.4}$$

Let $\Gamma(s) = \cos s - (1/2)\cos(s/4)$. By the paper [31], we know that the sequence of positive eigenvalue of (4.4) is exactly given by $\lambda_n = s_n^2,\quad n = 1, 2, \ldots$, where s_n is the sequence of positive solutions of $\Gamma(s) = 0$. In [31], Han obtained that $s_1 = 1.0675$, moreover $\lambda_1 = s_1^2 = 1.1396$. It is easy to know that

$$\lim_{|x|\to\infty} \frac{f_1(x) + f_2(y)}{x} = \lim_{|x|\to\infty} \frac{\left(x^3/30000\right) + 5x^{1/3} + (1/6)\cdot\sqrt[5]{100} + (1/12)\arctan 6y}{x}$$
$$= +\infty \geq 2\cdot 1.1396 = 2\lambda_1, \tag{4.5}$$

uniformly for $y \in \mathbb{R}$,

$$\lim_{|y|\to\infty} \frac{g_1(y) + g_2(x)}{y} = \lim_{|y|\to\infty} \frac{\left(y^3/30000\right) + 4y^{1/5} + (1/16)\cdot\sqrt[3]{100} + (1/30)\arctan 7x}{y}$$
$$= +\infty \geq 2\cdot 1.1396 = 2\lambda_1, \tag{4.6}$$

uniformly for $x \in \mathbb{R}$. Therefore, condition (H_4) is also satisfied. Consequently, all conditions of Theorem 3.1 are satisfied, and we get the system of differential equations (4.1) has at least six solutions in $C[0,1] \times C[0,1]$.

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