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Research Article

An Iterative Algorithm on Approximating Fixed Points of Pseudocontractive Mappings

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Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let K be a nonempty bounded closed convex subset of E, and every nonempty closed convex bounded subset of K has the fixed point property for non-expansive self-mappings. Let $f : K \to K$ a contractive mapping and $T : K \to K$ be a uniformly continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\} \subset (0, 1/2)$ be a sequence satisfying the following conditions: (i) $\lim_{n\to\infty}\lambda_n = 0$; (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$. Define the sequence $\{x_n\}$ in K by $x_0 \in K$, $x_{n+1} = \lambda_n f(x_n) + (1 - 2\lambda_n)x_n + \lambda_n T x_n$, for all $n \geq 0$. Under some appropriate assumptions, we prove that the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F(T)$ which is the unique solution of the following variational inequality: $\langle f(p) - p, j(z - p)\rangle \leq 0$, for all $z \in F(T)$.

1. Introduction

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{ f^* \in E^* : \left\langle x, f^* \right\rangle = \|x\|^2 = \left\| f^* \right\|^2 \right\}, \quad \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

It is well known that, if E is smooth, then J is single-valued. In the sequel, we will denote the single-valued normalized duality mapping by j. We use $D(T), R(T)$ to denote the domain and range of T, respectively.

An operator $T : D(T) \to R(T)$ is called pseudocontractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \le \|x-y\|^2, \quad \forall x, y \in D(T). \tag{1.2}$$

A point $x \in K$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$.

Within the past 40 years or so, many authors have been devoted to the iterative construction of fixed points of pseudocontractive mappings (see [1–10]).

In 1974, Ishikawa [11] introduced an iterative scheme to approximate the fixed points of Lipschitzian pseudocontractive mappings and proved the following result.

Theorem 1.1 (see [11]). *If K is a compact convex subset of a Hilbert space $H, T : K \to K$ is a Lipschitzian pseudocontractive mapping. Define the sequence $\{x_n\}$ in K by*

$$\begin{gathered} x_0 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad \forall n \ge 0, \end{gathered} \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions

(i) $0 \le \alpha_n \le \beta_n < 1$,

(ii) $\lim_{n\to\infty} \beta_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

In connection with the iterative approximation of fixed points of pseudo-contractions, in 2001, Chidume and Mutangadura [12] provided an example of a Lipschitz pseudocontractive mapping with a unique fixed point for which the Mann iterative algorithm failed to converge. Chidume and Zegeye [13] introduced a new iterative scheme for approximating the fixed points of pseudocontractive mappings.

Theorem 1.2 (see [13]). *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let K be a nonempty closed convex subset of E. Let $T : K \to K$ be a L-Lipschitzian pseudocontractive mapping such that $F(T) \neq \emptyset$. Suppose that every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive self-mappings. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences in $(0,1]$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \theta_n = 0$,

(ii) $\lambda_n(1+\theta_n) \le 1$, $\sum_{n=0}^{\infty} \lambda_n \theta_n = \infty$, $\lim_{n\to\infty}(\lambda_n/\theta_n) = 0$,

(iii) $\lim_{n\to\infty}((\theta_{n-1}/\theta_n - 1)/\lambda_n \theta_n) = 0$.

For given $x_1 \in K$ arbitrarily, let the sequence $\{x_n\}$ be defined iteratively by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1), \quad \forall n \geq 1. \tag{1.4}$$

Then, the sequence $\{x_n\}$ defined by (1.4) *converges strongly to a fixed point of T.*

Prototypes for the iteration parameters are, for example, $\lambda_n = 1/(n+1)^a$ and $\theta_n = 1/(n+1)^b$ for $0 < b < a$ and $a + b < 1$. But we observe that the canonical choices of $\lambda_n = 1/n$ and $\theta_n = 1/n$ are impossible. This bring us a question.

Question 1. Under what conditions, $\lim_{n\to\infty}\lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$ are sufficient to guarantee the strong convergence of the iterative scheme (1.4) to a fixed point of T?

In this paper, we explore an iterative scheme to approximate the fixed points of pseudocontractive mappings and prove that, under some appropriate assumptions, the proposed iterative scheme converges strongly to a fixed point of T, which solves some variational inequality. Our results improve and extend many results given in the literature.

2. Preliminaries

Let K be a nonempty closed convex subset of a real Banach space E. Recall that a mapping $f : K \to K$ is called contractive if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in K. \tag{2.1}$$

Let μ be a continuous linear functional on l^∞ and $s = (a_0, a_1, \ldots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu(s)$. We call μ a Banach limit if μ satisfies $\|\mu\| = \mu(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \ldots) \in l^\infty$.

If μ is a Banach limit, then we have the following.

(1) For all $n \geq 1$, $a_n \leq c_n$ implies $\mu_n(a_n) \leq \mu_n(c_n)$.

(2) $\mu_n(a_{n+r}) = \mu_n(a_n)$ for any fixed positive integer r.

(3) $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, \ldots) \in l^\infty$.

(4) If $s = (a_0, a_1, \ldots) \in l^\infty$ with $a_n \to a$, then $\mu(s) = \mu_n(a_n) = a$ for any Banach limit μ.

For more details on Banach limits, we refer readers to [14]. We need the following lemmas for proving our main results.

Lemma 2.1 (see [15]). *Let E be a Banach space. Suppose that K is a nonempty closed convex subset of E and $T : K \to E$ is a continuous pseudocontractive mapping satisfying the weakly inward condition: $T(x) \in \overline{I_K(x)}$ ($\overline{I_K(x)}$ is the closure of $I_K(x)$) for each $x \in K$, where $I_K(x) = \{x + c(u - x) : u \in E \text{ and } c \geq 1\}$. Then, for each $z \in K$, there exists a unique continuous path $t \mapsto z_t \in K$ for all $t \in [0,1)$, satisfying the following equation*

$$z_t = tTz_t + (1-t)z. \tag{2.2}$$

Furthermore, if E is a reflexive Banach space with a uniformly Gâteaux differentiable norm and every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive self-mappings, then, as $t \to 1$, z_t converges strongly to a fixed point of T.

Lemma 2.2 (see [16]). (1) *If E is smooth Banach space, then the duality mapping J is single valued and strong-weak* continuous.*

(2) *If E is a Banach space with a uniformly Gâteaux differentiable norm, then the duality mapping $J : E \to E^*$ is single valued and norm to weak star uniformly continuous on bounded sets of E.*

Lemma 2.3 (see [17]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n$ for all $n \geq 0$, where $\{\alpha_n\} \subset (0,1)$, and $\{\beta_n\}$ two sequences of real numbers such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \beta_n \leq 0$. Then $\{a_n\}$ converges to zero as $n \to \infty$.*

Lemma 2.4 (see [18]). *Let E be a real Banach space, and let J be the normalized duality mapping. Then, for any given $x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y). \tag{2.3}$$

Lemma 2.5 (see [14]). *Let a be a real number, and let $(x_0, x_1, \ldots,) \in l^\infty$ such that $\mu_n x_n \leq a$ for all Banach limits. If $\limsup_{n\to\infty}(x_{n+1} - x_n) \leq 0$, then $\limsup_{n\to\infty} x_n \leq a$.*

3. Main Results

Now, we are ready to give our main results in this paper.

Theorem 3.1. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let K be a nonempty bounded closed convex subset of E, and every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive self-mappings. Let $f : K \to K$ a contractive mapping and $T : K \to K$ be a uniformly continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\} \subset (0, 1/2]$ be a sequence satisfying the conditions:*

(i) $\lim_{n\to\infty}\lambda_n = 0$,

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Define the sequence $\{x_n\}$ in K by

$$\begin{gathered} x_0 \in K, \\ x_{n+1} = \lambda_n f(x_n) + (1 - 2\lambda_n)x_n + \lambda_n T x_n, \quad \forall n \geq 0. \end{gathered} \tag{3.1}$$

If $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle f(p) - p, j(z - p)\rangle \leq 0, \quad \forall z \in F(T). \tag{3.2}$$

Proof. Take $p \in F(T)$, and let $S = I - T$. Then, we have

$$\langle Sx - Sy, j(x-y) \rangle \geq 0. \tag{3.3}$$

From (3.1), we obtain

$$\begin{aligned}
x_n &= x_{n+1} + \lambda_n x_n - \lambda_n T x_n + \lambda_n x_n - \lambda_n f(x_n) \\
&= x_{n+1} + \lambda_n x_n + \lambda_n S x_n - \lambda_n f(x_n) \\
&= x_{n+1} + \lambda_n \left[x_{n+1} + \lambda_n x_n + \lambda_n S x_n - \lambda_n f(x_n) \right] + \lambda_n S x_n - \lambda_n f(x_n) \\
&= (1+\lambda_n) x_{n+1} + \lambda_n^2 (x_n + S x_n) - \lambda_n^2 f(x_n) + \lambda_n S x_n - \lambda_n f(x_n) \\
&= (1+\lambda_n) x_{n+1} + \lambda_n S x_{n+1} + \lambda_n^2 (x_n + S x_n) - \lambda_n^2 f(x_n) \\
&\quad + \lambda_n (S x_n - S x_{n+1}) - \lambda_n f(x_n).
\end{aligned} \tag{3.4}$$

By (3.4), we have

$$\begin{aligned}
x_n - p = {} & (1+\lambda_n)(x_{n+1} - p) + \lambda_n (S x_{n+1} - Sp) + \lambda_n^2 (x_n + S x_n) \\
& - \lambda_n^2 f(x_n) + \lambda_n (S x_n - S x_{n+1}) + \lambda_n (p - f(x_n)).
\end{aligned} \tag{3.5}$$

Combining (3.3) and (3.5), we have

$$\begin{aligned}
& \left\langle x_n - p - \lambda_n^2 (x_n + S x_n) + \lambda_n^2 f(x_n) - \lambda_n (S x_n - S x_{n+1}) + \lambda_n (f(x_n) - p), j(x_{n+1} - p) \right\rangle \\
&\quad = (1+\lambda_n) \| x_{n+1} - p \|^2 + \lambda_n \langle S x_{n+1} - Sp, j(x_{n+1} - p) \rangle \\
&\quad \geq (1+\lambda_n) \| x_{n+1} - p \|^2.
\end{aligned} \tag{3.6}$$

Next, we prove that $\limsup_{n \to \infty} \langle f(p) - p, j(x_n - p) \rangle \leq 0$. Indeed, taking $z = f(p)$ in Lemma 2.1, we have

$$z_t - x_n = (1-t)(T z_t - x_n) + t(f(p) - x_n), \tag{3.7}$$

and, hence,

$$\begin{aligned}
\| z_t - x_n \|^2 &= (1-t) \langle T z_t - x_n, j(z_t - x_n) \rangle + t \langle f(p) - x_n, j(z_t - x_n) \rangle \\
&= (1-t) \langle T z_t - T x_n, j(z_t - x_n) \rangle + (1-t) \langle T x_n - x_n, j(z_t - x_n) \rangle \\
&\quad + t \langle f(p) - z_t, j(z_t - x_n) \rangle + t \| z_t - x_n \|^2 \\
&\leq \| z_t - x_n \|^2 + (1-t) \| T x_n - x_n \| \| z_t - x_n \| \\
&\quad + t \langle f(p) - z_t, j(z_t - x_n) \rangle.
\end{aligned} \tag{3.8}$$

Therefore, we have

$$\begin{aligned}\left\langle z_t - f(p), j(z_t - x_n)\right\rangle &\le \frac{1-t}{t}\|Tx_n - x_n\|\|z_t - x_n\| \\ &\le M_1\frac{1-t}{t}\|Tx_n - x_n\|,\end{aligned} \tag{3.9}$$

where $M_1 > 0$ is some constant such that $\|z_t - x_n\| \le M_1$ for all $t \in (0,1]$ and $n \ge 1$. Letting $n \to \infty$, we have

$$\limsup_{n\to\infty}\left\langle z_t - f(p), j(z_t - x_n)\right\rangle \le 0. \tag{3.10}$$

From Lemma 2.1, we know $z_t \to p$ as $t \to 0$. Since the duality mapping $J : E \to E^*$ is norm to weak star uniformly continuous from Lemma 2.2, we have

$$\limsup_{n\to\infty}\left\langle f(p) - p, j(x_n - p)\right\rangle \le 0. \tag{3.11}$$

From (3.6), we have

$$\begin{aligned}(1+\lambda_n)\left\|x_{n+1} - p\right\|^2 &\le \left\langle x_n - p - \lambda_n^2(x_n + Sx_n) + \lambda_n^2 f(x_n) - \lambda_n(Sx_n - Sx_{n+1}), j(x_{n+1} - p)\right\rangle \\ &\le \left\|x_n - p\right\|\left\|x_{n+1} - p\right\| + M_2\lambda_n^2 + M_2\lambda_n\|Sx_{n+1} - Sx_n\| \\ &\quad + \lambda_n\left\|f(x_n) - f(p)\right\|\left\|x_{n+1} - p\right\| + \lambda_n\left\langle f(p) - p, j(x_{n+1} - p)\right\rangle \\ &\le \left\|x_n - p\right\|\left\|x_{n+1} - p\right\| + M_2\lambda_n^2 + M_2\lambda_n\|Sx_{n+1} - Sx_n\| \\ &\quad + \lambda_n\alpha\left\|x_n - p\right\|\left\|x_{n+1} - p\right\| + \lambda_n\left\langle f(p) - p, j(x_{n+1} - p)\right\rangle \\ &\le \frac{1+\lambda_n\alpha}{2}\left(\left\|x_n - p\right\|^2 + \left\|x_{n+1} - p\right\|^2\right) + M_2\lambda_n^2 \\ &\quad + M_2\lambda_n\|Sx_{n+1} - Sx_n\| + \lambda_n\left\langle f(p) - p, j(x_{n+1} - p)\right\rangle,\end{aligned} \tag{3.12}$$

where M_2 is a constant such that

$$\sup\left\{\|x_n + Sx_n\|\left\|x_{n+1} - p\right\| + \left\|f(x_n)\right\|\left\|x_{n+1} - p\right\| + \left\|x_{n+1} - p\right\|,\ n \ge 0\right\} \le M_2. \tag{3.13}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \frac{1+\lambda_n\alpha}{1+(2-\alpha)\lambda_n}\|x_n-p\|^2 + M_2\lambda_n^2 + M_2\lambda_n\|Sx_{n+1}-Sx_n\| \\
&\quad + \frac{\lambda_n}{1+(2-\alpha)\lambda_n}\langle f(p)-p, j(x_{n+1}-p)\rangle \\
&= \left[1-\frac{2(1-\alpha)}{1+(2-\alpha)\lambda_n}\lambda_n\right]\|x_n-p\|^2 + \frac{2(1-\alpha)\lambda_n}{1+(2-\alpha)\lambda_n} \\
&\quad \times\left\{\frac{1+(2-\alpha)\lambda_n}{2(1-\alpha)}M_2\lambda_n + \frac{1+(2-\alpha)\lambda_n}{2(1-\alpha)}M_2\|Sx_{n+1}-Sx_n\|\right. \\
&\quad\quad \left. + \frac{1}{2(1-\alpha)}\langle f(p)-p, j(x_{n+1}-p)\rangle\right\} \\
&= (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\beta_n,
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
\alpha_n &= \frac{2(1-\alpha)}{1+(2-\alpha)\lambda_n}\lambda_n, \\
\beta_n &= \frac{1+(2-\alpha)\lambda_n}{2(1-\alpha)}M_2\lambda_n + \frac{1+(2-\alpha)\lambda_n}{2(1-\alpha)}M_2\|Sx_{n+1}-Sx_n\| \\
&\quad + \frac{1}{2(1-\alpha)}\langle f(p)-p, j(x_{n+1}-p)\rangle.
\end{aligned} \tag{3.15}$$

Note that

$$\|x_{n+1}-x_n\| \le \lambda_n\|x_n\| + \lambda_n\|Tx_n\| + \lambda_n\|x_n - f(x_n)\| \longrightarrow 0 \quad (n\longrightarrow\infty). \tag{3.16}$$

By the uniformly continuity of T, we have

$$\|Sx_{n+1}-Sx_n\| \longrightarrow 0 \quad (n\longrightarrow\infty). \tag{3.17}$$

Hence, it is clear that $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\limsup_{n\to\infty}\beta_n \le 0$.

Finally, applying Lemma 2.3 to (3.14), we can conclude that $x_n \to p$. This completes the proof. □

From Theorem 3.1, we can prove the following corollary.

Corollary 3.2. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm. Let K be a nonempty bounded closed convex subset of E, and every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive self-mappings. Let $T : K \to K$ be a uniformly continuous pseudocontractive mapping with $F(T)\neq\emptyset$. Let $\{\lambda_n\}\subset(0,1/2]$ be a sequence satisfying the conditions:*

(i) $\lim_{n\to\infty}\lambda_n = 0$,

(ii) $\sum_{n=0}^{\infty}\lambda_n = \infty$.

Define the sequence $\{x_n\}$ *in* K *by*

$$u, x_0 \in K,$$
$$x_{n+1} = \lambda_n u + (1-2\lambda_n)x_n + \lambda_n T x_n, \quad \forall n \geq 0. \tag{3.18}$$

Then, the sequence $\{x_n\}$ *converges strongly to a fixed point of* T *if and only if* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Theorem 3.3. *Let* E *be a uniformly smooth Banach space and* K *a nonempty bounded closed convex subset of* E*. Let* $f : K \to K$ *be a contractive mapping and* $T : K \to K$ *a uniformly continuous pseudocontractive mapping with* $F(T) \neq \emptyset$*. Let* $\{\lambda_n\} \subset (0, 1/2]$ *be a sequence satisfying the conditions:*

(i) $\lim_{n\to\infty}\lambda_n = 0$,

(ii) $\sum_{n=0}^{\infty}\lambda_n = \infty$.

If $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$*, then the sequence* $\{x_n\}$ *defined by* (3.1) *converges strongly to a fixed point* $p \in F(T)$*, which is the unique solution of the following variational inequality:*

$$\langle f(p) - p, j(z-p)\rangle \leq 0, \quad \forall z \in F(T). \tag{3.19}$$

Proof. Since every uniformly smooth Banach space E is reflexive and whose norm is uniformly Gâteaux differentiable, at the same time, every closed convex and bounded subset of K has the fixed point property for nonexpansive mappings. Hence, from Theorem 3.1, we can obtain the result. This completes the proof. □

From Theorem 3.3, we can prove the following corollary.

Corollary 3.4. *Let* E *be a uniformly smooth Banach space and* K *a nonempty bounded closed convex subset of* E*. Let* $T : K \to K$ *be a uniformly continuous pseudocontractive mapping with* $F(T) \neq \emptyset$*. Let* $\{\lambda_n\} \subset (0, 1/2]$ *be a sequence satisfying the conditions:*

(i) $\lim_{n\to\infty}\lambda_n = 0$,

(ii) $\sum_{n=0}^{\infty}\lambda_n = \infty$.

Define the sequence $\{x_n\}$ *in* K *by*

$$u, x_0 \in K,$$
$$x_{n+1} = \lambda_n u + (1-2\lambda_n)x_n + \lambda_n T x_n, \quad \forall n \geq 0. \tag{3.20}$$

Then, the sequence $\{x_n\}$ *converges strongly to a fixed point of* T *if and only if* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Theorem 3.5. *Let* K *be a nonempty bounded closed convex subset of a real reflexive Banach space* E *with a uniformly Gâteaux differentiable norm. Let* $f : K \to K$ *a contractive mapping and* $T : K \to K$ *be a uniformly continuous pseudocontractive mapping. Let* $\{\lambda_n\} \subset (0, 1/2]$ *be a sequence satisfying the conditions:*

(i) $\lim_{n\to\infty} \lambda_n = 0$,

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

If $D \cap F(T) \neq \emptyset$, where D is defined as (3.22) *below, then the sequence $\{x_n\}$ defined by* (3.1) *converges strongly to a fixed point $p \in F(T)$, which is the unique solution of the following variational inequality:*

$$\langle f(p) - p, j(z-p)\rangle \leq 0, \quad \forall z \in F(T). \tag{3.21}$$

Proof. First, we note that the sequence $\{x_n\}$ is bounded. Now, if we define $g(x) = \mu_n \|x_n - x\|^2$, then $g(x)$ is convex and continuous. Also, we can easily prove that $g(x) \to \infty$ as $\|x\| \to \infty$. Since E is reflexive, there exists $y \in K$ such that $g(y) = \inf_{x\in K} g(x)$. So the set

$$D = \left\{ y \in K : g(y) = \inf_{x\in K} g(x) \right\} \neq \emptyset. \tag{3.22}$$

Clearly, D is closed convex subset of K.

Now, we can take $p \in D \cap F(T)$ and $t \in (0,1)$. By the convexity of K, we have that $(1-t)p + tf(p) \in K$. It follows that

$$g(p) \leq g((1-t)p + tf(p)). \tag{3.23}$$

By Lemma 2.4, we have

$$\|x_n - p - t(f(p) - p)\|^2 \leq \|x_n - p\|^2 - 2t\langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle. \tag{3.24}$$

Taking the Banach limit in (3.24), we have

$$\mu_n\|x_n - p - t(f(p) - p)\|^2 \leq \mu_n\|x_n - p\|^2 - 2t\mu_n\langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle. \tag{3.25}$$

This implies

$$2t\mu_n\langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle \leq g(p) - g((1-t)p + tf(p)). \tag{3.26}$$

Therefore, it follows from (3.23) and (3.26) that

$$\mu_n\langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle \leq 0. \tag{3.27}$$

Since the normalized duality mapping j is single valued and norm-weak* uniformly continuous on bounded subset of E, we have

$$\langle f(p) - p, j(x_n - p)\rangle - \langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle \longrightarrow 0 \quad (t \longrightarrow 0). \tag{3.28}$$

This implies that, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for all $t \in (0, \delta)$ and $n \geq 1$,

$$\langle f(p) - p, j(x_n - p)\rangle - \langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle < \epsilon. \tag{3.29}$$

Taking the Banach limit and noting that (3.27), we have

$$\mu_n \langle f(p) - p, j(x_n - p)\rangle \leq \mu_n \langle f(p) - p, j(x_n - p - t(f(p) - p))\rangle + \epsilon \leq \epsilon. \tag{3.30}$$

By the arbitrariness of ϵ, we obtain

$$\mu_n \langle f(p) - p, j(x_n - p)\rangle \leq 0. \tag{3.31}$$

At the same time, we note that

$$\|x_{n+1} - x_n\| \leq \lambda_n \left(\|f(x_n)\| + 2\|x_n\| + \|Tx_n\| \right) \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.32}$$

Since $\{x_n - p\}$, $\{f(p) - p\}$ are bounded and the duality mapping j is single valued and norm topology to weak star topology uniformly continuous on bounded sets in Banach space E with a uniformly Gâteaux differentiable norm, it follows that

$$\lim_{n \to \infty} \{\langle f(p) - p, j(x_{n+1} - p)\rangle - \langle f(p) - p, j(x_n - p)\rangle\} = 0. \tag{3.33}$$

From (3.31), (3.33), and Lemma 2.5, we conclude that

$$\limsup_{n \to \infty} \langle f(p) - p, j(x_{n+1} - p)\rangle \leq 0. \tag{3.34}$$

Finally, by the similar arguments as that the proof in Theorem 3.1, it is easy prove that the sequence $\{x_n\}$ converges to a fixed point of T. This completes the proof. □

From Theorem 3.5, we can easily to prove the following result.

Corollary 3.6. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E with a uniformly Gâteaux differentiable norm. Let $f : K \to K$ be a contractive mapping and $T : K \to K$ a uniformly continuous pseudocontractive mapping. Let $\{\lambda_n\} \subset (0, 1/2]$ be a sequence satisfying the conditions:*

(i) $\lim_{n \to \infty} \lambda_n = 0$,

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Define the sequence $\{x_n\}$ in K by

$$\begin{gathered} u, x_0 \in K, \\ x_{n+1} = \lambda_n u + (1 - 2\lambda_n) x_n + \lambda_n T x_n, \quad \forall n \geq 0. \end{gathered} \tag{3.35}$$

If $D \cap F(T) \neq \emptyset$, where D is defined as (3.22)*, then the sequence $\{x_n\}$ defined by* (3.35) *converges strongly to a fixed point $p \in F(T)$.*

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