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Research Article

An Underactuated Drift-Free Left Invariant Control System on a Specific Lie Group

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The paper presents a geometrical overview on an optimal control problem on a special Lie group. The Hamilton-Poisson realization of the dynamics offers us the possibility to study the system from mechanical geometry point of view.

1. Introduction

Recent work in nonlinear control has drawn attention to drift-free systems with fewer degrees than state variables. These arise naturally in problems of motion planning for wheeled robots subject to nonholonomic controls [1], models of kinematic drift effects in space subjects to appendage vibrations or articulations [1], the molecular dynamics [2], the autonomous underwater vehicle dynamics [3], the car's dynamics [4], and spacecraft dynamics [5]. The purpose of our paper is to study a class of left-invariant, drift-free optimal control problems on a specific Lie group G. The class of all control-affine left-invariant, drift-free optimal control problems on G can be reduced to a class of two typical controllable left-invariant control systems on G. The left-invariant, drift-free optimal control problems involve finding a trajectory-control pair on G, which minimizes a cost function and satisfies the given dynamical constrains and boundary conditions in a fixed time. The problem is lifted to the cotangent bundle T^*G using the optimal Hamiltonian on $\mathcal{G}^*$, where the maximum principle yields the optimal control. The energy-Casimir method is used to give sufficient conditions for nonlinear stability of the equilibrium states. Around this equilibrium states, we are able to find the periodical orbits using Moser's theorem. In the last paragraph, we have studied the numerical integration via three methods: Lie-Trotter algorithm, Kahan's algorithms, and Runge-Kutta 4th method. Numerical simulations and a comparison between these three methods are presented too.

2. The Geometrical Picture of the Problem

Let G be the Lie group given by

$$G = \left\{ \begin{bmatrix} 1 & x_2 & x_4 \\ 0 & e^{-x_1} & x_3 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_3(\mathbb{R}) \mid x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}. \tag{2.1}$$

Proposition 2.1. *The Lie algebra $\mathcal{G}$ of G is generated by*

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.2}$$

and the Lie algebra structure of $\mathcal{G}$ is given by the following

$[\cdot,\cdot]$	A_1	A_2	A_3	A_4
A_1	0	A_2	$-A_3$	0
A_2	$-A_2$	0	A_4	0
A_3	A_3	$-A_4$	0	0
A_4	0	0	0	0

(2.3)

Proposition 2.2. *The minus-Lie-Poisson structure on $\mathcal{G}^* \simeq (R^4)^* \simeq R^4$ is generated by the matrix*

$$\Pi_- = \begin{bmatrix} 0 & -x_2 & x_3 & 0 \\ x_2 & 0 & -x_4 & 0 \\ -x_3 & x_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.4}$$

An easy computation leads one to the following.

Proposition 2.3. *The following two systems are drift-free-left invariant controllable systems on G, namely:*

$$\dot{X} = X(A_1u_1 + A_2u_2 + A_3u_3), \tag{2.5}$$

$$\dot{X} = X(A_1u_1 + A_2u_2 + A_3u_3 + A_4u_4), \tag{2.6}$$

where $X \in G$, A_i are the matrix defined above, and $u_i \in C^\infty(\mathbb{R}, \mathbb{R})$, $i = \overline{1,4}$.

Proof. Since the span of the set of Lie brackets generated by A_1, A_2, and A_3 coincides with $\mathcal{G}$ the proposition is a consequence of a result due to Jurdjevic and Sussmann [6]. □

3. An Optimal Control Problem for the System (2.5)

Let J be the cost function given by

$$J(u_1, u_2, u_3) = \frac{1}{2}\int_0^{t_f}\left[c_1 u_1^2(t) + c_2 u_2^2(t) + c_3 u_3^2(t)\right]dt \quad c_1 > 0,\ c_2 > 0,\ c_3 > 0. \tag{3.1}$$

Then we have the following.

Proposition 3.1. *The controls that minimize J and steer the system* (2.5) *from $X = X_0$ at $t = 0$ to $X = X_f$ at $t = t_f$ are given by*

$$u_1 = \frac{1}{c_1}x_1, \qquad u_2 = \frac{1}{c_2}x_2, \qquad u_3 = \frac{1}{c_3}x_3, \tag{3.2}$$

where x_i's are solutions of

$$\begin{aligned}
\dot{x}_1 &= -\frac{1}{c_2}x_2^2 + \frac{1}{c_3}x_3^2,\\
\dot{x}_2 &= \frac{1}{c_1}x_1x_2 - \frac{1}{c_3}x_3x_4,\\
\dot{x}_3 &= -\frac{1}{c_1}x_1x_3 + \frac{1}{c_2}x_2x_4,\\
\dot{x}_4 &= 0.
\end{aligned} \tag{3.3}$$

Proof. Let us consider the optimal Hamiltonian given by

$$H(x_1, x_2, x_2, x_4) = \frac{1}{2}\left(\frac{x_1^2}{c_1} + \frac{x_2^2}{c_2} + \frac{x_3^2}{c_3}\right). \tag{3.4}$$

It is in fact the controlled Hamiltonian H_{opt} given by

$$H_{\text{opt}} = x_1u_1 + x_2u_2 + x_3u_3 - \frac{1}{2}\left(c_1u_1^2 + c_2u_2^2 + c_3u_3^2\right), \tag{3.5}$$

which is reduced to $\mathcal{G}^*$ via Poisson reduction. Then the optimal controls are given by

$$u_1 = \frac{1}{c_1}x_1, \qquad u_2 = \frac{1}{c_2}x_2, \qquad u_1 = \frac{1}{c_3}x_3, \tag{3.6}$$

where x_i*'s* are solutions of the reduced Hamilton's equations given by

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^t = \Pi_- \cdot \nabla H, \tag{3.7}$$

which is nothing else than the required (3.3). □

Remark 3.2. It is easy to see from (3.3) that $x_4 =$ constant, and so the dynamics (3.3) can be put in the equivalent form

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{c_2}x_2^2 + \frac{1}{c_3}x_3^2, \\ \dot{x}_2 &= \frac{1}{c_1}x_1x_2 - \frac{k}{c_3}x_3, \\ \dot{x}_3 &= -\frac{1}{c_1}x_1x_3 + \frac{k}{c_2}x_2. \end{aligned} \tag{3.8}$$

The goal of our paper is to study some geometrical and dynamical properties for the system (3.8).

Proposition 3.3. *The dynamics* (3.8) *has the following Hamilton-Poisson realization:*

$$\left(\mathbb{R}^3, \Pi, H\right), \tag{3.9}$$

where

$$\Pi = \begin{bmatrix} 0 & -x_2 & x_2 \\ x_2 & 0 & -k \\ -x_3 & k & 0 \end{bmatrix} \tag{3.10}$$

and the Hamiltonian

$$H(x_1, x_2, x_3) = \frac{1}{2}\left(\frac{x_1^2}{c_1} + \frac{x_2^2}{c_2} + \frac{x_3^2}{c_3}\right). \tag{3.11}$$

Proof. Indeed, it is not hard to see that the dynamics (3.8) can be put in the equivalent form

$$[\dot{x}_1, \dot{x}_2, \dot{x}_3]^t = \Pi \cdot \nabla H, \tag{3.12}$$

as required. Moreover, the function C given by

$$C = kx_1 + x_2x_3 \tag{3.13}$$

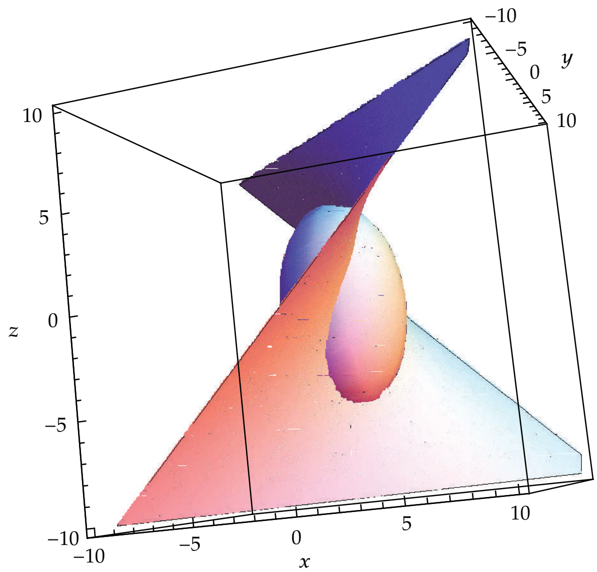

Figure 1: The phase curves of the system (3.8) ($c_1 = 1$, $c_2 = 2$, $c_3 = 3$, and $k = 10$).

is a Casimir of our configuration. Indeed,

$$(\nabla C)^t \Pi = 0, \tag{3.14}$$

as desired. □

Remark 3.4. The phase curves of the dynamics (3.8) are intersections of

$$\frac{x_1^2}{c_1} + \frac{x_2^2}{c_2} + \frac{x_3^2}{c_3} = \text{const.}, \tag{3.15}$$

with

$$kx_1 + x_2 x_3 = \text{const.}, \tag{3.16}$$

see Figure 1.

Proposition 3.5. *The dynamics* (3.8) *has an infinite number of Hamilton-Poisson realizations.*

Proof. An easy computation shows us that the triples

$$\left(\mathbb{R}^3, \{\cdot,\cdot\}_{ab}, H_{cd}\right), \tag{3.17}$$

where

$$\{f,g\}_{ab} = -\nabla C_{ab}\cdot(\nabla f\times\nabla g),\quad (\forall) f,g\in C^{\infty}\left(\mathbb{R}^3,\mathbb{R}\right),$$
$$C_{ab} = aC + bH,\qquad H_{cd} = cC + dH,\quad a,b,c,d\in\mathbb{R},\ ad-bc=1, \tag{3.18}$$

define Hamilton-Poisson realizations of the dynamics (3.8), as required. □

Remark 3.6. The above proposition tells us in fact that (3.8) is unchanged, so the trajectories of motion in $\mathbb{R}^3$ remain the same when H and C are replaced by linear combinations of H and C with coefficients which form a real matrix with det one.

4. Stability and Periodical Orbits

It is not hard to see that the equilibrium states of our dynamics (3.8) are

$$\begin{aligned} e_1^M &= (M,0,0),\quad M\in\mathbb{R},\\ e_2^M &= \left(-\frac{kc_1}{\sqrt{c_2c_3}},-\sqrt{\frac{c_2}{c_3}}M,M\right),\quad M\in\mathbb{R},\\ e_3^M &= \left(\frac{kc_1}{\sqrt{c_2c_3}},\sqrt{\frac{c_2}{c_3}}M,M\right),\quad M\in\mathbb{R}. \end{aligned} \tag{4.1}$$

Let A be the matrix of the linear part of the system (3.8), that is,

$$\mathrm{A} = \begin{pmatrix} 0 & -\dfrac{2}{c_2}x_2 & \dfrac{2}{c_3}x_3 \\ \dfrac{1}{c_1}x_2 & \dfrac{1}{c_1}x_1 & -\dfrac{k}{c_3} \\ -\dfrac{1}{c_1}x_3 & \dfrac{k}{c_2} & -\dfrac{1}{c_1}x_1 \end{pmatrix}, \tag{4.2}$$

then the characteristic roots of $A(e_1^M)$, respectively, $A(e_2^M)$, respectively, and $A(e_3^M)$ are given by

$$\lambda_1 = 0,\qquad \lambda_{2,3} = \pm\frac{\sqrt{c_2c_3M^2 - c_1k^2}}{c_1\sqrt{c_2c_3}}, \tag{4.3}$$

respectively,

$$\lambda_1 = 0,\qquad \lambda_{2,3} = \pm\frac{2i}{\sqrt{c_1c_3}}M, \tag{4.4}$$

respectively,

$$\lambda_1 = 0, \qquad \lambda_{2,3} = \pm \frac{2i}{\sqrt{c_1 c_3}} M, \tag{4.5}$$

so we can conclude with the following.

Proposition 4.1. *The equilibrium states e_1^M, $M \in \mathbb{R}$, are spectrally stable if $M \in (-(c_1 k/\sqrt{c_2 c_3}), (c_1 k/\sqrt{c_2 c_3}))$.*

Proposition 4.2. *The equilibrium states e_2^M and e_3^M, $M \in \mathbb{R}$, are spectrally stable for any $M \in R$.*

We can now pass to discuss the nonlinear stability of the equilibrium states e_1^M, e_2^M, and e_3^M, $M \in \mathbb{R}$.

Proposition 4.3. *The equilibrium states e_1^M, $M \in \mathbb{R}^*$, are nonlinearly stable if $M \in (-(c_1 k/\sqrt{c_2 c_3}), (c_1 k/\sqrt{c_2 c_3}))$.*

Proof. We will make the proof using energy-Casimir method (see [7]). Let

$$H_\varphi = H + \varphi(C) = \frac{x_1^2}{2c_1} + \frac{x_2^2}{2c_2} + \frac{x_3^2}{2c_3} + \varphi(kx_1 + x_2 x_3) \tag{4.6}$$

be the energy-Casimir function, where $\varphi : R \to R$ is a smooth real-valued function defined on R.

Now, the first variation of H_φ is given by

$$\delta H_\varphi = \frac{x_1}{c_1}\delta x_1 + \frac{x_2}{c_2}\delta x_2 + \frac{x_3}{c_3}\delta x_3 + \dot{\varphi}(kx_1 + x_2 x_3) \cdot (k\delta x_1 + x_2 \delta x_3 + x_3 \delta x_2), \tag{4.7}$$

where

$$\dot{\varphi} = \frac{\partial \varphi}{\partial (kx_1 + x_2 x_3)}. \tag{4.8}$$

This equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}(kM) = \frac{-M}{kc_1}. \tag{4.9}$$

The second variation of H_φ is given by

$$\delta^2 H_\varphi = \frac{1}{c_1}(\delta x_1)^2 + \frac{1}{c_2}(\delta x_2)^2 + \frac{1}{c_3}(\delta x_3)^2 + \ddot{\varphi} \cdot (k\delta x_1 + x_2 \delta x_3 + x_3 \delta x_2)^2 + 2\dot{\varphi} \cdot \delta x_2 \delta x_3. \tag{4.10}$$

Since $M \in (-(c_1k/\sqrt{c_2c_3}), (c_1k/\sqrt{c_2c_3}))$ and if we choose the function φ such that

$$\begin{aligned} \dot{\varphi}(kM) &= -\frac{M}{kc_1}, \\ \ddot{\varphi}(kM) &> 0, \end{aligned} \tag{4.11}$$

we can conclude that the second variation of H_φ at the equilibrium of interest is positive definite and, thus, e_1^M are nonlinearly stable. □

Similar arguments lead us to the following result.

Proposition 4.4. *The equilibrium states* $e_2^M, M \in \mathbb{R}^*$ *and* $e_3^M, M \in \mathbb{R}^*$ *are nonlinearly stable for any* $M \in \mathbb{R}^*$.

In order to find the periodical orbits around the equilibrium states e_1^M, we make use of the property that the dynamics described by a Hamilton-Poisson system takes place on the symplectic leaves of the Poisson configuration manifold, to prove the existence of periodic orbits by looking for periodic orbits of the symplectic Hamiltonian completely integrable system obtained by the restriction of the Lorenz system to the regular coadjoint orbits of G^*. This procedure will be implemented around nonlinearly stable equilibrium states. The procedure is the following: we consider the system restricted to a regular coadjoint orbit of G^* that contains a nonlinearly stable equilibrium, and then we will get the existence of periodic solutions for the restricted system. These periodic solutions are periodic solutions also for the unrestricted system.

Proposition 4.5. *Near* $e_1^M = (M, 0, 0)$, $M \in (-(c_1k/\sqrt{c_2c_3}), (c_1k/\sqrt{c_2c_3}))$, *the reduced dynamics has, for each sufficiently small value of the reduced energy, at least 1 periodic solution whose period is close to*

$$\frac{2\pi c_1\sqrt{c_2c_3}}{\sqrt{k^2c_1^2 - c_2c_3M^2}}. \tag{4.12}$$

Proof. Indeed, we have successively the following:

(i) the restriction of our dynamics (3.8) to the coadjoint orbit

$$kx_1 + x_2x_3 = kM \tag{4.13}$$

gives rise to a classical Hamiltonian system,

(ii) the matrix of the linear part of the reduced dynamics has purely imaginary roots. More exactly

$$\lambda_{2,3} = \pm i\frac{\sqrt{k^2c_1^2 - c_2c_3M^2}}{c_1\sqrt{c_2c_3}}, \tag{4.14}$$

(iii) consider the following:

$$\operatorname{span}\left(\nabla C\left(e_1^M\right)\right) = V_0, \tag{4.15}$$

where

$$V_0 = \ker\left(A\left(e_1^M\right)\right) = \operatorname{span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{4.16}$$

(iv) the reduced Hamiltonian has a local minimum at the equilibrium state e_1^M (see the proof of Proposition 4.3).

Then our assertion follows via the Moser-Weinstein theorem with zero eigenvalue; see [8] for details. □

Remark 4.6. The existence of the periodical orbits around the equilibrium points e_2 and e_3 remains an open problem, the Moser-Weinstein theorem with zero eigenvalue being inconclusive.

5. Lax Formulation and Numerical Integration of the Dynamics (3.8)

Proposition 5.1. *The dynamics* (3.8) *allows a formulation in terms of Lax pairs.*

Proof. Let us take the following:

$$L = \begin{bmatrix} 0 & l_{12} & l_{13} \\ -l_{12} & 0 & l_{23} \\ -l_{13} & -l_{23} & 0 \end{bmatrix},$$

$$l_{12} = \frac{2\sqrt{c_3}x_2}{c_2\sqrt{c_1}} + \frac{2x_3}{\sqrt{c_1c_2}} + \frac{\sqrt{2}\left(\sqrt{c_2} - 2\sqrt{2 - c_1c_2}\right)}{c_2\sqrt{c_1}}k - \frac{8\sqrt{c_3}}{c_2\sqrt{c_1}},$$

$$l_{13} = -\frac{2\sqrt{c_3}x_1}{\sqrt{c_1}} - \frac{\sqrt{2c_3(2 - c_1c_2)}}{c_2\sqrt{c_1}}x_2 - \sqrt{\frac{2(2 - c_1c_2)}{c_1c_2}}x_3 + \frac{c_2\sqrt{2 - c_1c_2} - 4}{c_2\sqrt{c_1c_2}}k - \frac{4\sqrt{2c_3(2 - c_1c_2)}}{c_2\sqrt{c_1}},$$

$$l_{23} = -\frac{2\sqrt{c_3(2 - c_1c_2)}}{c_1\sqrt{c_2}}x_1 + \sqrt{\frac{2c_3}{c_2}}x_2 + \sqrt{2}x_3 + 4\sqrt{\frac{2c_3}{c_2}} - k,$$

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{bmatrix},$$

$$b_{12} = \frac{c_2\left(2\sqrt{2}x_3 + k\right) - 2\sqrt{2 - c_1 c_2}k - 4\sqrt{2c_2 c_3}}{2c_2\sqrt{c_1 c_3}},$$

$$b_{13} = \frac{-2\sqrt{2}(c_2\sqrt{c_2 c_3}x_1 + 2k) + \sqrt{c_2(2 - c_1 c_2)}\left(-4\sqrt{c3}(2 + x_2) + 2\sqrt{c_2}k\right)}{4c_2\sqrt{c_1 c_3}},$$

$$b_{23} = -\frac{\sqrt{2 - c_1 c_2}}{c_1\sqrt{2}}x_1 + x_2 - \frac{\sqrt{c_2}}{2\sqrt{2c_3}}k + 2, \tag{5.1}$$

then, using MATHEMATICA 7.0, we can put the system (3.8) in the equivalent form

$$\dot{L} = [L, B], \tag{5.2}$$

as desired. □

We will discuss now the numerical integration of the dynamics (3.8) via the Kahan integrator, Lie-Trotter integrator [9], and also via Runge-Kutta 4th steps integrator, and we will point out some properties of Kahan and Lie-Trotter integrators.

It is easy to see that for the equations (3.8), Kahan's integrator can be written in the following form:

$$\begin{aligned} x_1^{n+1} - x_1^n &= -\frac{h}{c_2}x_2^{n+1}x_2^n + \frac{h}{c_3}x_3^{n+1}x_3^n, \\ x_2^{n+1} - x_2^n &= \frac{h}{2c_1}\left(x_1^{n+1}x_2^n + x_2^{n+1}x_1^n\right) - \frac{hk}{2c_3}\left(x_3^n + x_3^{n+1}\right), \\ x_3^{n+1} - x_3^n &= -\frac{h}{2c_1}\left(x_1^{n+1}x_3^n + x_3^{n+1}x_1^n\right) + \frac{hk}{2c_2}\left(x_2^n + x_2^{n+1}\right). \end{aligned} \tag{5.3}$$

Using MATHEMATICA 8.0, we can prove the following proposition which shows the incompatibility of the Kahan's integrator with the Poisson structure of the system (3.8).

Proposition 5.2. *Kahan's integrator* (5.3) *has the following properties:*

(i) *it is not Poisson preserving;*

(ii) *it does not preserve the Casimir C of the Poisson configuration* $(\mathbb{R}^3, \Pi)$*;*

(iii) *it does not preserve the Hamiltonian H of the system* (3.8)*.*

We will discuss now the numerical integration of the dynamics (3.8) via the Lie-Trotter integrator.

To begin with, let us observe that the Hamiltonian vector field X_H splits as follows:

$$X_H = X_{H_1} + X_{H_2} + X_{H_3}, \tag{5.4}$$

where

$$H_1 = \frac{1}{2c_1}x_1^2, \qquad H_2 = \frac{1}{2c_3}x_2^2, \qquad H_3 = \frac{1}{2c_3}x_3^2. \tag{5.5}$$

Their corresponding integral curves are, respectively, given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = A_i \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}, \quad i = 1, 3, \tag{5.6}$$

where

$$\begin{gathered} A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{(a/c_1)t} & 0 \\ 0 & 0 & e^{-(a/c_1)t} \end{bmatrix}, \\ a = x_1(0), \\ A_2 = \begin{bmatrix} 1 & -\dfrac{b}{c_2}t & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{k}{c_2}t & 1 \end{bmatrix}, \\ b = x_2(0), \\ A_3 = \begin{bmatrix} 1 & 0 & \dfrac{c}{c_3}t \\ 0 & 1 & -\dfrac{k}{c_3}t \\ 0 & 0 & 1 \end{bmatrix}, \\ c = x_3(0). \end{gathered} \tag{5.7}$$

Then the Lie-Trotter integrator is given by

$$\begin{aligned} x_1^{n+1} &= x_1^n - \frac{b}{c_2}tx_2^n + \left(\frac{c}{c_3}t + \frac{kb}{c_2c_3}t^2\right)x_3^n, \\ x_2^{n+1} &= e^{(a/c_1)t}x_2^n - \frac{k}{c_3}te^{(a/c_1)t}x_3^n, \\ x_3^{n+1} &= \frac{k}{c_2}te^{(-a/c_1)t}x_2^n - \frac{k^2}{c_2c_3}t^2e^{(-a/c_1)t}x_3^n. \end{aligned} \tag{5.8}$$

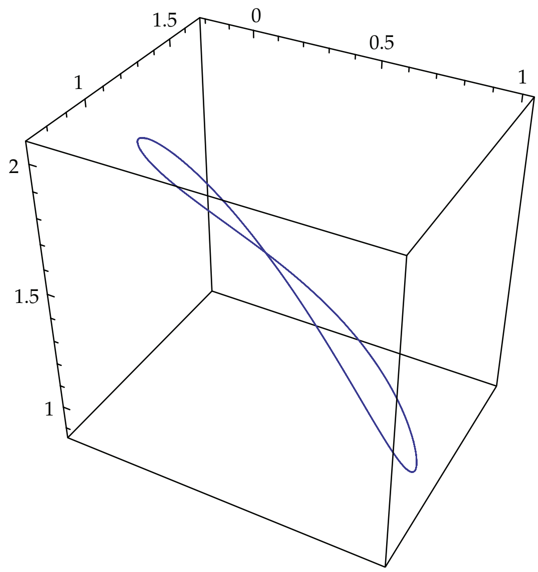

Figure 2: The 4th-step Runge-Kutta integrator of the system (3.8) ($c_1 = 1, c_2 = 2, c_3 = 3, k = 10$, and $x_1(0) = x_2(0) = x_3(0) = 1$).

Now, a direct computation or using MATHEMATICA leads us to the following.

Proposition 5.3. *The Lie-Trotter integrator* (5.8) *has the following properties:*

(i) *it preserves the Poisson structure* Π*;*

(ii) *it preserves the Casimir C of the Poisson configuration* $(\mathbb{R}^3, \Pi)$*;*

(iii) *it does not preserve the Hamiltonian* H *of the system* (3.8)*;*

(iv) *its restriction to the coadjoint orbit* $(\mathcal{O}_k, \omega_k)$*, where*

$$\mathcal{O}_k = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid kx_1 + x_2 x_3 = \text{const.} \right\} \tag{5.9}$$

and ω_k *is the Kirillov-Kostant-Souriau symplectic structure on* $\mathcal{O}_k$*, gives rise to a symplectic integrator.*

Remark 5.4. If we compare these methods, see Figures 2, 3, and 4, with the 4th-step Runge-Kutta method, we can see that Lie-Trotter integrator and Kahan's integrator give us a weak approximation of our dynamics. However, Kahan's integrator and the Lie-Trotter integrator have the advantage of being more easily implemented.

6. Conclusion

The paper analyses a drift-free left invariant controllable system on a special Lie group. The Hamilton-Poisson realization of the system allows us to study the system from the

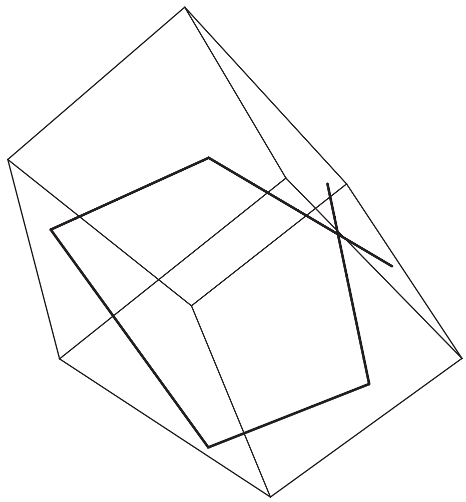

Figure 3: The Kahan's integrator of the system (3.8) ($c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $k = 10$, and $h = 1$, $x_1(0) = x_2(0) = x_3(0) = 1$).

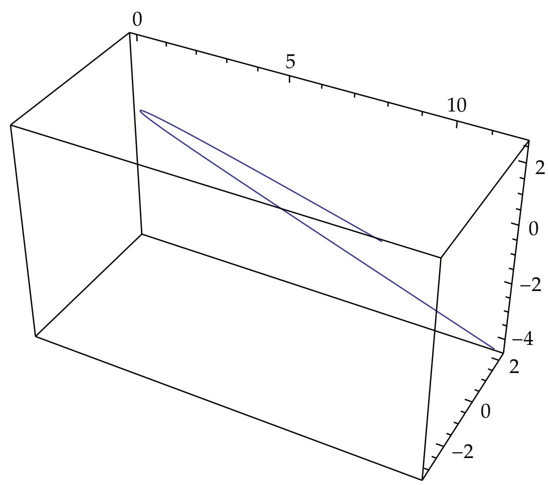

Figure 4: The Lie-Trotter integrator of the system (3.8) ($c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $k = 10$, and $x_1(0) = x_2(0) = x_3(0) = 1$).

mechanical geometry point of view. This means that we can use specific tools as energy-Casimir method for nonlinear stability, the Moser's theorem to find the periodical orbits, and Poisson integrators to make the numerical integration of the dynamics. In addition, we use non-Poisson integrators (Kahan's integrator and Runge-Kutta 4th-step integrator) to make a comparison between the obtained results. Numerical simulations via MATHEMATICA 8.0 are presented too. Similar problems have been studied on the Lie groups $SO(3)$, $SO(4)$ (see [10]), on the Heisenberg Lie groups $H(3)$ and $H(4)$, or $SE(2,\mathbb{R}) \times SO(3)$ (see [11]).

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References

[1] N. E. Leonard, *Averaging and motion control systems on Lie groups*, Ph.D. thesis, University of Maryland, College Park, Md, USA, 1994.

[2] M. Dahleh, A. Peirce, H. A. Rabitz, and V. Ramakrishna, "Control of molecular motion," *Proceedings of the IEEE*, vol. 84, no. 1, pp. 7–15, 1996.

[3] A. Aron, P. Birtea, L. Dobre, M. Puta, and R. Tudoran, "Some remarks on the underwater vehicle dynamics with a fixed control authority," in *Modern Trends in Geometry and Topology*, D. Andrica, P. A. Blaga, and S. Moroianu, Eds., pp. 55–61, Cluj University Press, 2006.

[4] M. Puta, "Geometrical aspects in the kinematics car's dynamics," in *Proceedings of the Mathematical Theory of Networks and Systems Symposium (MTNS '98)*, A. Beghi, L. Finesso, and G. Picci, Eds., Mathematical Theory of Networks and Systems, pp. 1063–1066, Il Poligrafo, Padova, Italy, July 1998.

[5] R. M. Murray and S. S. Sastry, "Nonholonomic motion planning: steering using sinusoids," *IEEE Transactions on Automatic Control*, vol. 38, no. 5, pp. 700–716, 1998.

[6] V. Jurdjevic and H. J. Sussmann, "Control systems on Lie groups," *Journal of Differential Equations*, vol. 12, pp. 313–329, 1972.

[7] P. Birtea and M. Puta, "Equivalence of energy methods in stability theory," *Journal of Mathematical Physics*, vol. 48, no. 4, pp. 81–99, 2007.

[8] P. Birtea, M. Puta, and R. M. Tudoran, "Periodic orbits in the case of a zero eigenvalue," *Comptes Rendus Mathématique: Académie des Sciences. Paris*, vol. 344, no. 12, pp. 779–784, 2007.

[9] M. Puta, "Lie-Trotter formula and Poisson dynamics," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 9, no. 3, pp. 555–559, 1999.

[10] A. Aron, I. Mos, A. Csaky, and M. Puta, "An optimal control problem on the Lie group $SO(4)$," *International Journal of Geometric Methods in Modern Physics*, vol. 5, no. 3, pp. 319–327, 2008.

[11] A. Aron, C. Pop, and M. Puta, "An optimal control problem on the Lie group $SE(2,\mathbb{R})\times SO(2)$," *Boletin de la Sociedad Matematica Mexicana*, vol. 15, no. 2, pp. 129–140, 2009.