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Research Article

Almost Sure Central Limit Theory for Self-Normalized Products of Sums of Partial Sums

Qunying Wu[1, 2]

[1] *College of Science, Guilin University of Technology, Guilin 541004, China*
[2] *Guangxi Key Laboratory of Spatial Information and Geomatics, Guilin University of Technology, Guilin 541004, China*

Correspondence should be addressed to Qunying Wu, wqy666@glite.edu.cn

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Let $X, X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables in the domain of attraction of a normal law. An almost sure limit theorem for the self-normalized products of sums of partial sums is established.

1. Introduction

Let $\{X, X_n\}_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed (i.i.d.) positive random variables with a nondegenerate distribution function and $\mathbb{E}X = \mu > 0$. For each $n \geq 1$, the symbol S_n/V_n denotes self-normalized partial sums, where $S_n = \sum_{i=1}^{n} X_i$ and $V_n^2 = \sum_{i=1}^{n}(X_i - \mu)^2$. We say that the random variable X belongs to the domain of attraction of the normal law if there exist constants $a_n > 0$, $b_n \in \mathbb{R}$ such that

$$\frac{S_n - b_n}{a_n} \xrightarrow{d} \mathcal{N}, \quad \text{as } n \longrightarrow \infty, \tag{1.1}$$

here and in the sequel $\mathcal{N}$ is a standard normal random variable. We say that $\{X, X_n\}_{n\in\mathbb{N}}$ satisfies the central limit theorem (CLT).

It is known that (1.1) holds if and only if

$$\lim_{x\to\infty} \frac{x^2 \mathbb{P}(|X| > x)}{\mathbb{E}X^2 I(|X| \leq x)} = 0. \tag{1.2}$$

In contrast to the well-known classical central limit theorem, Gine et al. [1] obtained the following self-normalized version of the central limit theorem: $(S_n - \mathbb{E}S_n)/V_n \xrightarrow{d} \mathcal{N}$ as $n \to \infty$ if and only if (1.2) holds.

Brosamler [2] and Schatte [3] obtained the following almost sure central limit theorem (ASCLT): let $\{X_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables with mean 0, variance $\sigma^2 > 0$, and partial sums S_n. Then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left\{\frac{S_k}{\sigma\sqrt{k}} < x\right\} = \Phi(x) \quad \text{a.s. } \forall x\in\mathbb{R}, \tag{1.3}$$

with $d_k = 1/k$ and $D_n = \sum_{k=1}^{n} d_k$; here and in the sequel I denotes an indicator function, and $\Phi(x)$ is the standard normal distribution function. Some ASCLT results for partial sums were obtained by Lacey and Philipp [4], Ibragimov and Lifshits [5], Miao [6], Berkes and Csáki [7], Hörmann [8], Wu [9, 10], and Ye and Wu [11]. Huang and Pang [12] and Zhang and Yang [13] obtained ASCLT results for self-normalized version. However, ASCLT results for self-normalized products of sums of partial sums have not been reported yet.

Under mild moment conditions, ASCLT follows from the ordinary CLT, but in general the validity of ASCLT is a delicate question of a totally different character as CLT. The difference between CLT and ASCLT lies in the weight in ASCLT. The terminology of summation procedures (see e.g., Chandrasekharan and Minakshisundaram [14], page 35) shows that the larger the weight sequence $\{d_k; k \ge 1\}$ in (1.3) is, the stronger the relation becomes. By this argument, one should also expect to get stronger results if we use larger weights. And it would be of considerable interest to determine the optimal weights.

On the other hand, by the Theorem 1 of Schatte [3], (1.3) fails for weight $d_k = 1$. The optimal weight sequence remains unknown.

The purpose of this paper is to study and establish the ASCLT for self-normalized products of sums of partial sums of random variables in the domain of attraction of the normal law, we will show that the ASCLT holds under a fairly general growth condition on $d_k = k^{-1}\exp((\ln k)^\alpha)$, $0 \le \alpha < 1/2$.

In the following, we assume that $\{X, X_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. positive random variables in the domain of attraction of the normal law with $\mathbb{E}X = \mu > 0$ and define $S_k = \sum_{i=1}^{k} X_i$, $V_k = \sum_{i=1}^{k}(X_i-\mu)^2$, $T_k = \sum_{i=1}^{k} S_i$. Let $b_{n,k} = \sum_{j=k}^{n} 1/j$, $c_{n,k} = 2\sum_{j=k}^{n}(j+1-k)/(j(j+1))$, and $d_{n,k} = (n+1-k)/(n+1)$ for $1 \le k \le n$. $I(A)$ denotes the indicator function of set A, and $a_n \sim b_n$ denotes $a_n/b_n \to 1, n\to\infty$. The symbol c stands for a generic positive constant, which may differ from one place to another. Let

$$\begin{gathered} l(x) = \mathbb{E}(X-\mu)^2 I\{|X-\mu| \le x\}, \qquad b = \inf\{x \ge 1; l(x) > 0\}, \\ \eta_j = \inf\left\{s; s \ge b+1, \frac{l(s)}{s^2} \le \frac{1}{j}\right\} \quad \text{for } j \ge 1. \end{gathered} \tag{1.4}$$

By the definition of η_j, we have $jl(\eta_j) \le \eta_j^2$ and $jl(\eta_j - \varepsilon) > (\eta_j - \varepsilon)^2$ for any $\varepsilon > 0$. It implies that

$$nl(\eta_n) \sim \eta_n^2, \quad \text{as } n \to \infty, \eta_n < n+1. \tag{1.5}$$

Our theorem is formulated in a more general setting.

Theorem 1.1. *Let $\{X, X_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. positive random variables in the domain of attraction of the normal law with $\mathbb{E}X = \mu > 0$. Suppose*

$$l(\eta_n) \sim l\left(\frac{\eta_n}{\ln n}\right). \tag{1.6}$$

For $0 \le \alpha < 1/2$, set

$$d_k = \frac{\exp(\ln^\alpha k)}{k}, \qquad D_n = \sum_{k=1}^{n} d_k. \tag{1.7}$$

Then

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left(\left(\prod_{j=1}^{k}\left(\frac{T_j}{j(j+1)\mu/2}\right)\right)^{\mu/V_k} \le x\right) = F(x) \quad a.s., \tag{1.8}$$

for any $x \in \mathbb{R}$, where F is the distribution function of the random variable $\exp(\sqrt{10/3}\mathcal{N})$.

By the terminology of summation procedures, we have the following corollary.

Corollary 1.2. *Theorem 1.1 remains valid if we replace the weight sequence $\{d_k\}_{k\in\mathbb{N}}$ by $\{d_k^*\}_{k\in\mathbb{N}}$ such that $0 \le d_k^* \le d_k$, $\sum_{k=1}^{\infty} d_k^* = \infty$.*

Remark 1.3. If $\mathbb{E}X^2 = \sigma^2 < \infty$, then X is in the domain of attraction of the normal law and $l(x) \to \sigma^2, \eta_n^2 \sim \sigma^2 n$, thus (1.6) *holds. Therefore, the class of random variables in Theorems* 1.1 *is of very broad range.*

Remark 1.4. Whether Theorem 1.1 *holds for $1/2 \le \alpha < 1$ remains open.*

2. Proofs

Furthermore, the following four lemmas will be useful in the proof, and the first is due to Csörgo et al. [15].

Lemma 2.1. *Let X be a random variable with $\mathbb{E}X = \mu$, and denote $l(x) = \mathbb{E}(X-\mu)^2 I\{|X-\mu| \le x\}$. The following statements are equivalent:*

(i) *$l(x)$ is a slowly varying function at ∞;*

(ii) *X is in the domain of attraction of the normal law;*

(iii) *$x^2\mathbb{P}(|X-\mu| > x) = o(l(x))$;*

(iv) *$x\mathbb{E}(|X-\mu| I(|X-\mu| > x)) = o(l(x))$;*

(v) *$\mathbb{E}(|X-\mu|^\alpha I(|X-\mu| \le x)) = o(x^{\alpha-2} l(x))$ for $\alpha > 2$.*

Lemma 2.2. *Let $\{\xi, \xi_n\}_{n\in\mathbb{N}}$ be a sequence of uniformly bounded random variables. If there exist constants $c > 0$ and $\delta > 0$ such that*

$$\left|\mathbb{E}\xi_k\xi_j\right| \le c\left(\frac{k}{j}\right)^{\delta}, \quad \text{for } 1 \le k < j, \tag{2.1}$$

then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k\xi_k = 0 \quad a.s., \tag{2.2}$$

where d_k and D_n are defined by (1.7).

Proof. We can easily apply the similar arguments of (2.1) in Wu [16] to get Lemma 2.2, and we omit the details here. □

The following Lemma 2.3 can be directly verified.

Lemma 2.3. (i) $c_{n,i} = 2(b_{n,i} - d_{n,i})$;

(ii) $\sum_{i=1}^{n} b_{n,i}^2 = 2n - b_{n,1} \sim 2n$;

(iii) $\sum_{i=1}^{n} d_{n,i}^2 = \frac{n}{3} - \frac{n}{6(n+1)} \sim \frac{n}{3}$;

(iv) $\sum_{i=1}^{n} c_{n,i}^2 = \frac{10n}{3} - 4b_{n,1} + \frac{10n}{3(n+1)} \sim \frac{10n}{3}$.

For every $1 \le i \le k \le n$, let

$$\overline{X}_{ki} = (X_i - \mu)I(|X_i - \mu| \le \eta_k), \qquad \overline{S}_{k,i} = \sum_{j=1}^{i} c_{k,j}\overline{X}_{kj}, \qquad \overline{V}_k^2 = \sum_{j=1}^{k} \overline{X}_{kj}^2. \tag{2.3}$$

Lemma 2.4. *Suppose that the assumptions of Theorem 1.1 hold. Then*

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left\{\frac{\overline{S}_{k,k} - \mathbb{E}\overline{S}_{k,k}}{\sqrt{10kl(\eta_k)/3}} \le x\right\} = \Phi(x) \quad a.s. \text{ for any } x \in \mathbb{R}, \tag{2.4}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k\left(I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \eta_k)\right) - \mathbb{E}I\left(\bigcup_{i=1}^{k}(|X_i - \mu| > \eta_k)\right)\right) = 0 \quad a.s., \tag{2.5}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k\left(f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right) - \mathbb{E}f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right)\right) = 0 \quad a.s., \tag{2.6}$$

where d_k and D_n are defined by (1.7) *and f is a nonnegative, bounded Lipschitz function.*

Proof. By $\mathbb{E}(X-\mu)=0$, Lemma 2.1 (iv), we have

$$\left|\mathbb{E}\overline{X}_{ni}\right| \le \mathbb{E}\left|X-\mu\right| I\left(\left|X-\mu\right| > \eta_n\right) = \frac{o(l(\eta_n))}{\eta_n}. \tag{2.7}$$

Thus, by (1.5) and Lemma 2.3 (iv),

$$\operatorname{Var}\overline{X}_{ni} = \mathbb{E}\overline{X}_{ni}^2 - \left(\mathbb{E}\overline{X}_{ni}\right)^2 \sim l(\eta_n), \qquad \sum_{i=1}^{n}\operatorname{Var}\left(c_{n,i}\overline{X}_{ni}\right) \sim \frac{10n}{3} l(\eta_n) := B_n^2. \tag{2.8}$$

By (1.5) and Lemma 2.3 (i),

$$\max_{1\le i\le n} c_{n,i} \le 2\max_{1\le i\le n} b_{n,i} \le 2b_{n,1} \sim 2\ln n, \qquad \frac{\ln n\ \max_{1\le i\le n}\left|\mathbb{E}\overline{X}_{ni}\right|}{B_n} \longrightarrow 0. \tag{2.9}$$

Thus by combining Lemma 2.3 (iv), (1.6), and (2.8), Lindeberg condition

$$\begin{aligned}
\frac{1}{B_n^2}\sum_{i=1}^{n}\mathbb{E}\left(c_{n,i}\overline{X}_{ni}\right)^2 I_{(|c_{n,i}\overline{X}_{ni}-\mathbb{E}c_{n,i}\overline{X}_{ni}|>\varepsilon B_n)} &\sim \frac{c}{nl(\eta_n)}\sum_{i=1}^{n} c_{n,i}^2\mathbb{E}\overline{X}_{ni}^2 I_{(|\overline{X}_{ni}-\mathbb{E}\overline{X}_{ni}|>\varepsilon B_n/2\ln n)} \\
&\le \frac{c}{nl(\eta_n)}\sum_{i=1}^{n} c_{n,i}^2\mathbb{E}\overline{X}_{ni}^2 I_{(|\overline{X}_{ni}|>\varepsilon B_n/4\ln n)} \\
&= \frac{c}{nl(\eta_n)}\sum_{i=1}^{n} c_{n,i}^2\mathbb{E}(X-\mu)^2 I_{(c\eta_n/\ln n<|X-\mu|\le\eta_n)} \\
&= c\frac{l(\eta_n)-l(c\eta_n/\ln n)}{l(\eta_n)} \longrightarrow 0, \quad \text{as } n\longrightarrow\infty
\end{aligned} \tag{2.10}$$

hold.

Hence, it follows that

$$\frac{\overline{S}_{n,n}-\mathbb{E}\overline{S}_{n,n}}{B_n} \xrightarrow{d} \mathcal{N}, \quad \text{as } n\longrightarrow\infty. \tag{2.11}$$

This implies that for any $g(x)$, which is a nonnegative, bounded Lipschitz function,

$$\mathbb{E}g\left(\frac{\overline{S}_{n,n}-\mathbb{E}\overline{S}_{n,n}}{B_n}\right) \longrightarrow \mathbb{E}g(\mathcal{N}), \quad \text{as } n\longrightarrow\infty. \tag{2.12}$$

Hence, we obtain

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\mathbb{E}g\left(\frac{\overline{S}_{k,k}-\mathbb{E}\overline{S}_{k,k}}{B_k}\right)=\mathbb{E}g(\mathcal{N}) \tag{2.13}$$

from the Toeplitz lemma.

On the other hand, note that (2.4) is equivalent to

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k g\left(\frac{\overline{S}_{k,k}-\mathbb{E}\overline{S}_{k,k}}{B_k}\right)=\mathbb{E}g(\mathcal{N}) \quad \text{a.s.,} \tag{2.14}$$

from Theorem 7.1 of Billingsley [17] and Section 2 of Peligrad and Shao [18]. Hence, to prove (2.4), it suffices to prove

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k\left(g\left(\frac{\overline{S}_{k,k}-\mathbb{E}\overline{S}_{k,k}}{B_k}\right)-\mathbb{E}g\left(\frac{\overline{S}_{k,k}-\mathbb{E}\overline{S}_{k,k}}{B_k}\right)\right)=0 \quad \text{a.s.,} \tag{2.15}$$

for any $g(x)$ which is a nonnegative, bounded Lipschitz function.

Let

$$\xi_k=g\left(\frac{\overline{S}_{k,k}-\mathbb{E}\overline{S}_{k,k}}{B_k}\right)-\mathbb{E}g\left(\frac{\overline{S}_{k,k}-\mathbb{E}\overline{S}_{k,k}}{B_k}\right), \quad \text{for } k\geq 1. \tag{2.16}$$

Clearly, there is a constant $c>0$ such that

$$|g(x)|\leq c, \qquad |g(x)-g(y)|\leq c|x-y| \quad \text{for any } x,y\in\mathbb{R}, \qquad |\xi_k|\leq 2c, \ \text{for any } k. \tag{2.17}$$

For any $1\leq k<l$, note that

$$\overline{S}_{l,l}-\overline{S}_{l,k}=\sum_{i=1}^{l}c_{l,i}\overline{X}_{li}-\sum_{i=1}^{k}c_{l,i}\overline{X}_{li}=\sum_{i=k+1}^{l}c_{l,i}\overline{X}_{li}. \tag{2.18}$$

For any $1 \le k < j$, note that $g((\overline{S}_{k,k}-\mathbb{E}\overline{S}_{k,k})/B_k)$ and $g((\overline{S}_{j,j}-\mathbb{E}\overline{S}_{j,j}-(\overline{S}_{j,k}-\mathbb{E}\overline{S}_{j,k}))/B_j)$ are independent, $g(x)$ is a nonnegative, bounded Lipschitz function, $\sum_{i=1}^{k} c_{k,i}^2 \sim 10k/3$, $c_{j,i}^2 \le 4b_{j,i}^2$, $\sum_{i=1}^{k} b_{k,i}^2 \sim 2k$, and $\ln x \le 4x^{1/4}$, $x \ge 1$. By the definition of η_j, we get

$$\begin{aligned} \left|\mathbb{E}\xi_k\xi_j\right| &= \left|\operatorname{Cov}\left(g\left(\frac{\overline{S}_{k,k}-\mathbb{E}\overline{S}_{k,k}}{B_k}\right), g\left(\frac{\overline{S}_{j,j}-\mathbb{E}\overline{S}_{j,j}}{B_j}\right)\right)\right| \\ &= \left|\operatorname{Cov}\left(g\left(\frac{\overline{S}_{k,k}-\mathbb{E}\overline{S}_{k,k}}{B_k}\right), g\left(\frac{\overline{S}_{j,j}-\mathbb{E}\overline{S}_{j,j}}{B_j}\right) - g\left(\frac{\overline{S}_{j,j}-\mathbb{E}\overline{S}_{j,j}-\left(\overline{S}_{j,k}-\mathbb{E}\overline{S}_{j,k}\right)}{B_j}\right)\right)\right| \\ &\le c\frac{\mathbb{E}\left|\sum_{i=1}^{k} c_{j,i}\left(\overline{X}_{ji}-\mathbb{E}\overline{X}_{ji}\right)\right|}{\sqrt{jl(\eta_j)}} \le c\frac{\sqrt{\mathbb{E}\left(\sum_{i=1}^{k} c_{j,i}(\overline{X}_{ji}-\mathbb{E}\overline{X}_{ji})\right)^2}}{\sqrt{jl(\eta_j)}} \\ &\le c\frac{\sqrt{\sum_{i=1}^{k} b_{j,i}^2\mathbb{E}\overline{X}_{ji}^2}}{\sqrt{jl(\eta_j)}} \le c\frac{\sqrt{\sum_{i=1}^{k}\left(b_{k,i}+b_{j,k+1}\right)^2 l(\eta_j)}}{\sqrt{jl(\eta_j)}} \\ &= c\frac{\sqrt{\sum_{i=1}^{k} b_{k,i}^2+\sum_{i=1}^{k} b_{j,k+1}^2}}{\sqrt{j}} \le c\frac{\sqrt{k+k\ln^2(j/k)}}{\sqrt{j}} \\ &\le c\left(\frac{k}{j}\right)^{1/4}. \end{aligned} \tag{2.19}$$

By Lemma 2.2, (2.15) holds.

Now we prove (2.5). Let

$$Z_k = I\left(\bigcup_{i=1}^{k}\left(\left|X_i-\mu\right| > \eta_k\right)\right) - \mathbb{E}I\left(\bigcup_{i=1}^{k}\left(\left|X_i-\mu\right| > \eta_k\right)\right) \quad \text{for any } k \ge 1. \tag{2.20}$$

It is known that $I(A\cup B)-I(B) \le I(A)$ for any sets A and B; then for $1 \le k < j$, by Lemma 2.1 (iii) and (1.5), we get

$$\mathbb{P}\left(\left|X-\mu\right| > \eta_j\right) = o(1)\frac{l(\eta_j)}{\eta_j^2} = \frac{o(1)}{j}. \tag{2.21}$$

Hence for $1 \le k < j$,

$$\begin{aligned}
\left|\mathbb{E}Z_k Z_j\right| &= \left|\operatorname{Cov}\left(I\left(\bigcup_{i=1}^{k}(|X_i-\mu|>\eta_k)\right), I\left(\bigcup_{i=1}^{j}(|X_i-\mu|>\eta_j)\right)\right)\right| \\
&= \left|\operatorname{Cov}\left(I\left(\bigcup_{i=1}^{k}(|X_i-\mu|>\eta_k)\right), I\left(\bigcup_{i=1}^{j}(|X_i-\mu|>\eta_j)\right)\right.\right. \\
&\qquad \left.\left. - I\left(\bigcup_{i=k+1}^{j}(|X_i-\mu|>\eta_j)\right)\right)\right| \\
&\le \mathbb{E}\left| I\left(\bigcup_{i=1}^{j}(|X_i-\mu|>\eta_j)\right) - I\left(\bigcup_{i=k+1}^{j}(|X_i-\mu|>\eta_j)\right)\right| \\
&\le \mathbb{E}I\left(\bigcup_{i=1}^{k}(|X_i-\mu|>\eta_j)\right) \le k\mathbb{P}(|X-\mu|>\eta_j) \\
&\le \frac{k}{j}.
\end{aligned} \tag{2.22}$$

By Lemma 2.2, (2.5) holds.

Finally, we prove (2.6). Let

$$\zeta_k = f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right) - \mathbb{E}f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right) \quad \text{for any } k \ge 1. \tag{2.23}$$

For $1 \le k < j$,

$$\begin{aligned}
\left|\mathbb{E}\zeta_k\zeta_j\right| &= \left|\operatorname{Cov}\left(f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right), f\left(\frac{\overline{V}_j^2}{jl(\eta_j)}\right)\right)\right| \\
&= \left|\operatorname{Cov}\left(f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right), f\left(\frac{\overline{V}_j^2}{jl(\eta_j)}\right) - f\left(\frac{\overline{V}_j^2 - \sum_{i=1}^{k}(X_i-\mu)^2 I(|X_i-\mu|\le\eta_j)}{jl(\eta_j)}\right)\right)\right| \\
&\le c\frac{\mathbb{E}\left(\sum_{i=1}^{k}(X_i-\mu)^2 I(|X_i-\mu|\le\eta_j)\right)}{jl(\eta_j)} = c\frac{k\mathbb{E}(X-\mu)^2 I(|X-\mu|\le\eta_j)}{jl(\eta_j)} = c\frac{kl(\eta_j)}{jl(\eta_j)} \\
&= c\frac{k}{j}.
\end{aligned} \tag{2.24}$$

By Lemma 2.2, (2.6) holds. This completes the proof of Lemma 2.4. □

Proof of Theorem 1.1. Let $Z_j = T_j/(j(j+1)\mu/2)$; then (1.8) is equivalent to

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k I\left(\frac{\sqrt{3}\mu}{\sqrt{10}V_k}\sum_{j=1}^{k}\ln Z_j \le x\right) = \Phi(x), \quad \text{a.s. for any } x, \tag{2.25}$$

where $\Phi(x)$ is the distribution function of the standard normal random variable $\mathcal{N}$.

Let $q \in (4/3, 2)$, then $\mathbb{E}|X|^q < \infty$. Using Marcinkiewicz-Zygmund strong large number law, we have

$$S_k - \mu k = o\left(k^{1/q}\right) \quad \text{a.s.} \tag{2.26}$$

Thus,

$$\begin{aligned}|Z_i - 1| &= \frac{\left|\sum_{j=1}^{i} S_j - i(i+1)\mu/2\right|}{i(i+1)\mu/2} \le \frac{\sum_{j=1}^{i}\left|S_j - \mu j\right|}{i(i+1)\mu/2} \le \frac{\sum_{j=1}^{i} j^{1/q}}{i(i+1)\mu/2} \le c\frac{i^{1/q+1}}{i^2} \\ &= i^{1/q-1} \longrightarrow 0, \quad \text{a.s.}\end{aligned} \tag{2.27}$$

Hence by $|\ln(1+x) - x| = O(x^2)$ for $|x| < 1/2$, for any $0 < \varepsilon < 1$,

$$\begin{aligned}\left|\frac{1}{\sqrt{1\pm\varepsilon}B_k}\sum_{i=1}^{k}\ln Z_i - \frac{1}{\sqrt{1\pm\varepsilon}B_k}\sum_{i=1}^{k}(Z_i - 1)\right| &\le c\frac{1}{\sqrt{kl(\eta_k)}}\sum_{i=1}^{k}(Z_i-1)^2 \le \frac{c}{\sqrt{kl(\eta_k)}}\sum_{i=1}^{k} i^{2(1/q-1)} \\ &\le c\frac{1}{k^{3/2-2/q}\sqrt{l(\eta_k)}} \longrightarrow 0 \quad \text{a.s. } k \longrightarrow \infty,\end{aligned} \tag{2.28}$$

from $3/2 - 2/q > 0$, $l(x)$ is a slowly varying function at ∞, and $\eta_k \le k+1$.

Hence for almost every event ω and any $\delta > 0$, there exists $k_0 = k_0(\omega, \delta, x)$ such that for $k > k_0$,

$$\begin{aligned}I\left(\frac{\mu}{\sqrt{1\pm\varepsilon}B_k}\sum_{i=1}^{k}(Z_i-1) \le x - \delta\right) &\le I\left(\frac{\mu}{\sqrt{1\pm\varepsilon}B_k}\sum_{i=1}^{k}\ln Z_i \le x\right) \\ &\le I\left(\frac{\mu}{\sqrt{1\pm\varepsilon}B_k}\sum_{i=1}^{k}(Z_i-1) \le x + \delta\right).\end{aligned} \tag{2.29}$$

Note that under condition $|X_j - \mu| \le \eta_k, 1 \le j \le k$,

$$\begin{aligned}
\mu\sum_{i=1}^{k}(Z_i - 1) &= \sum_{i=1}^{k}\frac{\sum_{l=1}^{i}S_l - \mu\sum_{l=1}^{i} l}{i(i+1)/2} = \sum_{i=1}^{k}\frac{1}{i(i+1)/2}\sum_{l=1}^{i}\sum_{j=1}^{l}(X_j-\mu) \\
&= \sum_{i=1}^{k}\frac{1}{i(i+1)/2}\sum_{j=1}^{i}\sum_{l=j}^{i}(X_j-\mu) = \sum_{i=1}^{k}\sum_{j=1}^{i}\frac{2(i+1-j)}{i(i+1)}(X_j-\mu) \\
&= \sum_{j=1}^{k}\sum_{i=j}^{k}\frac{2(i+1-j)}{i(i+1)}\overline{X}_{kj} = \sum_{j=1}^{k}c_{k,j}\overline{X}_{kj} \\
&= \overline{S}_{k,k}.
\end{aligned} \tag{2.30}$$

Thus, for any given $0 < \varepsilon < 1$, $\delta > 0$, combining (2.29), we have for $k > k_0$

$$\begin{aligned}
I\left(\frac{\sqrt{3}\mu\sum_{i=1}^{k}\ln Z_i}{\sqrt{10}V_k} \le x\right) &\le I\left(\frac{\sqrt{3}\mu\sum_{i=1}^{k}\ln Z_i}{\sqrt{10(1+\varepsilon)kl(\eta_k)}} \le x\right) + I\left(\overline{V}_k^2 > (1+\varepsilon)kl(\eta_k)\right) \\
&\quad + I\left(\bigcup_{i=1}^{k}(|X_i-\mu| > \eta_k)\right) \\
&\le I\left(\frac{\mu\sum_{i=1}^{k}(Z_i-1)}{\sqrt{1+\varepsilon}B_k} \le x+\delta\right) + I\left(\overline{V}_k^2 > (1+\varepsilon)kl(\eta_k)\right) \\
&\quad + I\left(\bigcup_{i=1}^{k}(|X_i-\mu| > \eta_k)\right) \\
&\le I\left(\frac{\overline{S}_{k,k}}{\sqrt{1+\varepsilon}B_k} \le x+\delta\right) + I\left(\overline{V}_k^2 > (1+\varepsilon)kl(\eta_k)\right) \\
&\quad + 2I\left(\bigcup_{i=1}^{k}(|X_i-\mu| > \eta_k)\right) \quad \text{for } x \ge 0, \\
I\left(\frac{\sqrt{3}\mu\sum_{i=1}^{k}\ln Z_i}{\sqrt{10}V_k} \le x\right) &\le I\left(\frac{\overline{S}_{k,k}}{\sqrt{1-\varepsilon}B_k} \le x+\delta\right) + I\left(\overline{V}_k^2 < (1-\varepsilon)kl(\eta_k)\right) \\
&\quad + 2I\left(\bigcup_{i=1}^{k}(|X_i-\mu| > \eta_k)\right) \quad \text{for } x < 0, \\
I\left(\frac{\sqrt{3}\mu\sum_{i=1}^{k}\ln Z_i}{\sqrt{10}V_k} \le x\right) &\ge I\left(\frac{\overline{S}_{k,k}}{\sqrt{1-\varepsilon}B_k} \le x-\delta\right) - I\left(\overline{V}_k^2 < (1-\varepsilon)kl(\eta_k)\right) \\
&\quad - 2I\left(\bigcup_{i=1}^{k}(|X_i-\mu| > \eta_k)\right), \quad \text{for } x \ge 0,
\end{aligned}$$

$$I\left(\frac{\sqrt{3}\mu\sum_{i=1}^{k}\ln Z_i}{\sqrt{10}V_k}\le x\right)\ge I\left(\frac{\overline{S}_{k,k}}{\sqrt{1+\varepsilon}B_k}\le x-\delta\right)-I\left(\overline{V}_k^2>(1+\varepsilon)kl(\eta_k)\right)$$
$$-2I\left(\bigcup_{i=1}^{k}(|X_i-\mu|>\eta_k)\right),\quad \text{for } x<0. \tag{2.31}$$

Therefore, to prove (2.25), it suffices to prove

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left(\frac{\overline{S}_{k,k}}{B_k}\le\sqrt{1\pm\varepsilon}x\pm\delta_1\right)=\Phi\left(\sqrt{1\pm\varepsilon}x\pm\delta_1\right)\quad \text{a.s.}, \tag{2.32}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left(\bigcup_{i=1}^{k}(|X_i-\mu|>\eta_k)\right)=0\quad \text{a.s.}, \tag{2.33}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left(\overline{V}_k^2>(1+\varepsilon)kl(\eta_k)\right)=0\quad \text{a.s.}, \tag{2.34}$$

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left(\overline{V}_k^2<(1-\varepsilon)kl(\eta_k)\right)=0\quad \text{a.s}, \tag{2.35}$$

for any $0<\varepsilon<1$ and $\delta_1>0$.

Firstly, we prove (2.32). Let $0<\beta<1/2$ and $h(\cdot)$ be a real function, such that for any given $x\in\mathbb{R}$,

$$I\left(y\le\sqrt{1\pm\varepsilon}x\pm\delta_1-\beta\right)\le h(y)\le I\left(y\le\sqrt{1\pm\varepsilon}x\pm\delta_1+\beta\right). \tag{2.36}$$

By $\mathbb{E}(X_i-\mu)=0$, Lemma 2.1(iv) and (1.5), we have

$$\begin{aligned}\left|\mathbb{E}\overline{S}_{k,k}\right|&=\left|\mathbb{E}\sum_{i=1}^{k}c_{k,i}(X_i-\mu)I(|X_i-\mu|\le\eta_k)\right|\le 2\sum_{i=1}^{k}b_{k,i}\mathbb{E}|X_i-\mu|I(|X_i-\mu|>\eta_k)\\&=2\sum_{i=1}^{k}\sum_{j=i}^{k}\frac{1}{j}\mathbb{E}|X-\mu|I(|X-\mu|>\eta_k)=\sum_{j=1}^{k}\sum_{i=1}^{j}\frac{1}{j}\frac{o(l(\eta_k))}{\eta_k}\\&=o\left(\sqrt{kl(\eta_k)}\right).\end{aligned} \tag{2.37}$$

This, combining with (2.4), (2.36) and the arbitrariness of β in (2.36), (2.32) holds.

By (2.5), (2.21) and the Toeplitz lemma,

$$
\begin{aligned}
0 &\leq \frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\bigcup_{i=1}^{k}(|X_i-\mu|>\eta_k)\right) \sim \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{E} I\left(\bigcup_{i=1}^{k}(|X_i-\mu|>\eta_k)\right) \\
&\leq \frac{1}{D_n}\sum_{k=1}^{n} d_k k \mathbb{P}(|X-\mu|>\eta_k) \longrightarrow 0 \quad \text{a.s.}
\end{aligned}
\tag{2.38}
$$

That is, (2.33) holds.

Now we prove (2.34). For any given $\varepsilon>0$, let f be a nonnegative, bounded Lipschitz function such that

$$
I(x>1+\varepsilon) \leq f(x) \leq I\left(x>1+\frac{\varepsilon}{2}\right). \tag{2.39}
$$

From $\mathbb{E}\overline{V}_k^2 = kl(\eta_k)$, $\overline{X}_{ki}$ is i.i.d., $\mathbb{E}(\overline{X}_{ki}^2 - \mathbb{E}\overline{X}_{ki}^2)=0$, Lemma 2.1 (v), and (1.5),

$$
\begin{aligned}
\mathbb{P}\left(\overline{V}_k^2 > \left(1+\frac{\varepsilon}{2}\right)kl(\eta_k)\right) &= \mathbb{P}\left(\overline{V}_k^2 - \mathbb{E}\overline{V}_k^2 > \frac{\varepsilon}{2}kl(\eta_k)\right) \\
&\leq c\frac{\mathbb{E}\left(\overline{V}_k^2 - \mathbb{E}\overline{V}_k^2\right)^2}{k^2 l^2(\eta_k)} \leq c\frac{\mathbb{E}|X-\mu|^4 I(|X-\mu|\leq\eta_k)}{kl^2(\eta_k)} \\
&= \frac{o(1)\eta_k^2}{kl(\eta_k)} = o(1) \longrightarrow 0.
\end{aligned}
\tag{2.40}
$$

Therefore, combining (2.6) and the Toeplitz lemma,

$$
\begin{aligned}
0 &\leq \frac{1}{D_n}\sum_{k=1}^{n} d_k I\left(\overline{V}_k^2 > (1+\varepsilon)kl(\eta_k)\right) \leq \frac{1}{D_n}\sum_{k=1}^{n} d_k f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right) \\
&\sim \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{E} f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right) \leq \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{E} I\left(\overline{V}_k^2 > \left(1+\frac{\varepsilon}{2}\right)kl(\eta_k)\right) \\
&= \frac{1}{D_n}\sum_{k=1}^{n} d_k \mathbb{P}\left(\overline{V}_k^2 > \left(1+\frac{\varepsilon}{2}\right)kl(\eta_k)\right) \\
&\longrightarrow 0 \quad \text{a.s.}
\end{aligned}
\tag{2.41}
$$

Hence, (2.34) holds. By similar methods used to prove (2.34), we can prove (2.35). This completes the proof of Theorem 1.1. □

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