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Research Article

Common Fixed Points for Asymptotic Pointwise Nonexpansive Mappings in Metric and Banach Spaces

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Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X. We prove that the common fixed point set of any commuting family of asymptotic pointwise nonexpansive mappings on C is nonempty closed and convex. We also show that, under some suitable conditions, the sequence $\{x_k\}_{k=1}^{\infty}$ defined by $x_{k+1} = (1-t_{mk})x_k \oplus t_{mk}T_m^{n_k}y_{(m-1)k}$, $y_{(m-1)k} = (1-t_{(m-1)k})x_k \oplus t_{(m-1)k}T_{m-1}^{n_k}y_{(m-2)k}, y_{(m-2)k} = (1-t_{(m-2)k})x_k \oplus t_{(m-2)k}T_{m-2}^{n_k}y_{(m-3)k}, \ldots, y_{2k} = (1-t_{2k})x_k \oplus t_{2k}T_2^{n_k}y_{1k}, y_{1k} = (1-t_{1k})x_k \oplus t_{1k}T_1^{n_k}y_{0k}, y_{0k} = x_k$, $k \in \mathbb{N}$, converges to a common fixed point of $T_1, T_2, \ldots, T_m$ where they are asymptotic pointwise nonexpansive mappings on C, $\{t_{ik}\}_{k=1}^{\infty}$ are sequences in $[0,1]$ for all $i = 1, 2, \ldots, m$, and $\{n_k\}$ is an increasing sequence of natural numbers. The related results for uniformly convex Banach spaces are also included.

1. Introduction

A mapping T on a subset C of a Banach space X is said to be asymptotic pointwise nonexpansive if there exists a sequence of mappings $\alpha_n : C \to [0,\infty)$ such that

$$\left\|T^n x - T^n y\right\| \le \alpha_n(x)\left\|x-y\right\|, \tag{1.1}$$

where $\limsup_{n\to\infty}\alpha_n(x) \le 1$, for all $x, y \in C$. This class of mappings was introduced by Kirk and Xu [1], where it was shown that if C is a bounded closed convex subset of a uniformly convex Banach space X, then every asymptotic pointwise nonexpansive mapping $T : C \to C$ always has a fixed point. In 2009, Hussain and Khamsi [2] extended Kirk-Xu's result to the case of metric spaces, specifically to the so-called CAT(0) spaces. Recently, Kozlowski [3] defined an iterative sequence for an asymptotic pointwise nonexpansive mapping $T : C \to C$ by $x_1 \in C$ and

$$\begin{aligned} x_{k+1} &= (1 - t_k)x_k + t_k T^{n_k} y_k, \\ y_k &= (1 - s_k)x_k + s_k T^{n_k} x_k, \quad k \in \mathbb{N}, \end{aligned} \tag{1.2}$$

where $\{t_k\}$ and $\{s_k\}$ are sequences in $[0,1]$ and $\{n_k\}$ is an increasing sequence of natural numbers. He proved, under some suitable assumptions, that the sequence $\{x_k\}$ defined by (1.2) converges weakly to a fixed point of T where X is a uniformly convex Banach space which satisfies the Opial condition and $\{x_k\}$ converges strongly to a fixed point of T provided T^r is a compact mapping for some $r \in \mathbb{N}$. On the other hand, Khan et al. [4] studied the iterative process defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_{mn})x_n + \alpha_{mn} T_m^n y_{(m-1)n}, \\ y_{(m-1)n} &= \left(1 - \alpha_{(m-1)n}\right)x_n + \alpha_{(m-1)n} T_{m-1}^n y_{(m-2)n}, \\ y_{(m-2)n} &= \left(1 - \alpha_{(m-2)n}\right)x_n + \alpha_{(m-2)n} T_{m-2}^n y_{(m-3)n}, \\ &\vdots \\ y_{2n} &= (1 - \alpha_{2n})x_n + \alpha_{2n} T_2^n y_{1n}, \\ y_{1n} &= (1 - \alpha_{1n})x_n + \alpha_{1n} T_1^n y_{0n}, \\ y_{0n} &= x_n, \quad n \in \mathbb{N}, \end{aligned} \tag{1.3}$$

where $T_1, \ldots, T_m$ are asymptotically quasi-nonexpansive mappings on C and $\{\alpha_{in}\}_{n=1}^{\infty}$ are sequences in $[0,1]$ for all $i = 1, 2, \ldots, m$.

In this paper, motivated by the results mentioned above, we ensure the existence of common fixed points for a family of asymptotic pointwise nonexpansive mappings in a CAT(0) space. Furthermore, we obtain $\triangle$ and strong convergence theorems of a sequence defined by

$$\begin{aligned} x_{k+1} &= (1 - t_{mk})x_k \oplus t_{mk} T_m^{n_k} y_{(m-1)k}, \\ y_{(m-1)k} &= \left(1 - t_{(m-1)k}\right)x_k \oplus t_{(m-1)k} T_{m-1}^{n_k} y_{(m-2)k}, \\ y_{(m-2)k} &= \left(1 - t_{(m-2)k}\right)x_k \oplus t_{(m-2)k} T_{m-2}^{n_k} y_{(m-3)k}, \\ &\vdots \\ y_{2k} &= (1 - t_{2k})x_k \oplus t_{2k} T_2^{n_k} y_{1k}, \\ y_{1k} &= (1 - t_{1k})x_k \oplus t_{1k} T_1^{n_k} y_{0k}, \\ y_{0k} &= x_k, \quad k \in \mathbb{N}, \end{aligned} \tag{1.4}$$

where $T_1, \ldots, T_m$ are asymptotic pointwise nonexpansive mappings on a subset C of a complete CAT(0) space and $\{t_{ik}\}_{k=1}^{\infty}$ are sequences in $[0,1]$ for all $i = 1, 2, \ldots, m$, and $\{n_k\}$ is an increasing sequence of natural numbers. We also note that our method can be used to prove the analogous results for uniformly convex Banach spaces.

2. Preliminaries

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. It is well-known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [5]), $\mathbb{R}$-trees (see [6]), Euclidean buildings (see [7]), and the complex Hilbert ball with a hyperbolic metric (see [8]). For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [5].

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [9, 10]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [2, 11–22] and the references therein). It is worth mentioning that fixed point theorems in CAT(0) spaces (specially in $\mathbb{R}$-trees) can be applied to graph theory, biology, and computer science (see, e.g., [6, 23–26]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subset X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

Let $x, y \in X$, by Lemma 2.1(iv) of [14] for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = t d(x, y), \qquad d(y, z) = (1 - t) d(x, y). \tag{2.2}$$

We will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (2.2). We now collect some elementary facts about CAT(0) spaces.

Lemma 2.1. *Let X be a complete CAT(0) space.*

(i) *[5, Proposition 2.4] If C is a nonempty closed convex subset of X, then, for every $x \in X$, there exists a unique point $P(x) \in C$ such that $d(x, P(x)) = \inf\{d(x, y) : y \in C\}$. Moreover, the map $x \mapsto P(x)$ is a nonexpansive retract from X onto C.*

(ii) *[14, Lemma 2.4] For $x, y, z \in X$ and $t \in [0,1]$, we have*

$$d\big((1-t)x \oplus ty, z\big) \leq (1-t)d(x,z) + td\big(y,z\big). \tag{2.3}$$

(iii) *[14, Lemma 2.5] For $x, y, z \in X$ and $t \in [0,1]$, we have*

$$d\big((1-t)x \oplus ty, z\big)^2 \leq (1-t)d(x,z)^2 + td\big(y,z\big)^2 - t(1-t)d\big(x,y\big)^2. \tag{2.4}$$

We now give the concept of Δ-convergence and collect some of its basic properties. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n). \tag{2.5}$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}, \tag{2.6}$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \tag{2.7}$$

It is known from Proposition 7 of [27] that, in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

Definition 2.2 (see [28, 29]). A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-*limit* of $\{x_n\}$.

Lemma 2.3. *Let X be a complete CAT(0) space.*

(i) *[28, page 3690] Every bounded sequence in X has a Δ-convergent subsequence.*

(ii) *[30, Proposition 2.1] If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

(iii) *[14, Lemma 2.8] If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Recall that a mapping $T : X \to X$ is said to be *nonexpansive* [31] if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in X, \tag{2.8}$$

where T is called *asymptotically nonexpansive* [32] if there is a sequence $\{k_n\}$ of positive numbers with the property $\lim_{n\to\infty} k_n = 1$ and such that

$$d(T^n x, T^n y) \leq k_n d(x, y), \quad \forall n \geq 1,\ x, y \in X, \tag{2.9}$$

where T is called an *asymptotic pointwise nonexpansive mapping* [1] if there exists a sequence of functions $\alpha_n : X \to [0, \infty)$ such that

$$d(T^n x, T^n y) \leq \alpha_n(x) d(x, y), \quad \forall n \geq 1,\ x, y \in X, \tag{2.10}$$

where $\limsup_{n\to\infty} \alpha_n(x) \leq 1$. The following implications hold.

$$\begin{aligned} T \ \text{ is nonexpansive} \Longrightarrow T \ &\text{ is asymptotically nonexpansive} \\ \Longrightarrow T \ &\text{ is asymptotic pointwise nonexpansive.} \end{aligned} \tag{2.11}$$

A point $x \in X$ is called a fixed point of T if $x = Tx$. We shall denote by $F(T)$ the set of fixed points of T. The existence of fixed points for asymptotic pointwise nonexpansive mappings in CAT(0) spaces was proved by Hussain and Khamsi [2] as the following result.

Theorem 2.4. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X. Suppose that $T : C \to C$ is an asymptotic pointwise nonexpansive mapping. Then, $F(T)$ is nonempty closed and convex.*

3. Existence Theorems

Let M be a metric space and $\mathcal{F}$ a family of subsets of M. Then, we say that $\mathcal{F}$ defines a *convexity structure* on M if it contains the closed balls and is stable by intersection.

Definition 3.1 (see [2]). Let $\mathcal{F}$ be a convexity structure on M. We will say that $\mathcal{F}$ is *compact* if any family $\{A_\alpha\}_{\alpha\in\Gamma}$ of elements of $\mathcal{F}$ has a nonempty intersection provided $\bigcap_{\alpha\in F} A_\alpha \neq \emptyset$ for any finite subset $F \subset \Gamma$.

Let X be a complete CAT(0) space. We denote by $\mathcal{C}(X)$ the family of all closed convex subsets of X. Then, $\mathcal{C}(X)$ is a compact convexity structure on X (see, e.g., [2]).

The following theorem is an extension of Theorem 4.3 in [33]. For an analog of this result in uniformly convex Banach spaces, see [34].

Theorem 3.2. *Let C be a nonempty bounded closed and convex subset of a complete CAT(0) space X. Then, for any commuting family $\mathcal{S}$ of asymptotic pointwise nonexpansive mappings on C, the set $\mathcal{F}(\mathcal{S})$ of common fixed points of $\mathcal{S}$ is nonempty closed and convex.*

Proof. Let $\mathcal{T}$ be the family of all finite intersections of the fixed point sets of mappings in the commutative family $\mathcal{S}$. We first show that $\mathcal{T}$ has the finite intersection property. Let $T_1, T_2, \ldots, T_n \in \mathcal{S}$. By Theorem 2.4, $F(T_1)$ is a nonempty closed and convex subset of C. We assume that $A := \bigcap_{j=1}^{k-1} F(T_j)$ is nonempty closed and convex for some $k \in \mathbb{N}$ with $1 < k \le n$. For $x \in A$ and $j \in \mathbb{N}$ with $1 \le j < k$, we have

$$T_k(x) = T_k \circ T_j(x) = T_j \circ T_k(x). \tag{3.1}$$

Thus, $T_k(x)$ is a fixed point of T_j, which implies that $T_k(x) \in A$; therefore, A is invariant under T_k. Again, by Theorem 2.4, T_k has a fixed point in A, that is,

$$\bigcap_{j=1}^{k} F(T_j) = F(T_k) \bigcap A \ne \emptyset. \tag{3.2}$$

By induction, $\bigcap_{j=1}^{n} F(T_j) \ne \emptyset$. Hence, $\mathcal{T}$ has the finite intersection property. Since $\mathcal{C}(X)$ is compact,

$$\mathcal{F}(\mathcal{S}) = \bigcap_{T \in \mathcal{T}} T \ne \emptyset. \tag{3.3}$$

Obviously, the set is closed and convex. □

As a consequence of Lemma 2.1(i) and Theorem 3.2, we obtain an analog of Bruck's theorem [35].

Corollary 3.3. *Let C be a nonempty bounded closed and convex subset of a complete CAT(0) space X. Then, for any commuting family $\mathcal{S}$ of nonexpansive mappings on C, the set $\mathcal{F}(\mathcal{S})$ of common fixed points of $\mathcal{S}$ is a nonempty nonexpansive retract of C.*

4. Convergence Theorems

Throughout this section, X stands for a complete CAT(0) space. Let C be a closed convex subset of X. We shall denote by $\mathcal{T}(C)$ the class of all asymptotic pointwise nonexpansive mappings from C into C. Let $T_1, \dots, T_m \in \mathcal{T}(C)$, without loss of generality, we can assume that there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ such that for all $x, y \in C$, $i = 1, \dots, m$, and $n \in \mathbb{N}$, we have

$$d(T_i^n x, T_i^n y) \le \alpha_n(x) d(x, y), \qquad \limsup_{n \to \infty} \alpha_n(x) \le 1. \tag{4.1}$$

Let $a_n(x) = \max\{\alpha_n(x), 1\}$. Again, without loss of generality, we can assume that

$$d(T_i^n x, T_i^n y) \le a_n(x) d(x, y), \quad \lim_{n \to \infty} a_n(x) = 1, \quad a_n(x) \ge 1, \tag{4.2}$$

for all $x, y \in C$, $i = 1, \dots, m$, and $n \in \mathbb{N}$. We define $b_n(x) = a_n(x) - 1$, then, for each $x \in C$, we have $\lim_{n \to \infty} b_n(x) = 0$.

The following definition is a mild modification of [3, Definition 2.3].

Definition 4.1. Define $\mathfrak{T}_r(C)$ as a class of all $T \in \mathfrak{T}(C)$ such that

$$\sum_{n=1}^{\infty} \sup_{x \in C} b_n(x) < \infty, \tag{4.3}$$

a_n is a bounded function for every $n \in \mathbb{N}$.

Let $T_1, \ldots, T_m \in \mathfrak{T}_r(C)$, and let $\{t_{ik}\}_{k=1}^{\infty} \subset (0,1)$ be bounded away from 0 and 1 for all $i = 1, 2, \ldots, m$, and $\{n_k\}$ an increasing sequence of natural numbers. Let $x_1 \in C$, and define a sequence $\{x_k\}$ in C as

$$\begin{aligned} x_{k+1} &= (1 - t_{mk}) x_k \oplus t_{mk} T_m^{n_k} y_{(m-1)k}, \\ y_{(m-1)k} &= \left(1 - t_{(m-1)k}\right) x_k \oplus t_{(m-1)k} T_{m-1}^{n_k} y_{(m-2)k}, \\ y_{(m-2)k} &= \left(1 - t_{(m-2)k}\right) x_k \oplus t_{(m-2)k} T_{m-2}^{n_k} y_{(m-3)k}, \\ &\;\;\vdots \\ y_{2k} &= (1 - t_{2k}) x_k \oplus t_{2k} T_2^{n_k} y_{1k}, \\ y_{1k} &= (1 - t_{1k}) x_k \oplus t_{1k} T_1^{n_k} y_{0k}, \\ y_{0k} &= x_k, \quad k \in \mathbb{N}. \end{aligned} \tag{4.4}$$

We say that the sequence $\{x_k\}$ in (4.4) is well defined if $\limsup_{k \to \infty} a_{n_k}(x_k) = 1$. As in [3], we observe that $\lim_{k \to \infty} a_k(x) = 1$ for every $x \in C$. Hence, we can always choose a subsequence $\{a_{n_k}\}$ which makes $\{x_k\}$ well defined.

Lemma 4.2 (see [36, Lemma 2.2]). *Let $\{a_n\}$ and $\{u_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \le (1 + u_n) a_n, \quad \forall n \in \mathbb{N}, \ \sum_{n=1}^{\infty} u_n < \infty. \tag{4.5}$$

Then, (i) $\lim_n a_n$ *exists, (ii) if* $\liminf_n a_n = 0$*, then* $\lim_n a_n = 0$*.*

Lemma 4.3 (see [37, 38]). *Suppose $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0,1)$ and $\{u_n\}$, $\{v_n\}$ are sequences in X such that* $\limsup_n d(u_n, w) \le r$, $\limsup_n d(v_n, w) \le r$, *and* $\lim_n d((1 - t_n) u_n \oplus t_n v_n, w) = r$ *for some $r \ge 0$. Then,*

$$\lim_{n \to \infty} d(u_n, v_n) = 0. \tag{4.6}$$

Lemma 4.4. *Let C be a nonempty closed convex subset of X and $T_1, \ldots, T_m \in \mathfrak{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [a, b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in* (4.4) *is well defined. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Then,*

(a) *there exists a sequence $\{v_k\}$ in $[0, \infty)$ such that $\sum_{k=1}^{\infty} v_k < \infty$ and $d(x_{k+1}, p) \le (1 + v_k)^m d(x_k, p)$, for all $p \in F$ and all $k \in \mathbb{N}$,*

(b) *there exists a constant* $M > 0$ *such that* $d(x_{k+l}, p) \leq M d(x_k, p)$, *for all* $p \in F$ *and* $k, l \in \mathbb{N}$.

Proof. (a) Let $p \in F$ and $v_k = \sup_{x \in C} b_{n_k}(x)$ for all $k \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} \sup_{x \in C} b_{n_k}(x) < \infty$, we have $\sum_{k=1}^{\infty} v_k < \infty$. Now,

$$\begin{aligned} d(y_{1k}, p) &\leq (1 - t_{1k}) d(x_k, p) + t_{1k} d(T_1^{n_k} x_k, p) \\ &\leq (1 - t_{1k}) d(x_k, p) + t_{1k}(1 + b_{n_k}(p)) d(x_k, p) \\ &= (1 + t_{1k} b_{n_k}(p)) d(x_k, p) \\ &\leq (1 + v_k) d(x_k, p). \end{aligned} \tag{4.7}$$

Suppose that $d(y_{jk}, p) \leq (1 + v_k)^j d(x_k, p)$ holds for some $1 \leq j \leq m - 2$. Then,

$$\begin{aligned} d(y_{(j+1)k}, p) &\leq (1 - t_{(j+1)k}) d(x_k, p) + t_{(j+1)k} d\left(T_{j+1}^{n_k} y_{jk}, p\right) \\ &\leq (1 - t_{(j+1)k}) d(x_k, p) + t_{(j+1)k}(1 + b_{n_k}(p)) d(y_{jk}, p) \\ &\leq (1 - t_{(j+1)k}) d(x_k, p) + t_{(j+1)k}(1 + v_k)^{j+1} d(x_k, p) \\ &= \left[1 - t_{(j+1)k} + t_{(j+1)k}\left(1 + \sum_{r=1}^{j+1} \frac{(j+1)j \cdots (j+2-r)}{r!} v_k^r\right)\right] d(x_k, p) \\ &= \left[1 + t_{(j+1)k} \sum_{r=1}^{j+1} \frac{(j+1)j \cdots (j+2-r)}{r!} v_k^r\right] d(x_k, p) \\ &\leq (1 + v_k)^{j+1} d(x_k, p). \end{aligned} \tag{4.8}$$

By induction, we have

$$d(y_{ik}, p) \leq (1 + v_k)^i d(x_k, p), \quad \forall i = 1, 2, \ldots, m - 1. \tag{4.9}$$

This implies

$$\begin{aligned} d(x_{k+1}, p) &\leq (1 - t_{mk}) d(x_k, p) + t_{mk} d(T_m^{n_k} y_{(m-1)k}, p) \\ &\leq (1 - t_{mk}) d(x_k, p) + t_{mk}(1 + b_{n_k}(p)) d(y_{(m-1)k}, p) \\ &\leq (1 - t_{mk}) d(x_k, p) + t_{mk}(1 + v_k)(1 + v_k)^{m-1} d(x_k, p) \\ &\leq (1 - t_{mk}) d(x_k, p) + t_{mk}(1 + v_k)^m d(x_k, p) \\ &= \left[1 - t_{mk} + t_{mk}\left(1 + \sum_{r=1}^{m} \frac{m(m-1) \cdots (m-r+1)}{r!} v_k^r\right)\right] d(x_k, p) \\ &= \left[1 + t_{mk} \sum_{r=1}^{m} \frac{m(m-1) \cdots (m-r+1)}{r!} v_k^r\right] d(x_k, p) \\ &\leq (1 + v_k)^m d(x_k, p). \end{aligned} \tag{4.10}$$

This completes the proof of (a).

(b) We observe that $(1+\alpha)^n \leq e^{n\alpha}$ holds for all $n \in \mathbb{N}$ and $\alpha \geq 0$. Thus, by (a), for $k, l \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{k+l}, p) &\leq (1+v_{k+l-1})^m d(x_{k+l-1}, p) \\ &\leq \exp\{m v_{k+l-1}\} d(x_{k+l-1}, p) \leq \cdots \leq \exp\left\{m \sum_{i=1}^{k+l-1} v_i\right\} d(x_k, p) \\ &\leq \exp\left\{m \sum_{i=1}^{\infty} v_i\right\} d(x_k, p). \end{aligned} \tag{4.11}$$

The proof is complete by setting $M = \exp\{m \sum_{i=1}^{\infty} v_i\}$. □

Theorem 4.5. *Let C be a nonempty closed convex subset of X and $T_1, \ldots, T_m \in \mathfrak{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [a, b] \subset (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.4) is well defined. Assume that $F \neq \emptyset$. Then, $\{x_k\}$ converges to some point in F if and only if $\liminf_{k\to\infty} d(x_k, F) = 0$, where $d(x, F) = \inf_{p\in F} d(x, p)$.*

Proof. The necessity is obvious. Now, we prove the sufficiency. From Lemma 4.4(a), we have

$$d(x_{k+1}, p) \leq (1+v_k)^m d(x_k, p), \quad \forall p \in F,\ \forall k \in \mathbb{N}. \tag{4.12}$$

This implies

$$d(x_{k+1}, F) \leq (1+v_k)^m d(x_k, F) = \left(1 + \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!} v_k^r\right) d(x_k, F). \tag{4.13}$$

Since $\sum_{k=1}^{\infty} v_k < \infty$, then $\sum_{k=1}^{\infty} \sum_{r=1}^{m} (m(m-1)\cdots(m-r+1)/r!) v_k^r < \infty$. By Lemma 4.2(ii), we get $\lim_{k\to\infty} d(x_k, F) = 0$. Next, we show that $\{x_k\}$ is a Cauchy sequence. From Lemma 4.4(b), there exists $M > 0$ such that

$$d(x_{k+l}, p) \leq M d(x_k, p), \quad \forall p \in F,\ k, l \in \mathbb{N}. \tag{4.14}$$

Since $\lim_{k\to\infty} d(x_k, F) = 0$, for each $\varepsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that

$$d(x_k, F) < \frac{\varepsilon}{2M}, \quad \forall k \geq k_1. \tag{4.15}$$

Hence, there exists $z_1 \in F$ such that

$$d(x_{k_1}, z_1) < \frac{\varepsilon}{2M}. \tag{4.16}$$

By (4.14) and (4.16), for $k \geq k_1$, we have

$$\begin{aligned} d(x_{k+l}, x_k) &\leq d(x_{k+l}, z_1) + d(x_k, z_1) \\ &\leq M d(x_{k_1}, z_1) + M d(x_{k_1}, z_1) \\ &< 2M\left(\frac{\varepsilon}{2M}\right) \\ &= \varepsilon. \end{aligned} \tag{4.17}$$

This shows that $\{x_k\}$ is a Cauchy sequence and so converges to some $q \in C$. We next show that $q \in F$. Let $L = \sup\{a_1(x) : x \in C\}$. Then, for each $\epsilon > 0$, there exists $k_2 \in \mathbb{N}$ such that

$$d(x_k, q) < \frac{\epsilon}{2(1+L)}, \quad \forall k \geq k_2. \tag{4.18}$$

Since $\lim_{k\to\infty} d(x_k, F) = 0$, there exists $k_3 \geq k_2$ such that

$$d(x_k, F) < \frac{\epsilon}{2(1+L)}, \quad \forall k \geq k_3. \tag{4.19}$$

Thus, there exists $z_2 \in F$ such that

$$d(x_{k_3}, z_2) < \frac{\epsilon}{2(1+L)}. \tag{4.20}$$

By (4.18) and (4.20), for each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} d(T_i q, q) &\leq d(T_i q, T_i x_{k_3}) + d(T_i x_{k_3}, z_2) + d(z_2, x_{k_3}) + d(x_{k_3}, q) \\ &\leq L d(x_{k_3}, q) + L d(x_{k_3}, z_2) + d(x_{k_3}, z_2) + d(x_{k_3}, q) \\ &\leq (1+L) d(x_{k_3}, q) + (1+L) d(x_{k_3}, z_2) \\ &< (1+L)\frac{\epsilon}{2(1+L)} + (1+L)\frac{\epsilon}{2(1+L)} \\ &= \epsilon. \end{aligned} \tag{4.21}$$

Since ϵ is arbitrary, we have $T_i q = q$ for all $i = 1, 2, \ldots, m$. Hence, $q \in F$. □

As an immediate consequence of Theorem 4.5, we obtain the following.

Corollary 4.6. *Let C be a nonempty closed convex subset of X and $T_1, \ldots, T_m \in \mathfrak{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [a, b] \subset (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.4) is well defined. Assume that $F \neq \emptyset$. Then, $\{x_k\}$ converges to a point $p \in F$ if and only if there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ which converges to p.*

Definition 4.7. A strictly increasing sequence $\{n_k\} \subset \mathbb{N}$ is called *quasiperiodic* [39] if the sequence $\{n_{k+1} - n_k\}$ is bounded or equivalently if there exists a number $p \in \mathbb{N}$ such that any block of p consecutive natural numbers must contain a term of the sequence $\{n_k\}$. The smallest of such numbers p will be called a quasiperiod of $\{n_k\}$.

Lemma 4.8. *Let C be a nonempty closed convex subset of X and $T_1, \ldots, T_m \in \mathcal{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.4) is well defined. Then,*

(i) $\lim_{k\to\infty} d(x_k, p)$ *exists for all* $p \in F$,

(ii) $\lim_{k\to\infty} d(x_k, T_j^{n_k} y_{(j-1)k}) = 0$, *for all* $j = 1, 2, \ldots, m$,

(iii) *if the set* $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ *is quasiperiodic, then* $\lim_{k\to\infty} d(x_k, T_j x_k) = 0$, *for all* $j = 1, 2, \ldots, m$.

Proof. (i) Follows from Lemmas 4.2(i) and 4.4(a).

(ii) Let $p \in F$, then, by (i), we have $\lim_{k\to\infty} d(x_k, p)$ exists. Let

$$\lim_{k\to\infty} d(x_k, p) = c. \tag{4.22}$$

By (4.9) and (4.22), we get that

$$\limsup_{k\to\infty} d(y_{jk}, p) \le c, \quad \text{for } 1 \le j \le m-1. \tag{4.23}$$

Note that

$$\begin{aligned} d(x_{k+1}, p) &\le (1 - t_{mk}) d(x_k, p) + t_{mk} d(T_m^{n_k} y_{(m-1)k}, p) \\ &\le (1 - t_{mk}) d(x_k, p) + t_{mk}(1 + v_k) d(y_{(m-1)k}, p) \\ &\vdots \\ &\le (1 - t_{mk} t_{(m-1)k} \cdots t_{(j+1)k})(1 + v_k)^{m-j} d(x_k, p) \\ &\quad + t_{mk} t_{(m-1)k} \cdots t_{(j+1)k}(1 + v_k)^{m-j} d(y_{jk}, p). \end{aligned} \tag{4.24}$$

Thus,

$$d(x_k, p) \le \frac{d(x_k, p)}{\delta^{m-j}} - \frac{d(x_{k+1}, p)}{\delta^{m-j}(1 + v_k)^{m-j}} + d(y_{jk}, p), \tag{4.25}$$

so that

$$c \le \liminf_{k\to\infty} d(y_{jk}, p), \quad \text{for } 1 \le j \le m-1. \tag{4.26}$$

From (4.23) and (4.26), we have

$$\lim_{k\to\infty} d(y_{jk}, p) = c, \quad \text{for each } j = 1, 2, \ldots, m-1. \tag{4.27}$$

That is

$$\lim_{k\to\infty} d\Big((1 - t_{jk}) x_k \oplus t_{jk} T_j^{n_k} y_{(j-1)k}, p\Big) = c, \tag{4.28}$$

for each $j = 1, 2, \ldots, m-1$.

We also obtain from (4.23) that

$$\limsup_{k\to\infty} d\left(T_j^{n_k} y_{(j-1)k}, p\right) \le c, \quad \text{for each } j = 1,2,\dots,m-1. \tag{4.29}$$

By Lemma 4.3, we get that

$$\lim_{k\to\infty} d\left(T_j^{n_k} y_{(j-1)k}, x_k\right) = 0, \quad \text{for each } j = 1,2,\dots,m-1. \tag{4.30}$$

For the case $j = m$, by (4.1), we have

$$d\left(T_m^{n_k} y_{(m-1)k}, p\right) \le \left(1 + b_{n_k}(p)\right) d\left(y_{(m-1)k}, p\right) \le \left(1 + b_{n_k}(p)\right)\left(1 + v_{n_k}\right)^{m-1} d(x_k, p). \tag{4.31}$$

But since $\lim_{k\to\infty} d(x_k, p) = c$, then

$$\limsup_{k\to\infty} d\left(T_m^{n_k} y_{(m-1)k}, p\right) \le c. \tag{4.32}$$

Moreover,

$$\lim_{k\to\infty} d\left((1 - t_{mk})x_k \oplus t_{mk} T_m^{n_k} y_{(m-1)k}, p\right) = \lim_{k\to\infty} d\left(x_{k+1}, p\right) = c. \tag{4.33}$$

Again, by Lemma 4.3, we get that

$$\lim_{k\to\infty} d\left(T_m^{n_k} y_{(m-1)k}, x_k\right) = 0. \tag{4.34}$$

Thus, (4.30) and (4.34) imply that

$$\lim_{k\to\infty} d\left(T_j^{n_k} y_{(j-1)k}, x_k\right) = 0, \quad \text{for each } j = 1,2,\dots,m. \tag{4.35}$$

(iii) For $j = 1$, from (ii), we have

$$\lim_{k\to\infty} d\left(T_1^{n_k} x_k, x_k\right) = 0. \tag{4.36}$$

If $j = 2,3,\dots,m$, then we have

$$\begin{aligned} d\left(T_j^{n_k} x_k, x_k\right) &\le d\left(T_j^{n_k} x_k, T_j^{n_k} y_{(j-1)k}\right) + d\left(T_j^{n_k} y_{(j-1)k}, x_k\right) \\ &\le a_{n_k}(x_k) d\left(x_k, y_{(j-1)k}\right) + d\left(T_j^{n_k} y_{(j-1)k}, x_k\right) \\ &\le a_{n_k}(x_k) t_{(j-1)k} d\left(x_k, T_{j-1}^{n_k} y_{(j-2)k}\right) + d\left(T_j^{n_k} y_{(j-1)k}, x_k\right). \end{aligned} \tag{4.37}$$

By (ii) and $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$, we get

$$\limsup_{k\to\infty} d\left(T_j^{n_k} x_k, x_k\right) = 0, \quad \text{for } j = 2, 3, \ldots, m. \tag{4.38}$$

By (4.36) and (4.38), we have

$$\lim_{k\to\infty} d\left(T_j^{n_k} x_k, x_k\right) = 0, \quad \forall j = 1, 2, \ldots, m. \tag{4.39}$$

By the construction of the sequence $\{x_k\}$, we have from (4.35) that

$$\lim_{k\to\infty} d(x_{k+1}, x_k) = 0. \tag{4.40}$$

Next, we show that

$$\lim_{k\to\infty} d(T_j x_k, x_k) = 0, \quad \forall j = 1, 2, \ldots, m. \tag{4.41}$$

It is enough to prove that $d(T_j x_k, x_k) \to 0$ as $k \to \infty$ though $\mathcal{Q}$. Indeed, let p be a quasiperiod of $\mathcal{Q}$, and let $\varepsilon > 0$ be given. Then, there exists $N_1 \in \mathbb{N}$ such that

$$\lim_{k\to\infty} d(T_j x_k, x_k) < \frac{\varepsilon}{3}, \quad \forall k \in \mathcal{Q} \text{ such that } k \geq N_1. \tag{4.42}$$

By the quasiperiodicity of $\mathcal{Q}$, for each $l \in \mathbb{N}$, there exists $i_l \in \mathcal{Q}$ such that $|l - i_l| \leq p$. Without loss of generality, we can assume that $l \leq i_l \leq l + p$ (the proof for the other case is identical). Let $M = \sup\{a_1(x) : x \in C\}$. Then, $M \geq 1$. Since $\lim_{l\to\infty} d(x_{l+1}, x_l) = 0$ by (4.40), there exists $N_2 \in \mathbb{N}$ such that

$$d(x_{l+1}, x_l) < \frac{\varepsilon}{3pM}, \quad \forall l \geq N_2. \tag{4.43}$$

This implies that

$$d(x_{i_l}, x_l) \leq d(x_{i_l}, x_{i_l-1}) + \cdots + d(x_{l+1}, x_l) \leq p\left(\frac{\varepsilon}{3pM}\right) = \frac{\varepsilon}{3M}. \tag{4.44}$$

By the definition of T, we have

$$d(T_j x_{i_l}, T_j x_l) \leq M d(x_{i_l}, x_l) \leq M\left(\frac{\varepsilon}{3M}\right) = \frac{\varepsilon}{3}. \tag{4.45}$$

Let $N = \max\{N_1, N_2\}$. Then, for $l \geq N$, we have from (4.42), (4.44), and (4.45) that

$$d(x_l, T_j x_l) \leq d(x_l, x_{i_l}) + d(x_{i_l}, T_j x_{i_l}) + d(T_j x_{i_l}, T_j x_l) < \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon. \tag{4.46}$$

To prove that $d(T_jx_k, x_k) \to 0$ as $k \to \infty$ though $\mathcal{Q}$. Since $\mathcal{Q} = \{k \in \mathbb{N} : n_{k+1} = n_k + 1\}$ is quasiperiodic, for each $k \in \mathcal{Q}$, we have

$$\begin{aligned} d(x_k, T_jx_k) &\leq d(x_k, x_{k+1}) + d\left(x_{k+1}, T_j^{n_{k+1}}x_{k+1}\right) + d\left(T_j^{n_{k+1}}x_{k+1}, T_j^{n_{k+1}}x_k\right) + d\left(T_j^{n_k+1}x_k, T_jx_k\right) \\ &\leq d(x_k, x_{k+1}) + d\left(x_{k+1}, T_j^{n_{k+1}}x_{k+1}\right) + a_{n_{k+1}}(x_{k+1})d(x_{k+1}, x_k) + a_1(x_k)d\left(T_j^{n_k}x_k, x_k\right). \end{aligned} \tag{4.47}$$

From this, together with (4.39) and (4.40), we can obtain that $d(T_jx_k, x_k) \to 0$ as $k \to \infty$ through $\mathcal{Q}$. □

The following lemmas can be found in [3] (see also [2]).

Lemma 4.9. *Let C be a nonempty closed convex subset of X, and let $T \in \mathfrak{T}_r(C)$. If $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $\lim_{n\to\infty} d(x_n, T^l x_n) = 0$ for every $l \in \mathbb{N}$.*

Lemma 4.10. *Let C be a nonempty closed convex subset of X, and let $T \in \mathfrak{T}_r(C)$. Suppose $\{x_n\}$ is a bounded sequence in C such that $\lim_n d(x_n, Tx_n) = 0$ and Δ-$\lim_n x_n = w$. Then, $Tw = w$.*

By using Lemmas 2.3 and 4.10, we can obtain the following result. We omit the proof because it is similar to the one given in [38].

Lemma 4.11. *Let C be a closed convex subset of X, and let $T : C \to C$ be an asymptotic pointwise nonexpansive mapping. Suppose $\{x_n\}$ is a bounded sequence in C such that $\lim_n d(x_n, T(x_n)) = 0$ and $d(x_n, v)$ converges for each $v \in F(T)$, then $\omega_w(x_n) \subset F(T)$. Here, $\omega_w(x_n) = \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Now, we are ready to prove our Δ-convergence theorem.

Theorem 4.12. *Let C be a nonempty closed convex subset of X and $T_1, \ldots, T_m \in \mathfrak{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.4) is well defined. Suppose that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{Q} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasiperiodic. Then, $\{x_k\}$ Δ-converges to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$.*

Proof. Let $p \in F$, by Lemma 4.8, $\lim_{k\to\infty} d(x_k, p)$ existsm and hence $\{x_k\}$ is bounded. Since $\lim_{k\to\infty} d(x_k, T_jx_k) = 0$ for all $j = 1, 2, \ldots, m$, then by Lemma 4.11 $\omega_w(x_k) \subset F(T_j)$ for all $j = 1, 2, \ldots, m$, and hence $\omega_w(x_k) \subset \bigcap_{j=1}^m F(T_j) = F$. Since $\omega_w(x_n)$ consists of exactly one point, then $\{x_k\}$ Δ-converges to an element of F. □

Before proving our strong convergence theorem, we recall that a mapping $T : C \to C$ is said to be *semicompact* if C is closed and, for any bounded sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $x \in C$ such that $\lim_{k\to\infty} x_{n_k} = x$.

Theorem 4.13. *Let C be a nonempty closed convex subset of X and $T_1, \ldots, T_m \in \mathfrak{T}_r(C)$ such that T_i^l is semicompact for some $i \in \{1, \ldots, m\}$ and $l \in \mathbb{N}$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.4) is well defined. Suppose that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{Q} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasiperiodic. Then, $\{x_k\}$ converges to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$.*

Proof. By Lemma 4.8, we have

$$\lim_{k\to\infty} d(x_k, T_i x_k) = 0, \quad \text{for } i = 1,\dots,m. \tag{4.48}$$

Let $i \in \{1,\dots,m\}$ be such that T_i^l is semicompact. Thus, by Lemma 4.9,

$$\lim_{k\to\infty} d\left(x_k, T_i^l x_k\right) = 0. \tag{4.49}$$

We can also find a subsequence $\{x_{n_j}\}$ of $\{x_k\}$ such that $\lim_{j\to\infty} x_{k_j} = q \in C$. Hence, from (4.48), we have

$$d(q, T_i q) = \lim_{j\to\infty} d\left(x_{k_j}, T_i x_{k_j}\right) = 0, \quad \forall i = 1,\dots,m. \tag{4.50}$$

Thus, $q \in F$, and, by Corollary 4.6, $\{x_k\}$ converges to q. This completes the proof. □

5. Concluding Remarks

One may observe that our method can be used to obtain the analogous results for uniformly convex Banach spaces. Let C be a nonempty closed convex subset of a Banach space X and fix $x_1 \in C$. Define a sequence $\{x_k\}$ in C as

$$\begin{aligned}
x_{k+1} &= (1 - t_{mk}) x_k + t_{mk} T_m^{n_k} y_{(m-1)k}, \\
y_{(m-1)k} &= \left(1 - t_{(m-1)k}\right) x_k + t_{(m-1)k} T_{m-1}^{n_k} y_{(m-2)k}, \\
y_{(m-2)k} &= \left(1 - t_{(m-2)k}\right) x_k + t_{(m-2)k} T_{m-2}^{n_k} y_{(m-3)k}, \\
&\vdots \\
y_{2k} &= (1 - t_{2k}) x_k + t_{2k} T_2^{n_k} y_{1k}, \\
y_{1k} &= (1 - t_{1k}) x_k + t_{1k} T_1^{n_k} y_{0k}, \\
y_{0k} &= x_k, \quad k \in \mathbb{N},
\end{aligned} \tag{5.1}$$

where $T_1,\dots,T_m \in \mathfrak{T}_r(C)$, $\{t_{ik}\}_{k=1}^{\infty}$ are sequences in $[0,1]$ for all $i = 1,2,\dots,m$, and $\{n_k\}$ is an increasing sequence of natural numbers.

Theorem 5.1. *Let X be a uniformly convex Banach space with the Opial property, and let C be a nonempty closed convex subset of X. Let $T_1,\dots,T_m \in \mathfrak{T}_r(C)$, $\{t_{ik}\}_{k=1}^{\infty} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1/2)$, and let $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (5.1) is well defined. Suppose that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasiperiodic. Then, $\{x_k\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1,2,\dots,m\}$.*

Theorem 5.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T_1,\dots,T_m \in \mathfrak{T}_r(C)$ such that T_i^l is semicompact for some $i \in \{1,\dots,m\}$ and $l \in \mathbb{N}$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1/2)$, and let $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (5.1) is well defined. Suppose that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasiperiodic. Then, $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1,2,\dots,m\}$.*

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