

## *Research Article*

# **Formulation and Solution of $n$ th-Order Derivative Fuzzy Integrodifferential Equation Using New Iterative Method with a Reliable Algorithm**

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Received 11 July 2012; Revised 9 September 2012; Accepted 10 September 2012

Academic Editor: Fazlollah Soleymani

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The  $n$ th-order derivative fuzzy integro-differential equation in parametric form is converted to its crisp form, and then the new iterative method with a reliable algorithm is used to obtain an approximate solution for this crisp form. The analysis is accompanied by numerical examples which confirm efficiency and power of this method in solving fuzzy integro-differential equations.

## **1. Introduction**

In 1975 Zadeh and then Dubois and Prade [1] introduced fuzzy numbers and fuzzy arithmetic. This concept propagated widely to various problems, for example, fuzzy linear systems [2, 3], fuzzy differential equations [4, 5], fuzzy integral equations [6–8], and fuzzy integro-differential equations [9–12]. Additional topics can be found in [13, 14]. Recently, several numerical methods were suggested to solve integro-differential equations, for example, Sine-Cosine wavelet used by Tavassoli Kajani et al. to obtain a solution of linear integro-differential equations [15] and variational iteration method used by Abbasbandy and Hashemi to formulate and solve fuzzy integro-differential equation [12], by Saberi-Nadjafi and Tamamgar for solving system of integro-differential equations [16], and by Shang and Han for solving  $n$ th-order integro-differential equations [17]. Some other worthwhile works can be found in [18, 19]. Recently, Daftardar-Gejji and Jafari [20, 21] proposed the new iterative method. This method has proven useful for solving a variety of linear and nonlinear equations such as algebraic equations, integral equations, ordinary and partial differential equations of integer and fractional order, and system of equations as well. The new iterative method is simple to understand and easy to implement using computer packages and

yields better results [21] than the existing Adomian decomposition method [22], homotopy perturbation method [23], or variational iteration method [24]. For more details, see [25–40].

In the present work we apply the new iterative method with a reliable algorithm to solve the  $n$ th-order derivative fuzzy integro-differential equation in its crisp form.

## 2. Preliminaries

In this section we set up the basic definitions of fuzzy numbers and fuzzy functions.

*Definition 2.1* (see [2, 3, 12]). A fuzzy number in parametric form is an ordered pair of functions  $(\underline{u}(r), \bar{u}(r))$ ,  $0 \leq r \leq 1$  which satisfy the following requirements:

- (1)  $\underline{u}(r)$  is a bounded left continuous nondecreasing function over  $[0, 1]$ ,
- (2)  $\bar{u}(r)$  is a bounded left continuous nonincreasing function over  $[0, 1]$ ,
- (3)  $\underline{u}(r) \leq \bar{u}(r)$ ,  $0 \leq r \leq 1$ .

*Remark 2.2* (see [6, 12]). Let  $u(r) = (\underline{u}(r), \bar{u}(r))$ ,  $0 \leq r \leq 1$  be a fuzzy number, and we can take

$$u^c(r) = \frac{1}{2}(\underline{u}(r) + \bar{u}(r)), \quad u^d(r) = \frac{1}{2}(\bar{u}(r) - \underline{u}(r)). \quad (2.1)$$

It is clear that  $u^d(r) \geq 0$ ,  $\underline{u}(r) = u^c(r) - u^d(r)$  and  $\bar{u}(r) = u^c(r) + u^d(r)$ , and also a fuzzy number  $u \in E$  is said symmetric if  $u^c(r)$  is independent of  $r$  for all  $0 \leq r \leq 1$ .

Let  $f : [a, b] \rightarrow E$  for each partition  $P = \{t_0, \dots, t_n\}$  of  $[a, b]$  and for arbitrary  $\xi_i \in [t_{i-1}, t_i]$ ,  $1 \leq i \leq n$  suppose that  $R_P = \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1})$  and  $\Delta = \max\{|t_i - t_{i-1}|, i = 1, \dots, n\}$ . The definite integral of  $f(t)$  over  $[a, b]$  is  $\int_a^b f(t) dt = \lim_{\Delta \rightarrow 0} R_P$  provided that this limit exists in the metric  $D$ . If the fuzzy function  $f(t)$  is continuous in the metric  $D$ , its definite integral exists. Also we have  $(\int_a^b \underline{f}(t, r) dt) = \int_a^b \underline{f}(t, r) dt$  and  $(\int_a^b \bar{f}(t, r) dt) = \int_a^b \bar{f}(t, r) dt$ .

*Definition 2.3* (see [33]). Let  $f : (a, b) \rightarrow R_F$  and  $t_0 \in (a, b)$ . We say that  $f'$  is Hukuhara differentiable at  $t_0$  if there exists an element  $f'(t_0) \in R_F$ , such that for all  $h > 0$  sufficiently small,  $\exists f(t_0 + h) \ominus_H f(t_0), f(t_0) \ominus_H f(t_0 - h)$  and

$$\lim_{h \rightarrow 0} \frac{f(t_0 + h) \ominus_H f(t_0)}{h} = \lim_{h \rightarrow 0} \frac{f(t_0) \ominus_H f(t_0 - h)}{h} = f'(t_0), \quad (2.2)$$

where  $\ominus_H$  is Hukuhara difference,  $R_F$  is the set of fuzzy numbers, and the limit is in the metric  $D$ .

In parametric form, if  $f = (\underline{f}, \bar{f})$ , then  $f' = (\underline{f}', \bar{f}')$ , where  $\underline{f}', \bar{f}'$  are Hukuhara differentiable of  $\underline{f}, \bar{f}$ , respectively.

From (2.2), the  $n$ th-order Hukuhara differentiable,  $f^{(n)}$ , of  $f$  at  $t_0$  can be defined as in the following definition.

*Definition 2.4.* Let  $f^{(n-1)} : (a, b) \rightarrow R_F$  and  $t_0 \in (a, b)$  where  $f^{(n-1)}$  is  $(n-1)$ th-order Hukuhara differentiable of  $f$  at  $t_0$  for all  $n > 1$ . We say that  $f^{(n)}$  is  $n$ th-order Hukuhara differentiable at

$t_0$  if there exists an element  $f^{(n)}(t_0) \in R_F$  such that for all  $h > 0$  sufficiently small,  $\exists f^{(n-1)}(t_0 + h) \ominus_H f^{(n-1)}(t_0), f^{(n-1)}(t_0) \ominus_H f^{(n-1)}(t_0 - h)$  and

$$\lim_{h \rightarrow 0} \frac{f^{(n-1)}(t_0 + h) \ominus_H f^{(n-1)}(t_0)}{h} = \lim_{h \rightarrow 0} \frac{f^{(n-1)}(t_0) \ominus_H f^{(n-1)}(t_0 - h)}{h} = f^{(n)}(t_0), \quad (2.3a)$$

$n = 1, 2, \dots$  In parametric form, as above, if  $f = (\underline{f}, \bar{f})$ , then  $f^{(n)} = (\underline{f}^{(n)}, \bar{f}^{(n)})$ , where  $\underline{f}^{(n)}, \bar{f}^{(n)}$  are  $n$ th-order Hukuhara differentiable of  $\underline{f}, \bar{f}$ , respectively. From (2.3a), in case  $n = 1$ , we obtain (2.2).

### 3. Fuzzy Integro-Differential Equation

Consider the  $n$ th-order derivative integro-differential equation [12] as follows:

$$u^{(n)}(t) + u(t) + \lambda \int_0^T k(s, t) u^{(m)}(s) ds = g(t), \quad (3.1a)$$

where  $m, n \in \mathbb{N}$ ,  $m < n$ ,  $t \in [0, T]$ ,  $\lambda \in \mathbb{R}$ ,  $g(t)$  is a known function and the kernel  $k(s, t) \geq 0$  with the initial conditions as follows:

$$u^{(k)}(0) = h_k, \quad k = 0, 1, \dots, n-1, \quad h_k \in \mathbb{R}, \quad (3.1b)$$

in the fuzzy case; that is,  $u$  and  $g$  be fuzzy functions. Let

$$\begin{aligned} u(t, r) &= (\underline{u}(t, r), \bar{u}(t, r)), & g(t, r) &= (\underline{g}(t, r), \bar{g}(t, r)), \\ u^{(n)}(t, r) &= (\underline{u}^{(n)}(t, r), \bar{u}^{(n)}(t, r)), & g^{(n)}(t, r) &= (\underline{g}^{(n)}(t, r), \bar{g}^{(n)}(t, r)), \end{aligned} \quad (3.2)$$

where all derivatives are, with respect to  $t$ , fuzzy functions. Therefore, related fuzzy integro-differential equation of (3.1a) can be written as follows:

$$\underline{u}^{(n)}(t, r) + \underline{u}(t, r) + \lambda \int_0^T k(s, t) \underline{u}^{(m)}(s, r) ds = \underline{g}(t, r), \quad (3.3a)$$

$$\bar{u}^{(n)}(t, r) + \bar{u}(t, r) + \lambda \int_0^T k(s, t) \bar{u}^{(m)}(s, r) ds = \bar{g}(t, r), \quad (3.3b)$$

and its two crisp equations can be written as follows:

$$u^{c(n)}(t, r) + u^c(t, r) + \lambda \int_0^T k(s, t) u^{c(m)}(s, r) ds = g^c(t, r), \quad (3.4a)$$

$$u^{d(n)}(t, r) + u^d(t, r) + \lambda \int_0^T k(s, t) u^{d(m)}(s, r) ds = g^d(t, r). \quad (3.4b)$$

#### 4. New Iterative Method

For simplicity, we present a review of the new iterative method [20, 21, 34–39, 41], and then we introduce a suitable algorithm of this method for solving the  $n$ th-order derivative fuzzy integro-differential equations.

Consider the following general functional equation:

$$u = f + N(u), \quad (4.1)$$

where  $N$  is a nonlinear operator from a Banach space  $B \rightarrow B$ , and  $f$  is a known function.

We are looking for a solution  $u$  of (4.1) having the series form:

$$u = \sum_{i=0}^{\infty} u_i. \quad (4.2)$$

The nonlinear operator  $N$  can be decomposed as follows:

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=0}^{\infty} \left\{ N\left(\sum_{j=0}^i u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\}. \quad (4.3)$$

From (4.2) and (4.3), (4.1) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^i u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\}. \quad (4.4)$$

We define the recurrence relation:

$$\begin{aligned} u_0 &= f, \\ u_1 &= N(u_0), \\ u_{n+1} &= N(u_0 + u_1 + \cdots + u_n) - N(u_0 + u_1 + \cdots + u_{n-1}), \quad n = 1, 2, \dots \end{aligned} \quad (4.5)$$

Then

$$\begin{aligned} (u_1 + \cdots + u_{n+1}) &= N(u_0 + \cdots + u_n), \quad n = 1, 2, \dots, \\ \sum_{i=0}^{\infty} u_i &= f + \sum_{i=1}^{\infty} u_i. \end{aligned} \quad (4.6)$$

The  $n$ -term approximate solution of (4.1) is given by  $u = u_0 + u_1 + \cdots + u_{n-1}$ .

### 4.1. Reliable Algorithm

After the above presentation of the new iterative method, we introduce a reliable algorithm of this method for solving any  $n$ th-order derivative fuzzy integro-differential equation. Consider the  $n$ th-order derivative integro-differential equation (3.1a) and (3.1b) defined as

$$u^{(n)}(t) + u(t) + \lambda \int_0^T k(s,t)u^{(m)}(s)ds = g(t), \quad (4.7a)$$

with the following initial conditions:

$$u^{(k)}(0) = h_k, \quad k = 0, 1, \dots, n-1, \quad h_k \in \mathbb{R}. \quad (4.7b)$$

The initial value problem (4.7a) and (4.7b) is equivalent to the following integral equation:

$$u(t) = u_0(t) - I_t^n \left[ u(t) + \lambda \int_0^T k(s,t)u^{(m)}(s)ds \right] = u_0(t) - N(u), \quad (4.8)$$

where  $u_0(t)$  is the solution of the  $n$ th-order differential equation:

$$\frac{d^n u_0}{dt^n} = g(t), \quad u_0^{(k)}(0) = h_k, \quad k = 0, 1, 2, \dots, n-1, \quad h_k \in \mathbb{R}, \quad (4.9a)$$

$$N(u) = I_t^n \left[ u(t) + \lambda \int_0^T k(s,t)u^{(m)}(s)ds \right], \quad (4.9b)$$

and  $I_t^n$  is an integral operator of order  $n$ . Also, the two crisp equations (3.4a) and (3.4b) are equivalent to the two integral equations:

$$u^c(t, r) = u_0^c(t, r) - I_t^n \left[ u^c(t, r) + \lambda \int_0^T k(s,t)u^{c(m)}(s, r)ds \right] = u_0^c(t, r) - N(u^c), \quad (4.10a)$$

$$u^d(t, r) = u_0^d(t, r) - I_t^n \left[ u^d(t, r) + \lambda \int_0^T k(s,t)u^{d(m)}(s, r)ds \right] = u_0^d(t, r) - N(u^d), \quad (4.10b)$$

where  $u_0^c(t, r)$  and  $u_0^d(t, r)$  are the solutions of the  $n$ th-order differential equations:

$$\frac{d^n u_0^c}{dt^n} = g^c(t), \quad u_0^{c(k)}(0) = h_k^c, \quad k = 0, 1, 2, \dots, n-1, \quad h_k^c \in \mathbb{R}, \quad (4.10c)$$

$$\frac{d^n u_0^d}{dt^n} = g^d(t), \quad u_0^{d(k)}(0) = h_k^d, \quad k = 0, 1, 2, \dots, n-1, \quad h_k^d \in \mathbb{R}. \quad (4.10d)$$

$N(u^c)$  and  $N(u^d)$  are the following two integral equations:

$$N(u^c) = I_t^n \left[ u^c(t, r) + \lambda \int_0^T k(s, t) u^{c(m)}(s, r) ds \right], \quad (4.10e)$$

$$N(u^d) = I_t^n \left[ u^d(t, r) + \lambda \int_0^T k(s, t) u^{c(m)}(s, r) ds \right]. \quad (4.10f)$$

We get the solution of (4.8) or the two (4.10a), (4.10b) by employing the recurrence relation (4.5).

## 4.2. Convergence of the New Iterative Method

Now, we introduce the condition of convergence of the new iterative method, which is proposed by Daftardar-Gejji and Jafari in (2006) [20], also called (DJM) [39]. From (4.3), the nonlinear operator  $N$  is decomposed as follows [39]:

$$N(u) = N(u_0) + [N(u_0 + u_1) - N(u_0)] + [N(u_0 + u_1 + u_2) - N(u_0 + u_1)] + \dots \quad (4.11)$$

Let  $G_0 = N(u_0)$  and

$$G_n = N\left(\sum_{i=0}^n u_i\right) - N\left(\sum_{i=0}^{n-1} u_i\right), \quad n = 1, 2, \dots \quad (4.12)$$

Then  $N(u) = \sum_{i=0}^{\infty} G_i$ .

Set

$$u_0 = f, \quad (4.13)$$

$$u_n = G_{n-1}, \quad n = 1, 2, \dots \quad (4.14)$$

Then

$$u = \sum_{i=0}^{\infty} u_i \quad (4.15)$$

is a solution of the general functional equation (4.1). Also, the recurrence relation (4.5) becomes

$$u_0 = f, \quad (4.16)$$

$$u_n = G_{n-1}, \quad n = 1, 2, \dots$$

Using Taylor series expansion for  $G_i$ ,  $i = 1, 2, \dots, n$ , we have

$$\begin{aligned}
 G_1 &= N(u_0 + u_1) - N(u_0) = N(u_0) + N'(u_0)u_1 + N''(u_0)\frac{u_1^2}{2!} \\
 &\quad + \dots - N(u_0) = \sum_{k=1}^{\infty} N^{(k)}(u_0)\frac{u_1^k}{k!}, \\
 G_2 &= N(u_0 + u_1 + u_2) - N(u_0 + u_1) = N'(u_0 + u_1)u_2 + N''(u_0 + u_1)\frac{u_2^2}{2!} \\
 &\quad + \dots = \sum_{j=1}^{\infty} \left[ \sum_{i=0}^{\infty} N^{(i+j)}(u_0)\frac{u_1^i}{i!} \right] \frac{u_2^j}{j!}, \\
 G_3 &= \sum_{i_3=1}^{\infty} \sum_{i_2=0}^{\infty} \sum_{i_1=0}^{\infty} N^{(i_1+i_2+i_3)}(u_0)\frac{u_3^{i_3}}{i_3!}\frac{u_2^{i_2}}{i_2!}\frac{u_1^{i_1}}{i_1!}.
 \end{aligned} \tag{4.17}$$

In general:

$$G_n = \sum_{i_n=1}^{\infty} \sum_{i_{n-1}=0}^{\infty} \dots \sum_{i_1=0}^{\infty} \left[ N^{(\sum_{k=1}^n i_k)}(u_0) \left( \prod_{j=1}^n \frac{u_j^{i_j}}{i_j!} \right) \right]. \tag{4.18}$$

In the following theorem we state and prove the condition of convergence of the method.

**Theorem 4.1.** *If  $N$  is  $C^{(\infty)}$  in a neighborhood of  $u_0$  and*

$$\left\| N^{(n)}(u_0) \right\| = \sup \left\{ N^{(n)}(u_0)(h_1, \dots, h_n) : \|h_i\| \leq 1, 1 \leq i \leq n \right\} \leq L, \tag{4.19a}$$

for any  $n$  and for some real  $L > 0$  and  $\|u_i\| \leq M < 1/e$ ,  $i = 1, 2, \dots$ , then the series  $\sum_{n=0}^{\infty} G_n$  is absolutely convergent, and, moreover,

$$\|G_n\| \leq LM^n e^{n-1}(e-1), \quad n = 1, 2, \dots \tag{4.19b}$$

*Proof.* In view of (4.18),

$$\|G_n\| \leq LM^n \sum_{i_n=1}^{\infty} \sum_{i_{n-1}=0}^{\infty} \dots \sum_{i_1=0}^{\infty} \left( \prod_{j=1}^n \frac{u_j^{i_j}}{i_j!} \right) = LM^n e^{n-1}(e-1). \tag{4.19c}$$

Thus, the series  $\sum_{n=1}^{\infty} \|G_n\|$  is dominated by the convergent series  $LM(e-1)\sum_{n=1}^{\infty} (Me)^{n-1}$ , where  $M < 1/e$ . Hence,  $\sum_{n=0}^{\infty} G_n$  is absolutely convergent, due to the comparison test.

For more details, see [39]. □

## 5. Numerical Examples

*Example 5.1.* Consider the following fuzzy integro-differential equation:

$$u'(t, r) + \lambda \int_0^1 k(s, t)u(s, r)ds = g(t, r), \quad 0 \leq t, r \leq 1, \quad (5.1)$$

where  $u(t, r) = (\underline{u}(t, r), \bar{u}(t, r)) = (-2rt, 4 + rt)$  is the exact solution for (5.1),  $g(t, r) = (\underline{g}(t, r), \bar{g}(t, r)) = (-2r - 2rt/3, r + 2t + rt/3)$ ,  $k(s, t) = st$ , and  $\lambda = 1$ , with the initial conditions  $\underline{u}(0, r) = 0$ ,  $\bar{u}(0, r) = 4$ . Taking  $g^c(t, r) = (1/2)(\underline{g}(t, r) + \bar{g}(t, r)) = -r/2 + t - rt/6$ ,  $g^d(t, r) = (1/2)(\bar{g}(t, r) - \underline{g}(t, r)) = 3r/2 + t + rt/2$ . The exact solution for related crisp equations is  $u^c(t, r) = 2 - rt/2$ ,  $u^d(t, r) = 2 + 3rt/2$ . At first, we identify  $u^c(t, r)$  and  $u^d(t, r)$ . From (3.4a) and (3.4b), (5.1), where  $n = 1$ , we have

$$u^c(t, r) + \int_0^1 stu^c(s, r)ds = g^c(t, r), \quad u^c(0, r) = 2, \quad (5.2a)$$

$$u^d(t, r) + \int_0^1 stu^d(s, r)ds = g^d(t, r), \quad u^d(0, r) = 2. \quad (5.2b)$$

From (4.10c), (4.10d), we obtain

$$u_0^c(t, r) = 2 - \frac{rt}{2} + \frac{t^2}{2} - \frac{rt^2}{12}, \quad u_0^d(t, r) = 2 + \frac{3rt}{2} + \frac{t^2}{2} + \frac{rt^2}{4}. \quad (5.3)$$

Therefore, from (4.10a), (4.10b) the fuzzy integro-differential equations (5.2a) and (5.2b) are equivalent to the following integral equations:

$$\begin{aligned} u^c(t, r) &= 2 - \frac{rt}{2} + \frac{t^2}{2} - \frac{rt^2}{12} - I_t \left[ \int_0^1 stu^c(s, r)ds \right], \\ u^d(t, r) &= 2 + \frac{3rt}{2} + \frac{t^2}{2} + \frac{rt^2}{4} - I_t \left[ \int_0^1 stu^d(s, r)ds \right]. \end{aligned} \quad (5.4)$$



Let  $N(u^c) = -I_t[\int_0^1 stu^c(s, r)ds]$ ,  $N(u^d) = -I_t[\int_0^1 stu^d(s, r)ds]$ . Therefore, from (4.5), we can obtain easily the following first few components of the new iterative solution for (5.1):

$$\begin{aligned} u_0^c(t, r) &= 2 - \frac{rt}{2} + \frac{t^2}{2} - \frac{rt^2}{12}, & u_0^d(t, r) &= 2 + \frac{3rt}{2} + \frac{t^2}{2} + \frac{rt^2}{4}, \\ u_1^c(t, r) &= N(u_0^c) = -\frac{9t^2}{16} + \frac{9rt^2}{96}, & u_1^d(t, r) &= N(u_0^d) = -\frac{9t^2}{16} - \frac{9rt^2}{32}, \\ u_2^c(t, r) &= \frac{9t^2}{128} - \frac{9rt^2}{768}, & u_2^d(t, r) &= \frac{9t^2}{128} + \frac{9rt^2}{256}, \\ & \vdots & & \\ u_5^c(t, r) &= -\frac{9t^2}{65536} + \frac{9rt^2}{393216}, & u_5^d(t, r) &= -\frac{9t^2}{65536} - \frac{9rt^2}{131072} \end{aligned} \quad (5.5)$$

and so on. The 6-term approximate solution is

$$\begin{aligned} u^c(t, r) &= \sum_{n=0}^5 u_n^c = 2 - \frac{rt}{2} - \frac{t^2}{65536} + \frac{rt^2}{393216}, \\ u^d(t, r) &= \sum_{n=0}^5 u_n^d = 2 + \frac{3rt}{2} - \frac{t^2}{65536} - \frac{rt^2}{131072}. \end{aligned} \quad (5.6)$$

It is clear that the iterations converge to the exact solution of the two crisp equations  $u^c, u^d$  as the number of iteration converges to  $\infty$ , that is,  $\lim_{n \rightarrow \infty} u_n^c(t, r) = 2 - rt/2 = u_{\text{Exact}}^c$ , and  $\lim_{n \rightarrow \infty} u_n^d(t, r) = 2 + 3rt/2 = u_{\text{Exact}}^d$ . From the relations between  $\underline{u}, \bar{u}, u^c, u^d$ , and  $u$  in Section 2, it follows immediately that this solution is the solution of the fuzzy integro-differential equation (5.1).

*Example 5.2.* Consider the following fuzzy integro-differential equation:

$$u''(t, r) + \lambda \int_0^1 k(s, t)u(s, r)ds = g(t, r), \quad 0 \leq t, r \leq 1. \quad (5.7)$$

In this example  $u(t, r) = (\underline{u}(t, r), \bar{u}(t, r)) = (2rt, 4rt)$  is the exact solution for (5.7),  $g(t, r) = (\underline{g}(t, r), \bar{g}(t, r)) = (2rt/3, 4rt/3)$ ,  $k(s, t) = st$ ,  $\lambda = 1$ , and the initial conditions are  $\underline{u}(0, r) = 0$ ,  $\underline{u}'(0, r) = 2r$ ,  $\bar{u}(0, r) = 0$ ,  $\bar{u}'(0, r) = 4r$ . Also, the exact solution for related crisp equations is  $u^c(t, r) = 3rt$ ,  $u^d(t, r) = rt$  with  $g^c(t, r) = rt$ ,  $g^d(t, r) = rt/3$ . At first, we identify  $u^c(t, r)$ ,  $u^d(t, r)$ . From (3.4a) and (3.4b), (5.7), where  $n = 2$ , we have

$$u^{c''}(t, r) + \int_0^1 stu^c(s, r)ds = g^c(t, r), \quad u^c(0, r) = 0, \quad u^c(0, r) = 3r, \quad (5.8a)$$

$$u^{d''}(t, r) + \int_0^1 stu^d(s, r)ds = g^d(t, r), \quad u^d(0, r) = 0, \quad u^d(0, r) = r. \quad (5.8b)$$

From (4.10c), (4.10d), we obtain

$$u_0^c(t, r) = 3rt + \frac{rt^3}{6}, \quad u_0^d(t, r) = rt + \frac{rt^3}{18}. \quad (5.9)$$

Therefore, from (4.10a), (4.10b), the fuzzy integro-differential equations (5.8a) and (5.8b) are equivalent to the following integral equations:

$$\begin{aligned} u^c(t, r) &= 3rt + \frac{rt^3}{6} - I_t^2 \left[ \int_0^1 stu^c(s, r) ds \right], \\ u^d(t, r) &= rt + \frac{rt^3}{18} - I_t^2 \left[ \int_0^1 stu^d(s, r) ds \right]. \end{aligned} \quad (5.10)$$

Let  $N(u^c) = -I_t^2 \left[ \int_0^1 stu^c(s, r) ds \right]$ ,  $N(u^d) = -I_t^2 \left[ \int_0^1 stu^d(s, r) ds \right]$ . Therefore, from (4.5), we can obtain easily the following first few components of the new iterative solution for (5.7):

$$\begin{aligned} u_0^c(t, r) &= 3rt + \frac{rt^3}{6}, & u_0^d(t, r) &= rt + \frac{rt^3}{18}, \\ u_1^c(t, r) &= N(u_0^c) = -\frac{31rt^3}{180}, & u_1^d(t, r) &= N(u_0^d) = -\frac{31rt^3}{540}, \\ u_2^c(t, r) &= \frac{31rt^3}{5400}, & u_2^d(t, r) &= \frac{31rt^2}{16200}, \\ &\vdots & & \\ u_5^c(t, r) &= -\frac{31rt^3}{145800000}, & u_5^d(t, r) &= -\frac{29rt^2}{437400000}, \end{aligned} \quad (5.11)$$

and so on. The 6-term approximate solution is

$$u^c(t, r) = \sum_{n=0}^5 u_n^c = 3rt - \frac{rt^3}{145800000}, \quad u^d(t, r) = \sum_{n=0}^5 u_n^d = rt - \frac{rt^3}{437400000}. \quad (5.12)$$

It is clear that the iterations converge to the exact solution of  $u^c, u^d$  as  $n \rightarrow \infty$ , that is,  $\lim_{n \rightarrow \infty} u_n^c(t, r) = 3rt = u_{\text{Exact}}^c$ ,  $\lim_{n \rightarrow \infty} u_n^d(t, r) = rt = u_{\text{Exact}}^d$ . As the above example the solution of (5.7) follows immediately.

*Example 5.3.* The third example is the fuzzy integro-differential equation:

$$u''(t, r) + \lambda \int_0^1 k(s, t) u'(s, r) ds = g(t, r), \quad 0 \leq t, r \leq 1, \quad (5.13)$$

where  $u(t, r) = (\underline{u}(t, r), \bar{u}(t, r)) = (rt, 4 - 3rt)$  is the exact solution for (5.13),  $g(t, r) = (\underline{g}(t, r), \bar{g}(t, r)) = (-rt/2, 3rt/2)$ ,  $k(s, t) = st$ ,  $\lambda = -1$ , and the initial conditions are  $\underline{u}(0, r) = 0$ ,

$\underline{u}'(0, r) = r$ ,  $\bar{u}(0, r) = 4$ ,  $\bar{u}'(0, r) = -3r$ . The exact solution for related crisp equations is  $u^c(t, r) = 2 - rt$ ,  $u^d(t, r) = 2 - 2rt$  with  $g^c(t, r) = rt/2$ ,  $g^d(t, r) = rt$ . At first, we identify  $u^c(t, r)$ ,  $u^d(t, r)$ , and from (3.4a) and (3.4b), (5.13), where  $n = 2$ , we have

$$u^{c'}(t, r) - \int_0^1 stu^{c'}(s, r)ds = g^c(t, r), \quad u^c(0, r) = 2, \quad u^{c'}(0, r) = -r, \quad (5.14a)$$

$$u^{d''}(t, r) - \int_0^1 stu^{d''}(s, r)ds = g^d(t, r), \quad u^d(0, r) = 2, \quad u^{d''}(0, r) = -2r. \quad (5.14b)$$

From (4.10c), (4.10d), we obtain

$$u_0^c(t, r) = 2 - rt + \frac{rt^3}{12}, \quad u_0^d(t, r) = 2 - 2rt + \frac{rt^3}{6}. \quad (5.15)$$

Therefore, from (4.10a) and (4.10b), the fuzzy integro-differential equations (5.14a) and (5.14b) are equivalent to the integral equations:

$$\begin{aligned} u^c(t, r) &= 2 - rt + \frac{rt^3}{12} + I_t^2 \left[ \int_0^1 stu^{c'}(s, r)ds \right], \\ u^d(t, r) &= 2 - 2rt + \frac{rt^3}{6} + I_t^2 \left[ \int_0^1 stu^{d''}(s, r)ds \right]. \end{aligned} \quad (5.16)$$

Let  $N(u^c) = I_t^2[\int_0^1 stu^{c'}(s, r)ds]$ ,  $N(u^d) = I_t^2[\int_0^1 stu^{d''}(s, r)ds]$ . Therefore, from (4.5), we can obtain easily the following first few components of the new iterative solution for (5.13):

$$\begin{aligned} u_0^c(t, r) &= 2 - rt + \frac{rt^3}{12}, & u_0^d(t, r) &= 2 - 2rt + \frac{rt^3}{6}, \\ u_1^c(t, r) &= N(u_0^c) = -\frac{7rt^3}{96}, & u_1^d(t, r) &= N(u_0^d) = -\frac{7rt^3}{48}, \\ u_2^c(t, r) &= -\frac{7rt^3}{768}, & u_2^d(t, r) &= -\frac{7rt^3}{384}, \\ &\vdots & & \\ u_5^c(t, r) &= -\frac{7rt^3}{393216}, & u_5^d(t, r) &= -\frac{7rt^3}{196608}, \end{aligned} \quad (5.17)$$

and so on. The 6-term approximate solution is

$$u^c(t, r) = \sum_{n=0}^5 u_n^c = 2 - rt + \frac{rt^3}{393216}, \quad u^d(t, r) = \sum_{n=0}^5 u_n^d = 2 - 2rt + \frac{rt^3}{196608}. \quad (5.18)$$

It is clear that the iterations converge to the exact solution of  $u^c, u^d$  as  $n \rightarrow \infty$ , that is,  $\lim_{n \rightarrow \infty} u_n^c(t, r) = 2 - rt = u_{\text{Exact}}^c$ ,  $\lim_{n \rightarrow \infty} u_n^d(t, r) = 2 - 2rt = u_{\text{Exact}}^d$ . Therefore the solution of (5.13) follows immediately.

*Example 5.4.* Let us consider the following fuzzy integro-differential equation:

$$u''(t, r) + u(t, r) + \lambda \int_0^1 k(s, t)u(s, r)ds = g(t, r), \quad 0 \leq t, r \leq 1, \quad (5.19)$$

with the exact solution  $u(t, r) = (\underline{u}(t, r), \bar{u}(t, r)) = (3rt, 8 - rt)$ ,  $g(t, r) = (\underline{g}, \bar{g}) = (2rt, 8 - 4t - (2rt/3))$ ,  $k(s, t) = st$ ,  $\lambda = -1$ , and the initial conditions  $\underline{u}(0, r) = 0$ ,  $\underline{u}'(0, r) = 3r$ ,  $\bar{u}(0, r) = 8$ ,  $\bar{u}'(0, r) = -r$ . The exact solution for related crisp equations is  $u^c(t, r) = 4 + rt$ ,  $u^d(t, r) = 4 - 2rt$  with  $g^c(t, r) = 4 - 2t + 2rt/3$ ,  $g^d(t, r) = 4 - 2t - 4rt/3$ . From (3.4a), (3.4b), and (5.19), where  $n = 2$ , the two crisp equations are

$$u^{c''}(t, r) + u^c(t, r) - \int_0^1 stu^c(s, r)ds = g^c(t, r), \quad u^c(0, r) = 4, \quad u^c(0, r) = r, \quad (5.20a)$$

$$u^{d''}(t, r) + u^d(t, r) - \int_0^1 stu^d(s, r)ds = g^d(t, r), \quad u^d(0, r) = 4, \quad u^d(0, r) = -2r. \quad (5.20b)$$

From (4.10c), (4.10d) we obtain

$$u_0^c(t, r) = 4 + rt + 2t^2 - \frac{t^3}{3} + \frac{rt^3}{9}, \quad u_0^d(t, r) = 4 - 2rt + 2t^2 - \frac{t^3}{3} - \frac{2rt^3}{9}. \quad (5.21)$$

Therefore, from (4.10a), (4.10b), the fuzzy integro-differential equations (5.20a) and (5.20b) are equivalent to the following integral equations:

$$\begin{aligned} u^c(t, r) &= 4 + rt + 2t^2 - \frac{t^3}{3} + \frac{rt^3}{9} + I_t^2 \left[ -u^c(t, r) + \int_0^1 stu^c(s, r)ds \right], \\ u^d(t, r) &= 4 - 2rt + 2t^2 - \frac{t^3}{3} - \frac{2rt^3}{9} + I_t^2 \left[ -u^d(t, r) + \int_0^1 stu^d(s, r)ds \right]. \end{aligned} \quad (5.22)$$

Let

$$\begin{aligned} N(u^c) &= I_t^2 \left[ -u^c(t, r) + \int_0^1 stu^c(s, r)ds \right], \\ N(u^d) &= I_t^2 \left[ -u^d(t, r) + \int_0^1 stu^d(s, r)ds \right]. \end{aligned} \quad (5.23)$$

Therefore, from (4.5), we can obtain the following first approximate solutions for (5.19):

$$\begin{aligned}
 u_0^c(t, r) &= 4 + rt + 2t^2 - \frac{t^3}{3} + \frac{rt^3}{9}, & u_0^d(t, r) &= 4 - 2rt + 2t^2 - \frac{t^3}{3} - \frac{2rt^3}{9}, \\
 u_1^c(t, r) &= -2t^2 + \frac{73t^3}{180} - \frac{29rt^3}{270} - \frac{t^4}{6} + \frac{t^5}{60} - \frac{rt^5}{180}, \\
 u_1^d(t, r) &= -2t^2 + \frac{73t^3}{180} + \frac{29rt^3}{135} - \frac{t^4}{6} + \frac{t^5}{60} + \frac{rt^5}{90}, \\
 u_2^c(t, r) &= -\frac{311t^3}{4200} - \frac{421rt^3}{113400} + \frac{t^4}{6} - \frac{73t^5}{3600} + \frac{29rt^5}{5400} + \frac{t^6}{180} - \frac{t^7}{2520} + \frac{rt^7}{7560}, \\
 u_2^d(t, r) &= -\frac{311t^3}{4200} + \frac{421rt^3}{56700} + \frac{t^4}{6} - \frac{73t^5}{3600} - \frac{29rt^5}{2700} + \frac{t^6}{180} - \frac{t^7}{2520} - \frac{rt^7}{3780}, \\
 &\vdots
 \end{aligned} \tag{5.24}$$

and so on. In the same manner the rest of components can be obtained. The 5-term approximate solution is

$$\begin{aligned}
 u^c(t, r) &= \sum_{n=0}^4 u_n^c = 4 + rt + \frac{2.8739t^3}{3143448} - \frac{0.1019rt^3}{9430344} + \frac{0.0523t^5}{27216} + \frac{0.0023rt^5}{20412} \\
 &\quad - \frac{0.023t^7}{10584} - \frac{0.001rt^7}{95256} - \frac{0.13t^9}{108864} - \frac{0.01rt^9}{163296} + \frac{0.01t^{10}}{9072} - \frac{0.01t^{11}}{199584} + \frac{0.01rt^{11}}{598752}, \\
 u^d(t, r) &= \sum_{n=0}^4 u_n^d = 4 - 2rt + \frac{2.8739t^3}{3143448} + \frac{0.1019rt^3}{4715172} + \frac{0.0523t^5}{27216} - \frac{0.0023rt^5}{10206} \\
 &\quad - \frac{0.023t^7}{10584} + \frac{0.001rt^7}{47628} - \frac{0.13t^9}{108864} + \frac{0.01rt^9}{81648} + \frac{0.01t^{10}}{9072} - \frac{0.01t^{11}}{199584} - \frac{0.01rt^{11}}{299376}.
 \end{aligned} \tag{5.25}$$

Now, let us define the absolute value of the  $n$ th-term error by  $|e_n(t, r)| = |u^*(t, r) - u_n(t, r)|$ , where  $u^*(t, r)$  is the exact solution and  $u_n(t, r)$  is the  $n$ th-term approximate solution respectively. It is clear from previous above solutions that the smallest value of the absolute  $n$ th-term error is at  $t = r = 0$ , that is,  $|e_n(0, 0)| = (|e_n^c(0, 0)|, |e_n^d(0, 0)|) = (0, 0)$ ,  $n = 0, 1, \dots, 4$  and the largest value is at  $t = r = 1$  as shown in Table 1. Therefore the iterations converge to the exact solution of  $u^c, u^d$ , that is,  $\lim_{n \rightarrow \infty} u_n^c(t, r) = 4 + rt = u_{\text{Exact}}^c$ ,  $\lim_{n \rightarrow \infty} u_n^d(t, r) = 4 - 2rt = u_{\text{Exact}}^d$  and the solution of (5.19) follows immediately.

*Example 5.5.* The final example is the following fuzzy integro-differential equation:

$$u''(t, r) + u(t, r) + \lambda \int_0^1 k(s, t)u'(s, r)ds = g(t, r), \quad 0 \leq t, r \leq 1, \tag{5.26}$$

Table 1

$n$	0	1	2	3	4
$ e_n^c $	1.777778	$7.9629629^{-2}$	$3.395065^{-4}$	$4.588^{-5}$	$5.67^{-7}$
$ e_n^d $	1.444444	$7.4074074^{-2}$	$1.543214^{-4}$	$4.3601^{-5}$	$3.96^{-7}$

with the exact solution  $u(t, r) = (\underline{u}(t, r), \bar{u}(t, r)) = (3rt, 4 + rt)$ ,  $g(t, r) = (\underline{g}, \bar{g}) = (2rt, 4 + 2rt/3)$ ,  $k(s, t) = s^2t$ ,  $\lambda = -1$  and the initial values  $\underline{u}(0, r) = 0$ ,  $\underline{u}'(0, r) = 3r$ ,  $\bar{u}(0, r) = 4$ ,  $\bar{u}'(0, r) = r$ . The exact solution for related crisp equations is  $u^c(t, r) = 2 + 2rt$ ,  $u^d(t, r) = 2 - rt$  with  $g^c(t, r) = 2 + 4rt/3$ ,  $g^d(t, r) = 2 - 2rt/3$ . As above, where  $n = 2$ , the two crisp equations are

$$u^{c''}(t, r) + u^c(t, r) - \int_0^1 s^2 t u^{c'}(s, r) ds = g^c(t, r), \quad u^c(0, r) = 2, \quad u^{c'}(0, r) = 2r, \quad (5.27a)$$

$$u^{d''}(t, r) + u^d(t, r) - \int_0^1 s^2 t u^{d'}(s, r) ds = g^d(t, r), \quad u^d(0, r) = 2, \quad u^{d'}(0, r) = -r. \quad (5.27b)$$

As the above examples, we obtain

$$u_0^c(t, r) = 2 + 2rt + t^2 + \frac{2rt^3}{9}, \quad u_0^d(t, r) = 2 - rt + t^2 - \frac{rt^3}{9}. \quad (5.28)$$

Therefore, the fuzzy integro-differential equations (5.27a) and (5.27b) are equivalent to the following integral equations:

$$u^c(t, r) = 2 + 2rt + t^2 + \frac{2rt^3}{9} + I_t^2 \left[ -u^c(t, r) + \int_0^1 s^2 t u^{c'}(s, r) ds \right], \quad (5.29)$$

$$u^d(t, r) = 2 - rt + t^2 - \frac{rt^3}{9} + I_t^2 \left[ -u^d(t, r) + \int_0^1 s^2 t u^{d'}(s, r) ds \right].$$

Let

$$N(u^c) = I_t^2 \left[ -u^c(t, r) + \int_0^1 s^2 t u^{c'}(s, r) ds \right], \quad (5.30)$$

$$N(u^d) = I_t^2 \left[ -u^d(t, r) + \int_0^1 s^2 t u^{d'}(s, r) ds \right].$$

Table 2

$n$	0	1	2	3	4
$ e_n^c $	1.2222222	1.1111111 <sup>-2</sup>	2.2222222 <sup>-3</sup>	1.757303 <sup>-4</sup>	4.3324 <sup>-6</sup>
$ e_n^d $	8.888889 <sup>-1</sup>	5.5555556 <sup>-3</sup>	2.23412699 <sup>-3</sup>	1.3038902 <sup>-4</sup>	1.19612 <sup>-6</sup>

Therefore, we obtain

$$\begin{aligned}
 u_0^c(t, r) &= 2 + 2rt + t^2 + \frac{2rt^3}{9}, & u_0^d(t, r) &= 2 - rt + t^2 - \frac{rt^3}{9}, \\
 u_1^c(t, r) &= -t^2 + \frac{t^3}{12} - \frac{rt^3}{5} - \frac{t^4}{12} - \frac{rt^5}{90}, \\
 u_1^d(t, r) &= -t^2 + \frac{t^3}{12} + \frac{rt^3}{10} - \frac{t^4}{12} + \frac{rt^5}{180}, \\
 u_2^c(t, r) &= -\frac{91t^3}{1080} - \frac{403rt^3}{18900} + \frac{t^4}{12} - \frac{t^5}{240} + \frac{rt^5}{100} + \frac{t^6}{360} + \frac{rt^7}{3780}, \\
 u_2^d(t, r) &= -\frac{91t^3}{1080} + \frac{403rt^3}{37800} + \frac{t^4}{12} - \frac{t^5}{240} - \frac{rt^5}{200} + \frac{t^6}{360} - \frac{rt^7}{7560}, \\
 &\vdots
 \end{aligned}
 \tag{5.31}$$

and so on. The 5-term approximate solution is

$$\begin{aligned}
 u^c(t, r) &= \sum_{n=0}^4 u_n^c = 2 + 2rt - \frac{0.337t^3}{27216} - \frac{5.0753rt^3}{1571724} + \frac{0.073t^5}{6048} + \frac{0.0041rt^5}{10206} \\
 &\quad - \frac{0.01t^7}{9072} + \frac{0.017rt^7}{15876} - \frac{0.1t^9}{72576} - \frac{0.01rt^9}{27216} + \frac{0.01t^{10}}{18144} + \frac{0.01t^{11}}{299376}, \\
 u^d(t, r) &= \sum_{n=0}^4 u_n^d = 2 - rt - \frac{0.337t^3}{27216} + \frac{5.0753rt^3}{3143448} + \frac{0.073t^5}{6048} - \frac{0.0041rt^5}{20412} \\
 &\quad - \frac{0.01t^7}{9072} - \frac{0.017rt^7}{31752} - \frac{0.1t^9}{72576} + \frac{0.01rt^9}{54432} + \frac{0.01t^{10}}{18144} - \frac{0.01t^{11}}{598752}.
 \end{aligned}
 \tag{5.32}$$

As the previous example, it is clear from the obtained results that the smallest value of the absolute  $n$ th-term error is  $|e_n(0, 0)| = (|e_n^c(0, 0)|, |e_n^d(0, 0)|) = (0, 0)$ ,  $n = 0, 1, \dots, 4$ , and the largest value is at  $t = r = 1$  as shown in Table 2. Therefore, the iterations converge to the exact solution of  $u^c, u^d$ , that is,  $\lim_{n \rightarrow \infty} u_n^c(t, r) = 2 + 2rt = u_{\text{Exact}}^c$ ,  $\lim_{n \rightarrow \infty} u_n^d(t, r) = 2 - rt = u_{\text{Exact}}^d$ . Therefore, the solution of (5.26) follows immediately.

## 6. Conclusion

In this work, the new iterative method used with a reliable algorithm to solve the  $n$ th-order derivative fuzzy integro-differential equations in crisp form and the cases of *first- and second-order* derivatives are taken into account. The obtained results concluded that the approximate solutions are in high agreement with corresponding exact solutions, which means that this method is suitable and effective to solve fuzzy integro-differential equations. Moreover, the solutions of the higher order fuzzy integro-differential equations can be calculated, as a future prospects, in a similar manner.

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