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Research Article

An Iterative Algorithm for a Hierarchical Problem

Yonghong Yao,[1] Yeol Je Cho,[2] and Pei-Xia Yang[1]

[1] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China*
[2] *Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Republic of Korea*

Correspondence should be addressed to Yeol Je Cho, yjcho@gsnu.ac.kr

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A general hierarchical problem has been considered, and an explicit algorithm has been presented for solving this hierarchical problem. Also, it is shown that the suggested algorithm converges strongly to a solution of the hierarchical problem.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. The hierarchical problem is of finding $\tilde{x} \in \mathrm{Fix}(T)$ such that

$$\langle S\tilde{x} - \tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \mathrm{Fix}(T), \tag{1.1}$$

where S, T are two nonexpansive mappings and $\mathrm{Fix}(T)$ is the set of fixed points of T. Recently, this problem has been studied by many authors (see, e.g., [1–15]). The main reason is that this problem is closely associated with some monotone variational inequalities and convex programming problems (see [16–19]).

Now, we briefly recall some historic results which relate to the problem (1.1).

For solving the problem (1.1), in 2006, Moudafi and Mainge [1] first introduced an implicit iterative algorithm:

$$x_{t,s} = sQ(x_{t,s}) + (1-s)[tS(x_{t,s}) + (1-t)T(x_{t,s})] \tag{1.2}$$

and proved that the net $\{x_{t,s}\}$ defined by (1.2) strongly converges to x_t as $s \to 0$, where x_t satisfies $x_t = \text{proj}_{\text{Fix}(P_t)} Q(x_t)$, where $P_t : C \to C$ is a mapping defined by

$$P_t(x) = tS(x) + (1-t)T(x), \quad \forall x \in C,\ t \in (0,1), \tag{1.3}$$

or, equivalently, x_t is the unique solution of the quasivariational inequality

$$0 \in (I-Q)x_t + N_{\text{Fix}(P_t)}(x_t), \tag{1.4}$$

where the normal cone to $\text{Fix}(P_t)$, $N_{\text{Fix}(P_t)}$, is defined as follows:

$$N_{\text{Fix}(P_t)} : x \longrightarrow \begin{cases} \{u \in H : \langle y - x, u \rangle \leq 0\}, & \text{if } x \in \text{Fix}(P_t), \\ \emptyset, & \text{otherwise.} \end{cases} \tag{1.5}$$

Moreover, as $t \to 0$, the net $\{x_t\}$ in turn weakly converges to the unique solution x_∞ of the fixed point equation $x_\infty = \text{proj}_\Omega Q(x_\infty)$ or, equivalently, x_∞ is the unique solution of the variational inequality

$$0 \in (I-Q)x_\infty + N_\Omega(x_\infty). \tag{1.6}$$

Recently, Moudafi [2] constructed an explicit iterative algorithm:

$$x_{n+1} = (1-\delta_n)x_n + \delta_n(\sigma_n S x_n + (1-\sigma_n)T x_n), \quad \forall n \geq 0, \tag{1.7}$$

where $\{\delta_n\}$ and $\{\sigma_n\}$ are two real numbers in $(0,1)$. By using this iterative algorithm, Moudafi [2] only proved a weak convergence theorem for solving the problem (1.1).

In order to obtain a strong convergence result, Mainge and Moudafi [3] further introduced the following iterative algorithm:

$$x_{n+1} = (1-\delta_n)Qx_n + \delta_n[\sigma_n S x_n + (1-\sigma_n)T x_n], \quad \forall n \geq 0, \tag{1.8}$$

where $\{\delta_n\}$ and $\{\sigma_n\}$ are two real numbers in $(0,1)$, and proved that, under appropriate conditions, the iterative sequence $\{x_n\}$ generated by (1.8) has strong convergence.

Subsequently, some authors have studied some algorithms on hierarchical fixed problems (see, e.g., [4–15]).

Motivated and inspired by the results in the literature, in this paper, we consider a general hierarchical problem of finding $\tilde{x} \in \text{Fix}(T)$ such that, for any $n \geq 1$,

$$\langle W_n \tilde{x} - \tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \text{Fix}(T), \tag{1.9}$$

where W_n is the W-mapping defined by (2.3) below and T is a nonexpansive mapping, and introduce an explicit iterative algorithm which converges strongly to a solution $\tilde{x}$ of the hierarchical problem (1.9).

2. Preliminaries

Let C a nonempty closed convex subset of a real Hilbert space H. Recall that a mapping $Q : C \to C$ is said to be contractive if there exists a constant $\gamma \in (0,1)$ such that

$$\|Qx - Qy\| \le \gamma \|x - y\|, \quad \forall x, y \in C. \tag{2.1}$$

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{2.2}$$

Forward, we use $\mathrm{Fix}(T)$ to denote the fixed points set of T.

Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings and $\{\xi_i\}_{i=1}^{\infty}$ a real number sequence such that $0 \le \xi_i \le 1$ for each $i \ge 1$.

For each $n \ge 1$, define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \xi_n T_n U_{n,n+1} + (1 - \xi_n) I, \\
U_{n,n-1} &= \xi_{n-1} T_{n-1} U_{n,n} + (1 - \xi_{n-1}) I, \\
&\cdots \\
U_{n,k} &= \xi_k T_k U_{n,k+1} + (1 - \xi_k) I, \\
U_{n,k-1} &= \xi_{k-1} T_{k-1} U_{n,k} + (1 - \xi_{k-1}) I, \\
&\cdots \\
U_{n,2} &= \xi_2 T_2 U_{n,3} + (1 - \xi_2) I, \\
W_n &= U_{n,1} = \xi_1 T_1 U_{n,2} + (1 - \xi_1) I.
\end{aligned} \tag{2.3}$$

Such W_n is called the W-mapping generated by $\{T_i\}_{i=1}^{\infty}$ and $\{\xi_i\}_{i=1}^{\infty}$.

Lemma 2.1 (see [20]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \ne \emptyset$. Let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \le b < 1$ for each $i \ge 1$. Then one has the following results:*

(1) *for any $x \in C$ and $k \ge 1$, the limit $\lim_{n \to \infty} U_{n,k} x$ exists;*

(2) $\mathrm{Fix}(W) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

Using Lemma 3.1 in [21], we can define a mapping W of C into itself by $Wx = \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n,1} x$ for all $x \in C$. Thus we have the following.

Lemma 2.2 (see [21]). *If $\{x_n\}$ is a bounded sequence in C, then one has*

$$\lim_{n \to \infty} \|W x_n - W_n x_n\| = 0. \tag{2.4}$$

Lemma 2.3 (see [22]). *Let C be a nonempty closed convex of a real Hilbert space H and $T : C \to C$ be nonexpansive mapping. Then T is demiclosed on C, that is, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.4 (see [23]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n\gamma_n + \eta_n, \quad \forall n \geq 1, \tag{2.5}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}, \{\eta_n\}$ are two sequences such that

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n\gamma_n| < \infty$;

(iii) $\sum_{n=1}^{\infty}|\eta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we introduce our algorithm and give its convergence analysis.

Algorithm 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H and $\{T_n\}_{n=1}^{\infty}$ be infinite family of nonexpansive mappings of C into itself. Let $Q : C \to C$ be a contraction with coefficient $\gamma \in [0,1)$. For any $x_0 \in C$, let $\{x_n\}$ the sequence generated iteratively by

$$x_{n+1} = \alpha_n W_n x_n + (1-\alpha_n)T\left(\beta_n Q x_n + (1-\beta_n)x_n\right), \quad \forall n \geq 0, \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real numbers in $(0,1)$ and W_n is the W-mapping defined by (2.3).

Now, we give the convergence analysis of the algorithm.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself. Let $Q : C \to C$ be a contraction with coefficient $\gamma \in [0,1)$. Assume that the set Ω of solutions of the hierarchical problem* (1.9) *is nonempty. Let $\{\alpha_n\}, \{\beta_n\}$ be two real numbers in $(0,1)$ and $\{x_n\}$ the sequence generated by* (3.1)*. Assume that the sequence $\{x_n\}$ is bounded and*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty}(\beta_n/\alpha_n) = 0$;

(ii) $\sum_{n=0}^{\infty}\beta_n = \infty$;

(iii) $\lim_{n\to\infty}(1/\beta_n)|(1/\alpha_n) - (1/\alpha_{n-1})| = 0$ *and* $\lim_{n\to\infty}(\prod_{i=1}^{n-1}\xi_i/\alpha_n\beta_n) = \lim_{n\to\infty}(1/\alpha_n)|1-(\beta_{n-1}/\beta_n)| = 0$.

Then $\lim_{n\to\infty}(\|x_{n+1} - x_n\|/\alpha_n) = 0$ *and every weak cluster point of the sequence $\{x_n\}$ solves the following variational inequality*

$$\begin{gathered} \tilde{x} \in \Omega, \\ \langle (I-Q)\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in \Omega. \end{gathered} \tag{3.2}$$

Proof. Set $y_n = \beta_n Q x_n + (1-\beta_n)x_n$ for each $n \ge 0$. Then we have

$$\begin{aligned} y_n - y_{n-1} &= \beta_n Q x_n + (1-\beta_n)x_n - \beta_{n-1}Qx_{n-1} - (1-\beta_{n-1})x_{n-1} \\ &= \beta_n(Qx_n - Qx_{n-1}) + (\beta_n - \beta_{n-1})Qx_{n-1} + (1-\beta_n)(x_n - x_{n-1}) \\ &\quad + (\beta_{n-1} - \beta_n)x_{n-1}. \end{aligned} \tag{3.3}$$

It follows that

$$\begin{aligned} \|y_n - y_{n-1}\| &\le \gamma\beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Qx_{n-1}\| + \|x_{n-1}\|) \\ &= [1-(1-\gamma)\beta_n]\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Qx_{n-1}\| + \|x_{n-1}\|). \end{aligned} \tag{3.4}$$

From (3.1), we have

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n W_n x_n + (1-\alpha_n)Ty_n - \alpha_{n-1}W_{n-1}x_{n-1} - (1-\alpha_{n-1})Ty_{n-1} \\ &= \alpha_n(W_n x_n - W_n x_{n-1}) + (\alpha_n - \alpha_{n-1})W_n x_{n-1} + \alpha_{n-1}(W_n x_{n-1} - W_{n-1}x_{n-1}) \\ &\quad + (1-\alpha_n)(Ty_n - Ty_{n-1}) + (\alpha_{n-1} - \alpha_n)Ty_{n-1}. \end{aligned} \tag{3.5}$$

Then we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &\le \alpha_n\|W_n x_n - W_n x_{n-1}\| + (1-\alpha_n)\|Ty_n - Ty_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}|(\|W_n x_{n-1}\| + \|Ty_{n-1}\|) + \alpha_{n-1}\|W_n x_{n-1} - W_{n-1}x_{n-1}\| \\ &\le \alpha_n\|x_n - x_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|W_n x_{n-1}\| + \|Ty_{n-1}\|) \\ &\quad + \alpha_{n-1}\|W_n x_{n-1} - W_{n-1}x_{n-1}\|. \end{aligned} \tag{3.6}$$

From (2.3), since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned} \|W_n x_{n-1} - W_{n-1}x_{n-1}\| &= \|\xi_1 T_1 U_{n,2}x_{n-1} - \xi_1 T_1 U_{n-1,2}x_{n-1}\| \\ &\le \xi_1\|U_{n,2}x_{n-1} - U_{n-1,2}x_{n-1}\| \\ &= \xi_1\|\xi_2 T_2 U_{n,3}x_{n-1} - \xi_2 T_2 U_{n-1,3}x_{n-1}\| \\ &\le \xi_1\xi_2\|U_{n,3}x_{n-1} - U_{n-1,3}x_{n-1}\| \\ &\le \cdots \\ &\le \xi_1\xi_2\cdots\xi_{n-1}\|U_{n,n}x_{n-1} - U_{n-1,n}x_{n-1}\| \\ &\le M_1\prod_{i=1}^{n-1}\xi_i, \end{aligned} \tag{3.7}$$

where M_1 is a constant such that $\sup_{n\geq 1}\{\|U_{n,n}x_{n-1} - U_{n-1,n}x_{n-1}\|\} \leq M_1$. Substituting (3.4) and (3.7) into (3.6), we get

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \alpha_n\|x_n - x_{n-1}\| + (1-\alpha_n)\left[1-(1-\gamma)\beta_n\right]\|x_n - x_{n-1}\| \\
&\quad + \left|\beta_n - \beta_{n-1}\right|(\|Qx_{n-1}\| + \|x_{n-1}\|) \\
&\quad + |\alpha_n - \alpha_{n-1}|\left(\|W_n x_{n-1}\| + \left\|Ty_{n-1}\right\|\right) + \alpha_{n-1}M_1\prod_{i=1}^{n-1}\xi_i \\
&= \left[1-(1-\gamma)\beta_n(1-\alpha_n)\right]\|x_n - x_{n-1}\| \\
&\quad + \left|\beta_n - \beta_{n-1}\right|(\|Qx_{n-1}\| + \|x_{n-1}\|) \\
&\quad + |\alpha_n - \alpha_{n-1}|\left(\|W_n x_{n-1}\| + \left\|Ty_{n-1}\right\|\right) + \alpha_{n-1}M_1\prod_{i=1}^{n-1}\xi_i.
\end{aligned} \tag{3.8}$$

Therefore, it follows that

$$\begin{aligned}
\frac{\|x_{n+1} - x_n\|}{\alpha_n} &\leq \left[1-(1-\gamma)\beta_n(1-\alpha_n)\right]\frac{\|x_n - x_{n-1}\|}{\alpha_n} \\
&\quad + \frac{\left|\beta_n - \beta_{n-1}\right|}{\alpha_n}(\|Qx_{n-1}\| + \|x_{n-1}\|) \\
&\quad + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n}\left(\|W_n x_{n-1}\| + \left\|Ty_{n-1}\right\|\right) + \alpha_{n-1}M_1\frac{\prod_{i=1}^{n-1}\xi_i}{\alpha_n} \\
&= \left[1-(1-\gamma)\beta_n(1-\alpha_n)\right]\frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \\
&\quad + \left[1-(1-\gamma)\beta_n(1-\alpha_n)\right]\left(\frac{\|x_n - x_{n-1}\|}{\alpha_n} - \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}}\right) \\
&\quad + \frac{\left|\beta_n - \beta_{n-1}\right|}{\alpha_n}(\|Qx_{n-1}\| + \|x_{n-1}\|) \\
&\quad + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n}\left(\|W_n x_{n-1}\| + \left\|Ty_{n-1}\right\|\right) + \alpha_{n-1}M_1\frac{\prod_{i=1}^{n-1}\xi_i}{\alpha_n} \\
&\leq \left[1-(1-\gamma)\beta_n(1-\alpha_n)\right]\frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \\
&\quad + \left(\left|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}\right| + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{\left|\beta_n - \beta_{n-1}\right|}{\alpha_n} + \frac{\prod_{i=1}^{n-1}\xi_i}{\alpha_n}\right)M
\end{aligned}$$

$$
\begin{aligned}
&= \left[1-(1-\gamma)\beta_n(1-\alpha_n)\right]\frac{\|x_n-x_{n-1}\|}{\alpha_{n-1}}+(1-\gamma)\beta_n(1-\alpha_n) \\
&\quad\times\left\{\frac{M}{(1-\gamma)(1-\alpha_n)}\left(\frac{1}{\beta_n}\left|\frac{1}{\alpha_n}-\frac{1}{\alpha_{n-1}}\right|+\frac{1}{\beta_n}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}\right.\right. \\
&\qquad\left.\left.+\frac{1}{\beta_n}\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}+\frac{\prod_{i=1}^{n-1}\xi_i}{\alpha_n\beta_n}\right)\right\},
\end{aligned}
\tag{3.9}
$$

where M is a constant such that

$$
\sup_{n\geq 1}\left\{M_1,\|x_n-x_{n-1}\|,\left(\|W_nx_{n-1}\|+\|Ty_{n-1}\|\right),\left(\|Qx_{n-1}\|+\|x_{n-1}\|\right)\right\}\leq M. \tag{3.10}
$$

From (iii), we note that $\lim_{n\to\infty}(1/\alpha_{n-1})|\alpha_n-\alpha_{n-1}/\beta_n\alpha_n|=0$, which implies that

$$
\lim_{n\to\infty}\frac{1}{\beta_n}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}=0. \tag{3.11}
$$

Thus it follows from (iii) and (3.11) that

$$
\lim_{n\to\infty}\left(\frac{1}{\beta_n}\left|\frac{1}{\alpha_n}-\frac{1}{\alpha_{n-1}}\right|+\frac{1}{\beta_n}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}+\frac{1}{\beta_n}\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}+\frac{\prod_{i=1}^{n-1}\xi_i}{\alpha_n\beta_n}\right)=0. \tag{3.12}
$$

Hence, applying Lemma 2.4 to (3.9), we immediately conclude that

$$
\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\alpha_n}=0. \tag{3.13}
$$

This implies that

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.14}
$$

Thus, from (3.1) and (3.14), we have

$$
\lim_{n\to\infty}\|x_n-Ty_n\|=0. \tag{3.15}
$$

At the same time, we note that

$$
y_n-x_n=\beta_n(Qx_n-x_n)\longrightarrow 0. \tag{3.16}
$$

Hence we get

$$
\lim_{n\to\infty}\|y_n-Ty_n\|=0. \tag{3.17}
$$

Since the sequence $\{x_n\}$ is bounded, $\{y_n\}$ is also bounded. Thus there exists a subsequence of $\{y_n\}$, which is still denoted by $\{y_n\}$ which converges weakly to a point $\tilde{x} \in H$. Therefore, $\tilde{x} \in \mathrm{Fix}(T)$ by (3.17) and Lemma 2.3. By (3.1), we observe that

$$x_{n+1} - x_n = \alpha_n (W_n x_n - x_n) + (1-\alpha_n)\big(T y_n - y_n\big) + (1-\alpha_n)\beta_n (Q x_n - x_n), \tag{3.18}$$

that is,

$$\frac{x_n - x_{n+1}}{\alpha_n} = (I - W_n) x_n + \frac{1-\alpha_n}{\alpha_n}(I-T) y_n + \frac{\beta_n(1-\alpha_n)}{\alpha_n}(I-Q) x_n. \tag{3.19}$$

Set $z_n = (x_n - x_{n+1})/\alpha_n$ for each $n \geq 1$, that is,

$$z_n = (I - W_n) x_n + \frac{1-\alpha_n}{\alpha_n}(I-T) y_n + \frac{\beta_n(1-\alpha_n)}{\alpha_n}(I-Q) x_n. \tag{3.20}$$

Using monotonicity of $I - T$ and $I - W_n$, we derive that, for all $u \in \mathrm{Fix}(T)$,

$$\begin{aligned}
&\langle z_n, x_n - u\rangle \\
&\quad = \langle (I - W_n) x_n, x_n - u\rangle + \frac{1-\alpha_n}{\alpha_n}\big\langle (I-T) y_n - (I-T) u, y_n - u\big\rangle \\
&\qquad + \frac{1-\alpha_n}{\alpha_n}\big\langle (I-T) y_n, x_n - y_n\big\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-Q) x_n, x_n - u\rangle \\
&\quad \geq \langle (I - W_n) u, x_n - u\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-Q) x_n, x_n - u\rangle + \frac{(1-\alpha_n)\beta_n}{\alpha_n}\big\langle (I-T) y_n, x_n - Q x_n\big\rangle \\
&\quad = \langle (I - W) u, x_n - u\rangle + \langle (W - W_n) u, x_n - u\rangle + \frac{\beta_n(1-\alpha_n)}{\alpha_n}\langle (I-Q) x_n, x_n - u\rangle \\
&\qquad + \frac{(1-\alpha_n)\beta_n}{\alpha_n}\big\langle (I-T) y_n, x_n - Q x_n\big\rangle.
\end{aligned} \tag{3.21}$$

But, since $z_n \to 0$, $\beta_n/\alpha_n \to 0$ and $\lim_{n\to\infty}\|W_n u - W u\| = 0$ (by Lemma 2.2), it follows from the above inequality that

$$\limsup_{n\to\infty}\langle (I-W) u, x_n - u\rangle \leq 0, \quad \forall u \in \mathrm{Fix}(T). \tag{3.22}$$

This suffices to guarantee that $\omega_w(x_n) \subset \Omega$. As a matter of fact, if we take any $x^* \in \omega_w(x_n)$, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$. Therefore, we have

$$\langle (I-W) u, x^* - u\rangle = \lim_{j\to\infty}\Big\langle (I-W) u, x_{n_j} - u\Big\rangle \leq 0, \quad \forall u \in \mathrm{Fix}(T). \tag{3.23}$$

Note that $x^* \in \text{Fix}(T)$. Hence x^* solves the following problem:

$$x^* \in \text{Fix}(T),$$
$$\langle (I-W)u, x^* - u\rangle \le 0, \quad \forall u \in \text{Fix}(T). \tag{3.24}$$

It is obvious that this is equivalent to the problem (1.9) since $W_n \to W$ uniformly in any bounded set (by Lemma 2.2). Thus $x^* \in \Omega$.

Let $\tilde{x}$ be the unique solution of the variational inequality (3.2). Now, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (I-Q)\tilde{x}, x_n - \tilde{x}\rangle = \lim_{i\to\infty}\langle (I-Q)\tilde{x}, x_{n_i} - \tilde{x}\rangle. \tag{3.25}$$

Without loss of generality, we may further assume that $x_{n_i} \rightharpoonup \bar{x}$. Then $\bar{x} \in \Omega$. Therefore, we have

$$\limsup_{n\to\infty}\langle (I-Q)\tilde{x}, x_n - \tilde{x}\rangle = \langle (I-Q)\tilde{x}, \bar{x} - \tilde{x}\rangle \ge 0. \tag{3.26}$$

This completes the proof. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be infinite family of nonexpansive mappings of C into itself. Let $Q : C \to C$ be a contraction with coefficient $\gamma \in [0,1)$. Assume that the set Ω of solutions of the hierarchical problem (1.9) is nonempty. Let $\{\alpha_n\}, \{\beta_n\}$ be two real numbers in $(0,1)$ and $\{x_n\}$ the sequence generated by (3.1). Assume that the sequence $\{x_n\}$ is bounded and*

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n/\alpha_n = 0$ *and* $\lim_{n\to\infty}\alpha_n^2/\beta_n = 0$;

(ii) $\sum_{n=0}^{\infty}\beta_n = \infty$;

(iii) $\lim_{n\to\infty}(1/\beta_n)|(1/\alpha_n)-(1/\alpha_{n-1})| = 0$ *and* $\lim_{n\to\infty}\prod_{i=1}^{n-1}\xi_i/\alpha_n\beta_n = \lim_{n\to\infty}(1/\alpha_n)|1-(\beta_{n-1}/\beta_n)| = 0$;

(iv) *there exists a constant $k > 0$ such that $\|x - Tx\| \ge k\,\text{Dist}(x, \text{Fix}(T))$, where*

$$\text{Dist}(x, \text{Fix}(T)) = \inf_{y\in \text{Fix}(T)}\|x - y\|. \tag{3.27}$$

Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a point $\tilde{x} \in \text{Fix}(T)$, which solves the variational inequality problem (3.2).

Proof. From (3.1), we have

$$x_{n+1} - \tilde{x} = \alpha_n(W_n x_n - W_n\tilde{x}) + \alpha_n(W_n\tilde{x} - \tilde{x}) + (1-\alpha_n)(Ty_n - \tilde{x}). \tag{3.28}$$

Thus we have

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &\le \left\|\alpha_n(W_nx_n-W_n\tilde{x})+(1-\alpha_n)(Ty_n-\tilde{x})\right\|^2+2\alpha_n\langle W_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&\le (1-\alpha_n)\left\|Ty_n-\tilde{x}\right\|^2+\alpha_n\|W_nx_n-W_n\tilde{x}\|^2+2\alpha_n\langle W_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&\le (1-\alpha_n)\left\|y_n-\tilde{x}\right\|^2+\alpha_n\|x_n-\tilde{x}\|^2+2\alpha_n\langle W_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle.
\end{aligned} \tag{3.29}$$

At the same time, we observe that

$$\begin{aligned}
\left\|y_n-\tilde{x}\right\|^2 &= \left\|(1-\beta_n)(x_n-\tilde{x})+\beta_n(Qx_n-Q\tilde{x})+\beta_n(Q\tilde{x}-\tilde{x})\right\|^2 \\
&\le \left\|(1-\beta_n)(x_n-\tilde{x})+\beta_n(Qx_n-Q\tilde{x})\right\|^2+2\beta_n\langle Q\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle \\
&\le (1-\beta_n)\|x_n-\tilde{x}\|^2+\beta_n\|Qx_n-Q\tilde{x}\|^2+2\beta_n\langle Q\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle \\
&\le (1-\beta_n)\|x_n-\tilde{x}\|^2+\beta_n\gamma^2\|x_n-\tilde{x}\|^2+2\beta_n\langle Q\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle \\
&= \left[1-\left(1-\gamma^2\right)\beta_n\right]\|x_n-\tilde{x}\|^2+2\beta_n\langle Q\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle.
\end{aligned} \tag{3.30}$$

Substituting (3.30) into (3.29), we get

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &\le \alpha_n\|x_n-\tilde{x}\|^2+(1-\alpha_n)\left[1-\left(1-\gamma^2\right)\beta_n\right]\|x_n-\tilde{x}\|^2 \\
&\quad +2\beta_n(1-\alpha_n)\langle Q\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle+2\alpha_n\langle W_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&= \left[1-\left(1-\gamma^2\right)\beta_n(1-\alpha_n)\right]\|x_n-\tilde{x}\|^2+2\beta_n(1-\alpha_n)\langle Q\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle \\
&\quad +2\alpha_n\langle W_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&= \left[1-\left(1-\gamma^2\right)\beta_n(1-\alpha_n)\right]\|x_n-\tilde{x}\|^2+\left(1-\gamma^2\right)\beta_n(1-\alpha_n) \\
&\quad \times\left\{\frac{2}{1-\gamma^2}\langle Q\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle+\frac{2}{(1-\gamma^2)(1-\alpha_n)}\times\frac{\alpha_n}{\beta_n}\langle W_n\tilde{x}-\tilde{x},x_{n+1}-\tilde{x}\rangle\right\}.
\end{aligned} \tag{3.31}$$

By Theorem 3.2, we note that every weak cluster point of the sequence $\{x_n\}$ is in Ω. Since $y_n-x_n\to 0$, then every weak cluster point of $\{y_n\}$ is also in Ω. Consequently, since $\tilde{x}=\operatorname{proj}_\Omega(Q\tilde{x})$, we easily have

$$\limsup_{n\to\infty}\langle Q\tilde{x}-\tilde{x},y_n-\tilde{x}\rangle\le 0. \tag{3.32}$$

On the other hand, we observe that

$$\langle W_n\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle = \left\langle W_n\tilde{x}-\tilde{x}, \mathrm{proj}_{\mathrm{Fix}(T)}x_{n+1}-\tilde{x}\right\rangle + \left\langle W_n\tilde{x}-\tilde{x}, x_{n+1}-\mathrm{proj}_{\mathrm{Fix}(T)}x_{n+1}\right\rangle. \tag{3.33}$$

Since $\tilde{x}$ is a solution of the problem (1.9) and $\mathrm{proj}_{\mathrm{Fix}(T)}x_{n+1} \in \mathrm{Fix}(T)$, we have

$$\left\langle W_n\tilde{x}-\tilde{x}, \mathrm{proj}_{\mathrm{Fix}(T)}x_{n+1}-\tilde{x}\right\rangle \le 0. \tag{3.34}$$

Thus it follows that

$$\begin{aligned}\langle W_n\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle &\le \left\langle W_n\tilde{x}-\tilde{x}, x_{n+1}-\mathrm{proj}_{\mathrm{Fix}(T)}x_{n+1}\right\rangle \\ &\le \|W_n\tilde{x}-\tilde{x}\|\left\|x_{n+1}-\mathrm{proj}_{\mathrm{Fix}(T)}x_{n+1}\right\| \\ &= \|W_n\tilde{x}-\tilde{x}\| \times \mathrm{Dist}(x_{n+1}, \mathrm{Fix}(T)) \\ &\le \frac{1}{k}\|W_n\tilde{x}-\tilde{x}\|\|x_{n+1}-Tx_{n+1}\|.\end{aligned} \tag{3.35}$$

We note that

$$\begin{aligned}\|x_{n+1}-Tx_{n+1}\| &\le \|x_{n+1}-Tx_n\| + \|Tx_n-Tx_{n+1}\| \\ &\le \alpha_n\|W_nx_n-Tx_n\| + (1-\alpha_n)\left\|Ty_n-Tx_n\right\| + \|x_{n+1}-x_n\| \\ &\le \alpha_n\|W_nx_n-Tx_n\| + \left\|y_n-x_n\right\| + \|x_{n+1}-x_n\| \\ &\le \alpha_n\|W_nx_n-Tx_n\| + \beta_n\|Qx_n-x_n\| + \|x_{n+1}-x_n\|.\end{aligned} \tag{3.36}$$

Hence we have

$$\begin{aligned}&\frac{\alpha_n}{\beta_n}\langle W_n\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle \\ &\quad\le \frac{\alpha_n^2}{\beta_n}\left(\frac{1}{k}\|W_n\tilde{x}-\tilde{x}\|\|W_nx_n-Tx_n\|\right) + \alpha_n\left(\frac{1}{k}\|W_n\tilde{x}-\tilde{x}\|\|Qx_n-x_n\|\right) \\ &\qquad + \frac{\alpha_n^2}{\beta_n}\frac{\|x_{n+1}-x_n\|}{\alpha_n}\left(\frac{1}{k}\|W_n\tilde{x}-\tilde{x}\|\right).\end{aligned} \tag{3.37}$$

From Theorem 3.2, we have $\lim_{n\to\infty}\|x_{n+1}-x_n\|/\alpha_n = 0$. At the same time, we note that $\{(1/k)\|W_n\tilde{x}-\tilde{x}\|\|W_nx_n-Tx_n\|\}$, $\{(1/k)\|W_n\tilde{x}-\tilde{x}\|\|Qx_n-x_n\|\}$, and $\{(1/k)\|W_n\tilde{x}-\tilde{x}\|\}$ are all bounded. Hence it follows from (i) and the above inequality that

$$\limsup_{n\to\infty}\frac{\alpha_n}{\beta_n}\langle W_n\tilde{x}-\tilde{x}, x_{n+1}-\tilde{x}\rangle \le 0. \tag{3.38}$$

Finally, by (3.31)–(3.38) and Lemma 2.4, we conclude that the sequence $\{x_n\}$ converges strongly to a point $\tilde{x} \in \mathrm{Fix}(T)$. This completes the proof. □

Remark 3.4. In the present paper, we consider the hierarchical problem (1.9) which includes the hierarchical problem (1.1) as a special case.

From the above discussion, we can easily deduce the following result.

Algorithm 3.5. Let C be a nonempty closed convex subset of a real Hilbert space H and S a nonexpansive mapping of C into itself. Let $Q : C \to C$ be a contraction with coefficient $\gamma \in [0,1)$. For any $x_0 \in C$, let$\{x_n\}$ the sequence generated iteratively by

$$x_{n+1} = \alpha_n S x_n + (1 - \alpha_n) T\left(\beta_n Q x_n + (1 - \beta_n) x_n\right), \quad \forall n \geq 0, \tag{3.39}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real numbers in $(0,1)$.

Corollary 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping. Let $Q : C \to C$ be a contraction with coefficient $\gamma \in [0,1)$. Assume that the set Ω' of solutions of the hierarchical problem* (1.1) *is nonempty. Let $\{\alpha_n\}, \{\beta_n\}$ be two real numbers in $(0,1)$ and $\{x_n\}$ the sequence generated by* (3.1)*. Assume that the sequence $\{x_n\}$ is bounded and*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n/\alpha_n = 0$ *and* $\lim_{n\to\infty} \alpha_n^2/\beta_n = 0$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\lim_{n\to\infty} (1/\beta_n)|(1/\alpha_n) - (1/\alpha_{n-1})| = 0$ *and* $\lim_{n\to\infty} (1/\alpha_n)|1 - (\beta_{n-1}/\beta_n)| = 0$;

(iv) *there exists a constant $k > 0$ such that $\|x - Tx\| \geq k\mathrm{Dist}(x, \mathrm{Fix}(T))$, where*

$$\mathrm{Dist}(x, \mathrm{Fix}(T)) = \inf_{y \in \mathrm{Fix}(T)} \|x - y\|. \tag{3.40}$$

Then the sequence $\{x_n\}$ defined by (3.39) *converges strongly to a point $\tilde{x} \in \mathrm{Fix}(T)$, which solves the hierarchical problem* (1.1)*.*

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