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Research Article

Fixed Points of Asymptotic Pointwise Nonexpansive Mappings in Modular Spaces

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Kirk and Xu studied the existence of fixed points of asymptotic pointwise nonexpansive mappings in the Banach space. In this paper, we investigate these kinds of mappings in modular spaces. Moreover, we prove the existence of fixed points of asymptotic pointwise nonexpansive mappings in modular spaces. The results improve and extend the corresponding results of Kirk and Xu (2008) to modular spaces.

1. Introduction

The theory of modular spaces was initiated by Nakano [1] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [2] in 1959. These spaces were developed following the successful theory of Orlicz spaces, which replaces the particular, integral form of the nonlinear functional, which controls the growth of members of the space, by an abstractly given functional with some good properties. In 2007, Razani et al. [3] studied some fixed points of nonlinear and asymptotic contractions in the modular spaces. In addition, quasi-contraction mappings in modular spaces without Δ_2-condition were considered by Khamsi [4] in 2008. Recently, Kuaket and Kumam [5] proved the existence of fixed points of asymptotic pointwise contractions in modular spaces. Even though a metric is not defined, many problems in fixed point theory for nonexpansive mappings can be reformulated in modular spaces.

The existence of fixed points of asymptotic pointwise nonexpansive mappings was studied by Kirk and Xu [6] in 2008, that is, mappings $T : C \to C$, such that

$$\left\|T^n(x) - T^n(y)\right\| \le \alpha_n(x)\left\|x - y\right\|, \quad \forall x, y \in C, \tag{1.1}$$

where $\limsup_{n\to\infty}\alpha_n(x) \le 1$. Their main result states that every asymptotic pointwise nonexpansive self-mapping of a nonempty, closed, bounded, and convex subset C of a uniformly convex Banach space X has a fixed point.

The above-mentioned result of Kirk and Xu is a generalization of the 1972 Geobel-Kirk fixed point theorem [7] for a narrower class of mappings—the class of asymptotic nonexpansive mappings, where-using our notation-every function α_n is a constant function. The latter result was in its own a generalization of the classical Browder-Gohde-Kirk fixed point theorem for nonexpansive mappings [8]. In 2009, the results of [6] were extended to the case of metric spaces by Hussain and Khamsi [9].

In this paper, we investigate asymptotic pointwise nonexpansive mappings in modular spaces. Moreover, we obtain similar results in the sense of modular spaces. The results presented in this paper extend and improve the corresponding results of Kirk and Xu [6].

2. Preliminaries

Definition 2.1. Let X be an arbitrary vector space over $K = (R$ or $C)$.

(a) A functional $\rho : X \to [0,\infty]$ is called modular if

(i) $\rho(x) = 0$ if and only if $x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, for all $x \in X$,

(iii) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ if $\alpha, \beta \ge 0$, for all $x, y \in X$. If (iii) is replaced by

(iii′) $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$, for $\alpha, \beta \ge 0$, $\alpha + \beta = 1$, for all $x, y \in X$, then the modular ρ is called convex modular.

(b) A modular ρ defines a corresponding modular space, that is, the space X_ρ given by

$$X_\rho = \{x \in X \mid \rho(\alpha x) \longrightarrow 0 \text{ as } \alpha \longrightarrow 0\}. \tag{2.1}$$

Remark 2.2. Note that ρ is an increasing function. Suppose $0 < a < b$. Then, property (iii′) with $y = 0$ shows that $\rho(ax) = \rho((a/b)(bx)) \le \rho(bx)$.

Remark 2.3. In general, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. But one can associate to a modular with a F-norm (see [10]).

The modular space X_ρ can be equipped with a F-norm defined by

$$\|x\|_\rho = \inf\left\{\alpha > 0; \rho\left(\frac{x}{\alpha}\right) \le \alpha\right\}. \tag{2.2}$$

Namely, if ρ is convex, then the functional $\|x\|_\rho = \inf\{\alpha > 0; \rho(x/\alpha) \le 1\}$ is a norm in X_ρ which is equivalent to the F-norm $\|\cdot\|_\rho$.

Definition 2.4. Let X_ρ be a modular space.

(a) A sequence $\{x_n\} \subset X_\rho$ is said to be ρ-convergent to $x \in X_\rho$ and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(b) A sequence $\{x_n\}$ is called ρ-Cauchy whenever $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(c) The modular ρ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

(d) A subset $B \subset X_\rho$ is called ρ-closed if for any sequence $\{x_n\} \subset B\rho$-convergent to $x \in X_\rho$, we have $x \in B$.

(e) A ρ-closed subset $B \subset X_\rho$ is called ρ-compact if any sequence $\{x_n\} \subset B$ has a ρ-convergent subsequence.

(f) ρ is said to satisfy the Δ_2-condition if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.

(g) We say that ρ has the Fâtou property if

$$(x-y) \leq \liminf_{n\to\infty}\rho(x_n - y_n) \tag{2.3}$$

whenever $x_n \xrightarrow{\rho} x$ and $y_n \xrightarrow{\rho} y$ as $n \to \infty$.

(h) A subset $B \subset X_\rho$ is said to be ρ-bounded if

$$\operatorname{diam}_\rho(B) < \infty, \tag{2.4}$$

where $\operatorname{diam}_\rho(B) = \sup\{\rho(x-y); x, y \in B\}$ is called the ρ-diameter of B.

(i) Define the ρ-distance between $x \in X_\rho$ and $B \subset X_\rho$ as

$$\operatorname{dis}_\rho(x, B) = \inf\{\rho(x-y); y \in B\}. \tag{2.5}$$

(j) Define the ρ-Ball, $B_\rho(x, r)$, centered at $x \in X_\rho$ with radius r as

$$B_\rho(x, r) = \{y \in X_\rho; \rho(x-y) \leq r\}. \tag{2.6}$$

Definition 2.5. Let X_ρ be a modular space. For any $\varepsilon > 0$ and $r > 0$, the modular of ρ_r-uniform convexity of X_ρ is defined by

$$\delta_\rho(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{x+y}{2}\right) : \rho(x) \leq r,\ \rho(y) \leq r, \rho\ (x-y) \geq r\varepsilon\right\}. \tag{2.7}$$

(i) We say ρ satisfies uniform convexity (UC) if for every $\varepsilon > 0$, $r > 0$, $\delta_\rho(r, \varepsilon) > 0$.

(ii) We say that ρ satisfies unique uniform convexity (UUC) if for every $s \geq 0$, $\varepsilon > 0$, there exists

$$\eta(s, \varepsilon) > 0, \tag{2.8}$$

depending on s and ε such that

$$\delta_\rho(r, \varepsilon) > \eta(s, \varepsilon) > 0, \quad \text{for } r > s. \tag{2.9}$$

Definition 2.6. Let $C \subset X_\rho$ be convex and ρ-bounded.

(a) A function $\tau : C \to [0, \infty]$ is called a (ρ)-type (or shortly a type) if there exists a sequence $\{y_m\}$ of elements of C such that for any $z \in C$ there holds

$$\tau(z) = \limsup_{m \to \infty} \rho(y_m - z). \tag{2.10}$$

(b) Let τ be a type. A sequence $\{g_n\}$ is called a minimizing sequence of τ if

$$\lim_{n \to \infty} \tau(g_n) = \inf\{\tau(h); h \in C\}. \tag{2.11}$$

The following definitions are straightforward generalizations of their norm and metric equivalents.

Definition 2.7. Let $C \subset X_\rho$ be nonempty and ρ-closed. A mapping $T : C \to C$ is called an asymptotic pointwise mapping if there exists a sequence of mapping $\alpha_n : C \to [0, \infty)$ such that

$$\rho(T^n(x) - T^n(y)) \leq \alpha_n(x)\rho(x - y), \quad \forall x, y \in X_\rho. \tag{2.12}$$

(i) If $\{\alpha_n\}$ converges pointwise to $\alpha : C \to [0, 1)$, then T is called an asymptotic pointwise contraction.

(ii) If $\limsup_{n \to \infty} \alpha_n(x) \leq 1$ for any $x \in X_\rho$, then T is called an asymptotic pointwise nonexpansive mapping.

(iii) If $\limsup_{n \to \infty} \alpha_n(x) \leq k$ for any $x \in X_\rho$ with $0 < k < 1$, then T is called a strongly asymptotic pointwise contraction.

3. Main Results

Lemma 3.1 (see [11]). *Let C be a ρ-closed ρ-bounded convex nonempty subset of X_ρ, let ρ satisfy (UUC), and let τ be a type defined on C. Then any minimizing sequence of τ is ρ-convergent and its limit is independent of the minimizing sequence.*

Theorem 3.2. *Let X_ρ be a modular space, let C be a ρ-bounded ρ-closed convex nonempty subset of X_ρ, and let ρ satisfy (UUC), and $T : C \to C$ is asymptotic pointwise nonexpansive. Then T has a fixed point. Moreover, the set of all fixed points* $\mathrm{Fix}(T)$ *is ρ-closed.*

Proof. For a fixed $x \in C$, define the type

$$\tau(h) = \limsup_{n \to \infty} \rho(T^n(x) - h), \quad h \in C. \tag{3.1}$$

Let $\tau_0 = \inf\{\tau(h); h \in C\}$. Let $\{f_n\} \subset C$ be a minimizing sequence of τ and $f_n \to f \in C$, which exists in view of Lemma 3.1.

Let us prove that f is a fixed point of T.

First notice that $\tau(T^m(h)) \leq \alpha_m(h)\tau(h)$, for any $h \in C$ and $m \geq 1$.

Indeed, for fixed $x \in C$,

$$\begin{aligned}\tau(T^m(h)) &= \limsup_{n\to\infty} \rho(T^n(x) - T^m(h)) \\ &= \limsup_{n\to\infty} \rho(T^{n+m}(x) - T^m(h)) \\ &= \limsup_{n\to\infty} \rho(T^m T^n(x) - T^m(h)) \\ &\leq \limsup_{n\to\infty} \alpha_m(h)\rho(T^n(x) - h) \\ &= \alpha_m(h)\tau(h).\end{aligned} \tag{3.2}$$

In particular, we have

$$\tau(T^m(f_n)) \leq \alpha_m(f_n)\tau(f_n), \quad \forall n, m \geq 1. \tag{3.3}$$

By induction, we build an increasing sequence $\{m_k\}$ such that

$$\alpha_{m_k+m}(f_k) \leq 1 + \frac{1}{k}, \quad \forall k, m \geq 1. \tag{3.4}$$

Indeed, since T is asymptotic pointwise nonexpansive mapping, we have $\limsup_{m\to\infty}\alpha_m(f_1) \leq 1$, so there exists $m_1 \geq 1$, such that for any $m \geq m_1$, we have $\alpha_m(f_1) \leq 1+1/1$. Since $\limsup_{m\to\infty}\alpha_m(f_2) \leq 1$, there exists $m_2 > m_1$ such that for any $m \geq m_2$, we have $\alpha_m(f_2) \leq 1+1/2$. Assume m_k is built, then since $\limsup_{m\to\infty}\alpha_m(f_{k+1}) \leq 1$, there exists $m_{k+1} > m_k$ such that for any $m \geq m_{k+1}$, we have $\alpha_m(f_{k+1}) \leq 1+1/(k+1)$, which completes our induction claim.

By (3.3) and Definition 2.6 (b), it is easy to observe that $\{T^{m_k+p}(f_k)\}$ is a minimizing sequence of τ, for any $p \geq 0$. Lemma 3.1 implies $\{T^{m_k+p}(f_k)\}$ is ρ-convergent to f, for any $p \geq 0$.

In particular, we have $\{T^{m_k+1}(f_k)\}$ is ρ-convergent to f. Since

$$\rho\Big(T^{m_k+1}(f_k) - T(f)\Big) \leq \alpha_1(f)\rho(T^{m_k}(f_k) - f), \tag{3.5}$$

we conclude that $\{T^{m_k+1}(f_k)\}$ is also ρ-convergent to $T(f)$.

Since the ρ-limit of any ρ-convergent sequence is unique by Lemma 3.1, we must have $T(f) = f$.

In the following, we will prove that $\mathrm{Fix}(T)$ is ρ-closed.

To prove that $\mathrm{Fix}(T)$ is ρ-closed, let $x_n \in \mathrm{Fix}(T)$ and $\rho(x_n - x) \to 0$. Observe that

$$\begin{aligned}\rho\left(\frac{1}{3}(T(x) - x)\right) &\leq \rho(T(x) - T(x_n)) + \rho(T(x_n) - x_n) + \rho(x_n - x) \\ &\leq \alpha_1\rho(x_n - x) + \rho(x_n - x) \longrightarrow 0.\end{aligned} \tag{3.6}$$

Hence, $x \in \mathrm{Fix}(T)$, so $\mathrm{Fix}(T)$ is ρ-closed.

This completes the proof. □

Remark 3.3. It is not hard to see that if C is ρ-bounded, then an asymptotic pointwise nonexpansive mapping T is of asymptotic nonexpansive type [12]; that is, there exists a sequence $\{k_n\}$ of positive numbers with the property $k_n \to 1$ as $n \to \infty$ and such that

$$\rho(T_n(x) - T_y(y)) \leq k_n \rho(x - y), \tag{3.7}$$

for all n and $x, y \in C$.

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