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Research Article

Harmonic Morphisms Projecting Harmonic Functions to Harmonic Functions

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For Riemannian manifolds M and N, admitting a submersion ϕ with compact fibres, we introduce the projection of a function via its decomposition into horizontal and vertical components. By comparing the Laplacians on M and N, we determine conditions under which a harmonic function on $U = \phi^{-1}(V) \subset M$ projects down, via its horizontal component, to a harmonic function on $V \subset N$.

1. Introduction and Preliminaries

Harmonic morphisms are the maps between Riemannian manifolds which preserve germs of harmonic functions, that is, these (locally) pull back harmonic functions to harmonic functions. The aim of this paper is to analyse the converse situation and to investigate the class of harmonic morphisms that (locally) projects or pushes forward harmonic functions to harmonic functions, in the sense of Definition 2.4. If such a class exists, another interesting question arises "*to what extent does the pull back of the projected function preserve the original function.*"

The formal theory of harmonic morphisms between Riemannian manifolds began with the work of Fuglede [1] and Ishihara [2].

Definition 1.1. A smooth map $\phi : M^m \to N^n$ between Riemannian manifolds is called a *harmonic morphism* if, for every real-valued function f which is harmonic on an open subset V of N with $\phi^{-1}(V)$ nonempty, $f \circ \phi$ is a harmonic function on $\phi^{-1}(V)$.

These maps are related to horizontally (weakly) conformal maps which are a natural generalization of Riemannian submersions.

For a smooth map $\phi : M^m \to N^n$, let $C_\phi = \{x \in M | \operatorname{rank} d\phi_x < n\}$ be its *critical set*. The points of the set $M \setminus C_\phi$ are called *regular points*. For each $x \in M \setminus C_\phi$, the *vertical space* at x is defined by $T_x^V M = \operatorname{Ker} d\phi_x$. The *horizontal space* $T_x^H M$ at x is given by the orthogonal complement of $T_x^V M$ in $T_x M$.

Definition 1.2 (see [3, Section 2.4]). A smooth map $\phi : (M^m, \mathbf{g}) \to (N^n, \mathbf{h})$ is called *horizontally (weakly) conformal* if $d\phi = 0$ on C_ϕ and the restriction of ϕ to $M \setminus C_\phi$ is a conformal submersion, that is, for each $x \in M \setminus C_\phi$, the differential $d\phi_x : T_x^H M \to T_{\phi(x)} N$ is conformal and surjective. This means that there exists a function $\lambda : M \setminus C_\phi \to \mathbb{R}^+$ such that

$$\mathbf{h}\big(d\phi(X), d\phi(Y)\big) = \lambda^2 \mathbf{g}(X, Y), \quad \forall X, Y \in T_x^H M. \tag{1.1}$$

By setting $\lambda = 0$ on C_ϕ, we can extend $\lambda : M \to \mathbb{R}_0^+$ to a continuous function on M such that λ^2 is smooth. The extended function $\lambda : M \to \mathbb{R}_0^+$ is called the *dilation* of the map.

For $x \in M \setminus C_\phi$, the assignments $x \mapsto T_x^H M$ and $x \mapsto T_x^V M$ define smooth distributions $T^H M$ and $T^V M$ on $M \setminus C_\phi$ or subbundles of $TM|_{M \setminus C_\phi}$, the tangent bundle of $M \setminus C_\phi$. The distributions $T^H M$ and $T^V M$ are, respectively, called horizontal distribution (or horizontal subbundle) and vertical distribution (or vertical subbundle) defined by ϕ.

Recall that a map $\phi : M^m \to N^n$ is said to be *harmonic* if it extremizes the associated energy integral $E(\phi) = (1/2)\int_\Omega \|\phi_*\|^2 dv^M$ for every compact domain $\Omega \subset M$. It is well known that a map (ϕ) is harmonic if and only if its tension field vanishes.

Harmonic morphisms can be viewed as a subclass of harmonic maps in the light of the following characterization, obtained in [1, 2].

A smooth map is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal.

What is special about this characterization of harmonic morphism is that it equips them with geometric as well as analytic features. For instance, the following result of Baird and Eells [4, Riemannian case] and Gudmundsson [5, semi-Riemannian case] reflects such properties of harmonic morphisms.

Theorem 1.3. *Let $\phi : M^m \to N^n$ be a horizontally conformal submersion with dilation λ. If*

(1) $n = 2$, *then ϕ is a harmonic map if and only if it has minimal fibres;*

(2) $n \geq 3$, *then two of the following imply the other:*

(a) *ϕ is a harmonic map,*

(b) *ϕ has minimal fibres,*

(c) $\mathbf{grad}^H \lambda^2 = 0$ *where $\mathbf{grad}^H \lambda^2$ denotes the projection of $\mathbf{grad}\, \lambda^2$ on the horizontal subbundle of TM, obtained through the unique orthogonal decomposition into vertical and horizontal parts.*

For the fundamental results and properties of harmonic morphisms, the reader is referred to [1, 3, 6, 7] and for an updated *online* bibliography to [8].

2. The Projection of a Function via a Submersion

Given a smooth map $\phi : M^m \to N^n$ with compact fibres $\phi^{-1}(\phi(x))$ for $x \in M \setminus C_\phi$, we can use fibre integration to define the horizontal and vertical components of every integrable function f on $U \subset M$ at regular points.

Definition 2.1. Let $\phi : M^m \to N^n$ be a smooth map between Riemannian manifolds with compact fibres. Define the horizontal component of an integrable function f, on M, at a regular point x as the average of f taken over the fibre $\phi^{-1}(\phi(x))$. Precisely, for any $V \subset N$ and integrable function $f : U = \phi^{-1}(V) \subset M \to \mathbb{R}$, the *horizontal component* of f at a regular point x is defined as

$$(\mathscr{H} f)(x) = \frac{1}{\operatorname{vol}(\phi^{-1}(y))} \left(\int_{\phi^{-1}(y)} f \, dv^{\phi^{-1}(y)} \right)(\phi(x)), \tag{2.1}$$

where $x \in U$, $\phi(x) = y$, $dv^{\phi^{-1}(y)}$ is the volume element of the fibre $\phi^{-1}(y)$, $\operatorname{vol}(\phi^{-1}(y))$ is the volume of the fibre $\phi^{-1}(y)$, and $(\int_{\phi^{-1}(y)} f dv^{\phi^{-1}(y)})(\phi(x))$ denotes the integral of $f|_{\phi^{-1}(\phi(x))}$.

The *vertical component* of f is given by

$$(\mathcal{V} f)(x) = (f - \mathscr{H} f)(x). \tag{2.2}$$

Note that the horizontal component of a function depends only on the fibre $\phi^{-1}(y)$ and not the choice of $x \in \phi^{-1}(y)$.

Definition 2.2. Let $\phi : M^m \to N^n$ be a submersion. A function $f : U \subset M \to \mathbb{R}$ is called *horizontally homothetic* if the vector field $\mathbf{grad}(f)$ is vertical, that is, at each point $\mathbf{grad}(f)$ is tangent to the fibre.

The components $\mathscr{H} f$ and $\mathcal{V} f$ have the following basic properties for submersions.

Lemma 2.3. *Let $\phi : M^m \to N^n$ be a submersion with compact fibres. Suppose that the fibres $\phi^{-1}(y)$, $y \in N$ are minimal submanifolds of M. Consider $x \in U$ and a function $f : U \subset M \to \mathbb{R}$.*

(1) *If f is horizontally homothetic at x, then $\mathscr{H} f$ is also horizontally homothetic at x.*

(2) *If $\mathscr{H} f$ is horizontally homothetic at x and either $X_i(f) \geq 0$ or $X_i(f) \leq 0$ (for all i) on the fibre through x, then f is horizontally homothetic, where $\{X_i\}_{i=1}^n$ is a local orthonormal frame for the horizontal distribution.*

(3) *If f is constant along the fibre through x then $\mathcal{V} f = 0$.*

Proof. The proof can be completed by following the calculations in Proposition 3.1. □

Definition 2.4. Let $\phi : M^m \to N^n$ be a submersion with compact fibres, and let $f : U = \phi^{-1}(V) \subset M \to \mathbb{R}$ be an integrable function. The horizontal component of f defines a function $\tilde{f} : V \subset N \to \mathbb{R}$ as

$$\tilde{f}(y) = (\mathscr{H} f)(x), \tag{2.3}$$

where $x \in U$ and $y = \phi(x)$. The function $\tilde{f}$ is called the *projection* of f on N, *via the map* ϕ.

We next focus on projection of harmonic functions to harmonic functions via harmonic morphisms.

3. Harmonic Morphisms Projecting Harmonic Functions

The conditions under which harmonic morphisms project harmonic functions to harmonic functions can be obtained by employing an identity relating the Laplacian on the fibre with the Laplacians on the domain and target manifolds.

Recalling that for a submersion $\phi : M^m \to N^n$, the vector fields X on M and Y on N are said to be *ϕ-related* if $d\phi(X_x) = Y_{\phi(x)}$ for every $x \in M$. A horizontal vector field X on M is called *basic* if it is ϕ-related to some vector field X' on N, and X is called *horizontal lift* of X'. It is well known that for a given vector field X' on N, there exists a unique horizontal lift X of X' such that X and X' are ϕ-related.

Proposition 3.1. *Let $\phi : (M^m, \mathbf{g}) \to (N^n, \mathbf{h})$ $(n > 2)$ be a nonconstant submersive harmonic morphism with dilation λ, having compact, connected, and minimal fibres. Then for any $V \subset N$ and $f : U = \phi^{-1}(V) \subset M \to \mathbb{R}$,*

$$\begin{aligned}\Delta^N \widetilde{f} &= \frac{1}{\operatorname{vol}(\phi^{-1}(y))}\int_{\phi^{-1}(y)} \frac{1}{\lambda^2}\left(\Delta^M f - \Delta^{\phi^{-1}(y)} f\right) dv^{\phi^{-1}(y)} \\ &\quad + \frac{1}{\operatorname{vol}(\phi^{-1}(y))}\sum_{i=1}^{n}\int_{\phi^{-1}(y)} \left(\nabla^M_{X_i} X_i\right)^V f\, dv^{\phi^{-1}(y)},\end{aligned} \tag{3.1}$$

where $x \in U$, $\phi(x) = y$, $\widetilde{f}$ is as defined in Definition 2.4 and Δ^M, Δ^N, $\Delta^{\phi^{-1}(y)}$ are the Laplacians on M, N, $\phi^{-1}(y)$, respectively, ∇^M is the Levi-Civita connection on M, $(\nabla^M_{X_i} X_i)^V$ denotes the vertical component of $\nabla^M_{X_i} X_i$, and $\{X_i\}_{i=1}^n$ denote the horizontal lift of a local orthonormal frame $\{X_i'\}_{i=1}^n$ for TN.

Proof. First notice from Theorem 1.3 that λ is horizontally homothetic, a fact which will be used repeatedly in the proof.

Choose a local orthonormal frame $\{X_i'\}_{i=1}^n$ for TN. If X_i denotes the horizontal lift of X_i' for $i = 1, \dots, n$, then $\{\lambda X_i\}_{i=1}^n$ is a local orthonormal frame for the horizontal distribution. Let $\{X_\alpha\}_{\alpha=n+1}^m$ be a local orthonormal frame for the vertical distribution. Then we can write the Laplacian Δ^M on M as

$$\begin{aligned}\Delta^M &= \sum_{i=1}^{n}\left\{\lambda X_i \circ \lambda X_i - \nabla^M_{\lambda X_i} \lambda X_i\right\} + \sum_{\alpha=n+1}^{m}\left\{X_\alpha \circ X_\alpha - \nabla^M_{X_\alpha} X_\alpha\right\} \\ &= \lambda^2 \sum_{i=1}^{n}\left\{X_i \circ X_i - \nabla^M_{X_i} X_i\right\} + \sum_{\alpha=n+1}^{m}\left\{X_\alpha \circ X_\alpha - \nabla^M_{X_\alpha} X_\alpha\right\}.\end{aligned} \tag{3.2}$$

Now the Laplacian of the fibre $\phi^{-1}(y)$ is

$$\Delta^{\phi^{-1}(y)} = \sum_{\alpha=n+1}^{m}\left\{X_\alpha \circ X_\alpha - \nabla^{\phi^{-1}(y)}_{X_\alpha} X_\alpha\right\}. \tag{3.3}$$

If B is the second fundamental form of the fibre $\phi^{-1}(y)$ as a submanifold in M, we can write $\nabla^M_{X_\alpha} X_\alpha$ as

$$\nabla^M_{X_\alpha} X_\alpha = \nabla^{\phi^{-1}(y)}_{X_\alpha} X_\alpha + B(X_\alpha, X_\alpha). \tag{3.4}$$

Let μ denote the mean curvature vector of $\phi^{-1}(y)$ given by

$$\mu = \frac{1}{m-n} \sum_{\alpha=n+1}^{m} B(X_\alpha, X_\alpha). \tag{3.5}$$

Setting $H = (m-n)\mu$, we obtain from (3.2)

$$\begin{aligned} \Delta^M &= \lambda^2 \sum_{i=1}^{n} \left\{ X_i \circ X_i - \left(\nabla^M_{X_i} X_i\right)^H \right\} + \Delta^{\phi^{-1}(y)} - H - \lambda^2 \sum_{i=1}^{n} \left(\nabla^M_{X_i} X_i\right)^V \\ &= \lambda^2 \sum_{i=1}^{n} \left\{ X_i \circ X_i - \left(\nabla^M_{X_i} X_i\right)^H \right\} + \Delta^{\phi^{-1}(y)} - \lambda^2 \sum_{i=1}^{n} \left(\nabla^M_{X_i} X_i\right)^V, \end{aligned} \tag{3.6}$$

where X^H, X^V denote the orthogonal projections of a vector field X on the horizontal and vertical subbundles of TM, respectively.

Since X_i is the horizontal lift of X_i' $(i = 1, \dots, n)$, we have

$$\begin{aligned} X_i'\left(\tilde{f}\right) &= \frac{1}{\operatorname{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i(f)\, dv^{\phi^{-1}(y)} + \int_{\phi^{-1}(y)} f \mathcal{L}_{X_i}\left(dv^{\phi^{-1}(y)}\right) \right\} \\ &= \frac{1}{\operatorname{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i(f)\, dv^{\phi^{-1}(y)} + \sum_{\alpha=n+1}^{m} \int_{\phi^{-1}(y)} f \mathbf{g}\left(\nabla^M_{X_\alpha} X_i, X_\alpha\right) dv^{\phi^{-1}(y)} \right\} \\ &= \frac{1}{\operatorname{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i(f)\, dv^{\phi^{-1}(y)} - \int_{\phi^{-1}(y)} f \mathbf{g}(H, X_i)\, dv^{\phi^{-1}(y)} \right\}, \end{aligned} \tag{3.7}$$

where $\mathcal{L}_{X_i}$ denotes the Lie derivative along X_i. The volume of the fibres does not vary in the horizontal direction because of the relation $X_i'(\operatorname{vol}(\phi^{-1}(y))) = -\int_{\phi^{-1}(y)} \mathbf{g}(H, X_i)\, dv^{\phi^{-1}(y)}$ and the fact that the fibres are minimal.

Similarly, we obtain

$$\begin{aligned} X_i' \circ X_i'\left(\tilde{f}\right) = {} & \frac{1}{\operatorname{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i \circ X_i(f)\, dv^{\phi^{-1}(y)} - \int_{\phi^{-1}(y)} X_i(f) \cdot \mathbf{g}(H, X_i)\, dv^{\phi^{-1}(y)} \right\} \\ & - \frac{1}{\operatorname{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i(f \mathbf{g}(H, X_i))\, dv^{\phi^{-1}(y)} - \int_{\phi^{-1}(y)} f (\mathbf{g}(H, X_i))^2\, dv^{\phi^{-1}(y)} \right\}. \end{aligned} \tag{3.8}$$

The horizontal homothety of the dilation implies that $(\nabla^M_{X_i}X_i)^H$ is the horizontal lift of $\nabla^N_{X'_i}X'_i$, cf. [9, Lemma 3.1]; therefore, we have

$$\nabla^N_{X'_i}X'_i\left(\tilde{f}\right) = \frac{1}{\mathrm{vol}(\phi^{-1}(y))} \times \left\{\int_{\phi^{-1}(y)}\left(\nabla^M_{X_i}X_i\right)^H(f)dv^{\phi^{-1}(y)} - \int_{\phi^{-1}(y)} f\cdot \mathbf{g}\left(H,\left(\nabla^M_{X_i}X_i\right)^H\right)dv^{\phi^{-1}(y)}\right\}. \tag{3.9}$$

Now using (3.7), (3.8), (3.9), along with the condition that the fibres are minimal, in (3.6) completes the proof. □

From the above proposition, we see that it suffices to take λ constant to have both f and $\tilde{f}$ harmonic on M and N, respectively. In this case, by a homothety of M we may suppose that $\lambda \equiv 1$ and ϕ is a harmonic Riemannian submersion. We then have the following consequence.

Theorem 3.2. *Let $\phi : (M^m, \mathbf{g}) \to (N^n, \mathbf{h})$ $(n \geq 2)$ be a harmonic Riemannian submersion with compact, connected fibres. Then the projection $\tilde{f} : V \subset N \to \mathbb{R}$ (via ϕ) of any harmonic function $f : U = \phi^{-1}(V) \subset M \to \mathbb{R}$ is a harmonic function. Moreover, $\mathscr{H}f = \tilde{f}\circ\phi$. If $[f_{\mathscr{H}}]$ denotes the class of harmonic functions on $U = \phi^{-1}(V)$ having the same horizontal component then each class $[f_{\mathscr{H}}]$ has a unique representative in the space of harmonic functions on V.*

Proof. Since $\Delta^M f = 0$ and the dilation $\lambda \equiv 1$, Proposition 3.1 leads to

$$\Delta^N\tilde{f} = \frac{1}{\mathrm{vol}(\phi^{-1}(y))}\sum_{i=1}^{n}\int_{\phi^{-1}(y)}\left(\nabla^M_{X_i}X_i\right)^V f dv^{\phi^{-1}(y)}, \tag{3.10}$$

where we have also used the fact that

$$\int_{\phi^{-1}(y)}\Delta^{\phi^{-1}(y)} f dv^{\phi^{-1}(y)} = 0 \tag{3.11}$$

for compact fibres.

Let $\{X'_i\}_{i=1}^n$ be a local orthonormal frame for TN and X_i be the horizontal lift of X'_i for $i = 1,\ldots,n$. Then $\{X_i\}_{i=1}^n$ is a local orthonormal frame for the horizontal distribution. Let $\{X_\alpha\}_{\alpha=n+1}^m$ be a local orthonormal frame for the vertical distribution. Then using the standard expression for Levi-Civita connection, we have

$$\begin{aligned}\left(\nabla^M_{X_i}X_i\right)^V &= \sum_{\alpha=n+1}^{m}\mathbf{g}\left(\nabla^M_{X_i}X_i, X_\alpha\right)X_\alpha \\ &= \frac{1}{2}\sum_{\alpha=n+1}^{m}\{X_i(\mathbf{g}(X_i,X_\alpha)) + X_i(\mathbf{g}(X_\alpha,X_i)) - X_\alpha(\mathbf{g}(X_i,X_i)) \\ &\qquad -\mathbf{g}(X_i,[X_i,X_\alpha]) + \mathbf{g}(X_i,[X_\alpha,X_i]) + \mathbf{g}(X_\alpha,[X_i,X_i])\}X_\alpha.\end{aligned} \tag{3.12}$$

Because X_i are basic, X_α are vertical we have $[X_i, X_\alpha]$ vertical and therefore

$$\left(\nabla^M_{X_i} X_i\right)^V = 0. \tag{3.13}$$

Hence, from (3.10), $\tilde{f}$ is harmonic. The rest of the proof follows from the construction of $\tilde{f}$. □

As an application, we give a description of harmonic functions on manifolds admitting harmonic Riemannian submersions with compact fibres.

Corollary 3.3. *Let M^m be a Riemannian manifold admitting a harmonic Riemannian submersion $\phi : M^m \to N^n$ with compact fibres. Then*

(1) *every horizontally homothetic harmonic function on $U \subset M$ is horizontal, that is, $\mathcal{V}f = 0$, and so in particular is constant;*

(2) *every nonhorizontally homothetic harmonic function f on $U \subset M$ satisfies one of the following:*

  (a) $\mathcal{V}f \neq 0$;

  (b) $\mathcal{V}f = 0$ *and* $X_i(\mathcal{H}f) \neq 0$ *for at least one* $i \in \{1, \ldots, n\}$;

  (c) $\mathcal{V}f = 0$, $X_i(\mathcal{H}f) = 0$ *(for all i) and $X_i(f)$ changes sign on the fibre, for at least one* $i \in \{1, \ldots, n\}$.

Proof. Equation (3.6) implies that a horizontally homothetic harmonic function on M is harmonic on the fibre and hence is constant on the fibre. Now using Lemma 2.3 we get the proof. □

Remark 3.4. (1) Since an $\mathbb{R}^N$-valued map $f = (f^1, \ldots, f^N)$ is harmonic if and only if each of its component is harmonic, we see that Riemannian submersions with compact fibres project $\mathbb{R}^N$-valued harmonic maps from $\phi^{-1}(V)$ to $\mathbb{R}^N$-valued harmonic maps from V.

(2) Given a Lie group G and a compact subgroup H of G, the standard projection $\phi : G \to G/H$ with G-invariant metric provides many examples satisfying the hypothesis of Theorem 3.2. Further examples can be obtained from Bergery's construction $\phi : G/K \to G/H$ with $K \subset H \subset G$ and K, H compact; see [10] for the details of the metrics for which ϕ is a harmonic morphism. Another reference for such examples is [11, Chapter 6].

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