

Research Article

Coupled Coincidence Points in Partially Ordered Cone Metric Spaces with a c -Distance

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Cho et al. (2012) proved some coupled fixed point theorems in partially ordered cone metric spaces by using the concept of a c -distance in cone metric spaces. In this paper, we prove some coincidence point theorems in partially ordered cone metric spaces by using the notion of a c -distance. Our results generalize several well-known comparable results in the literature. Also, we introduce an example to support the usability of our results.

1. Introduction and Preliminaries

Fixed point theory is an essential tool in functional nonlinear analysis. Consequently, fixed point theory has wide applications areas not only in the various branches of mathematics (see, e.g., [1, 2]) but also in many fields, such as, chemistry, biology, statistics, economics, computer science, and engineering (see, e.g., [3–11]). For example, fixed point results are incredibly useful when it comes to proving the existence of various types of Nash equilibria (see, e.g., [7]) in economics. On the other hand, fixed point theorems are vital for the existence and uniqueness of differential equations, matrix equations, integral equations (see, e.g., [1, 2]). Banach contraction mapping principle (Banach fixed point theorem) is one of the most powerful theorems of mathematics and hence fixed point theory. Huang and Zhang [12] generalized the Banach contraction principle by replacing the notion of usual metric spaces by the notion of cone metric spaces. Then many authors obtained many fixed and common fixed point theorems in cone metric spaces. For some works in cone metric spaces, we may refer the reader (as examples) to [13–24]. The concept of a coupled fixed point of a mapping $F : X \times X \rightarrow X$ was initiated by Bhaskar and Lakshmikantham [25], while Lakshmikantham

and Ćirić [26] initiated the notion of coupled coincidence point of mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ and studied some coupled coincidence point theorems in partially ordered metric spaces. For some coupled fixed point and coupled coincidence point theorems, we refer the reader to [27–34].

In the present paper, \mathbf{N}^* is the set of positive integers and E stands for a real Banach space. Let P be a subset of E . We will always assume that the cone P has a nonempty interior $\text{Int}(P)$ (such cones are called solid). Then P is called a cone if the following conditions are satisfied:

- (1) P is closed and $P \neq \{\theta\}$,
- (2) $a, b \in \mathbf{R}^+$, $x, y \in P$ implies $ax + by \in P$,
- (3) $x \in P \cap -P$ implies $x = \theta$.

For a cone P , define a partial ordering \leq with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{Int}(P)$. It can be easily shown that $\lambda \text{Int}(P) \subseteq \text{Int}(P)$ for all positive scalar λ .

Definition 1.1 (see [12]). Let X be a nonempty set. Suppose the mapping $d : X \times X \rightarrow E$ satisfies

- (1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
- (3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X , and (X, d) is called a cone metric space.

Bhaskar and Lakshmikantham [25] introduced the notion of mixed monotone property of the mapping $F : X \times X \rightarrow X$.

Definition 1.2 (see [25]). Let (X, \leq) be a partially ordered set and $F : X \times X \rightarrow X$ be a mapping. Then the mapping F is said to have mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y ; that is, for any $x, y \in X$,

$$\begin{aligned} x_1 \leq x_2 \text{ implies } F(x_1, y) \leq F(x_2, y), \quad \forall y \in X, \\ y_1 \leq y_2 \text{ implies } F(x, y_2) \leq F(x, y_1), \quad \forall x \in X. \end{aligned} \tag{1.1}$$

Inspired by Definition 1.2, Lakshmikantham and Ćirić in [26] introduced the concept of a g -mixed monotone mapping.

Definition 1.3 (see [26]). Let (X, \leq) be a partially ordered set and $F : X \times X \rightarrow X$. Then the mapping F is said to have mixed g -monotone property if $F(x, y)$ is monotone g -nondecreasing in x and is monotone g -nonincreasing in y ; that is:

$$\begin{aligned} gx_1 \leq gx_2 \text{ implies } F(x_1, y) \leq F(x_2, y), \quad \forall y \in X, \\ gy_1 \leq gy_2 \text{ implies } F(x, y_2) \leq F(x, y_1), \quad \forall x \in X. \end{aligned} \tag{1.2}$$

Definition 1.4 (see [25]). An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \rightarrow X$ if

$$F(x, y) = x, \quad F(y, x) = y. \quad (1.3)$$

Definition 1.5 (see [26]). An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if

$$F(x, y) = gx, \quad F(y, x) = gy. \quad (1.4)$$

Recently, Cho et al. [35] introduced the concept of c -distance on cone metric space (X, d) which is a generalization of w -distance of Kada et al. [36] (see also [37, 38]).

Definition 1.6 (see [35]). Let (X, d) be a cone metric space. Then a function $q : X \times X \rightarrow E$ is called a c -distance on X if the following is satisfied:

- (q1) $\theta \leq q(x, y)$ for all $x, y \in X$,
- (q2) $q(x, z) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$,
- (q3) for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \leq u$ for some $u = u_x \in P$, then $q(x, y) \leq u$ whenever (y_n) is a sequence in X converging to a point $y \in X$,
- (q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $0 \leq e$ such that $q(z, x) \ll e$, and $q(z, y) \ll e$ implies $d(x, y) \ll c$.

Cho et al. [35] noticed the following important remark in the concept of c -distance on cone metric spaces.

Remark 1.7 (see [35]). Let q be a c -distance on a cone metric space (X, d) . Then

- (1) $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$,
- (2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Very recently, Cho et al. [39] proved the following existence theorems.

Theorem 1.8 (see [39]). Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c -distance on X and let $F : X \times X \rightarrow X$ be a continuous function having the mixed monotone property such that

$$q(F(x, y), F(x^*, y^*)) \leq \frac{k}{2}(q(x, x^*) + q(y, y^*)), \quad (1.5)$$

for some $k \in [0, 1)$ and all $x, y, x^*, y^* \in X$ with $(x \sqsubseteq x^*) \wedge (y \supseteq y^*)$ or $(x \supseteq x^*) \wedge (y \sqsubseteq y^*)$. If there exist $x_0, y_0 \in X$ such that $x_0 \sqsubseteq F(x_0, y_0)$ and $y_0 \supseteq F(y_0, x_0)$, then F has a coupled fixed point (u, v) . Moreover, one has $q(v, v) = q(u, u) = \theta$.

Theorem 1.9 (see [39]). Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c -distance on X , and let $F : X \times X \rightarrow X$ be a function having the mixed monotone property such that

$$q(F(x, y), F(x^*, y^*)) \leq \frac{k}{4}(q(x, x^*) + q(y, y^*)), \quad (1.6)$$

for some $k \in [0, 1)$ and all $x, y, x^*, y^* \in X$ with $(x \sqsubseteq x^*) \wedge (y \supseteq y^*)$ or $(x \supseteq x^*) \wedge (y \sqsubseteq y^*)$. Also, suppose that X has the following properties:

- (1) if (x_n) is a nondecreasing sequence in X with $x_n \rightarrow x$, then $x_n \sqsubseteq x$ for all $n \geq 1$,
- (2) if (x_n) is a nonincreasing sequence in X with $x_n \rightarrow x$, then $x \sqsubseteq x_n$ for all $n \geq 1$.

If there exist $x_0, y_0 \in X$ such that $x_0 \sqsubseteq F(x_0, y_0)$ and $y_0 \supseteq F(y_0, x_0)$, then F has a coupled fixed point (u, v) . Moreover, one has $q(v, v) = q(u, u) = \theta$.

For other fixed point results using a c -distance, see [40].

In this paper, we prove some coincidence point theorems in partially ordered cone metric spaces by using the notion of c -distance. Our results generalize Theorems 1.8 and 1.9. We consider an application to illustrate our result is useful (see Section 3).

2. Main Results

The following lemma is essential in proving our results.

Lemma 2.1 (see [35]). *Let (X, d) be a cone metric space, and let q be a cone distance on X . Let (x_n) and (y_n) be sequences in X and $x, y, z \in X$. Suppose that (u_n) is a sequence in P converging to θ . Then the following holds.*

- (1) If $q(x_n, y) \leq u_n$ and $q(x_n, z) \leq u_n$, then $y = z$.
- (2) If $q(x_n, y_n) \leq u_n$ and $q(x_n, z) \leq u_n$, then (y_n) converges to z .
- (3) If $q(x_n, x_m) \leq u_n$ for $m > n$, then (x_n) is a Cauchy sequence in X .
- (4) If $q(y, x_n) \leq u_n$, then (x_n) is a Cauchy sequence in X .

In this section, we prove some coupled fixed point theorems by using c -distance in partially partially ordered cone metric spaces.

Theorem 2.2. *Let (X, \leq) be a partially ordered set and suppose that (X, d) is a cone metric space. Let q be a c -distance on X . Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that*

$$q(F(x, y), F(x^*, y^*)) + q(F(y, x), F(y^*, x^*)) \leq k(q(gx, gx^*) + q(gy, gy^*)), \quad (2.1)$$

for some $k \in [0, 1)$ and for all $x, y, x^*, y^* \in X$ with $(gx \leq gx^*) \wedge (gy \geq gy^*)$ or $(gx \geq gx^*) \wedge (gy \leq gy^*)$. Assume that F and g satisfy the following conditions:

- (1) F is continuous,
- (2) g is continuous and commutes with F ,
- (3) $F(X \times X) \subseteq gX$,
- (4) (X, d) is complete,
- (5) F has the mixed g -monotone property.

If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$, then F and g have a coupled coincidence point (u, v) . Moreover, one has $q(gu, gu) = \theta$ and $q(gv, gv) = \theta$.

Proof. Let $x_0, y_0 \in X$ be such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Since F has the mixed g -monotone property, we have $gx_0 \leq gx_1 \leq gx_2$ and $gy_2 \leq gy_1 \leq gy_0$. Continuing this process, we can construct two sequences (x_n) and (y_n) in X such that

$$\begin{aligned} gx_n &= F(x_{n-1}, y_{n-1}) \leq gx_{n+1} = F(x_n, y_n), \\ gy_{n+1} &= F(y_n, x_n) \leq gy_n = F(y_{n-1}, x_{n-1}). \end{aligned} \quad (2.2)$$

Let $n \in \mathbf{N}^*$. Then by (2.1), we have

$$\begin{aligned} & q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) \\ &= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + q(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq k(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n)). \end{aligned} \quad (2.3)$$

Repeating (2.3) n -times, we get

$$q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) \leq k^n (q(gx_0, gx_1) + q(gy_0, gy_1)). \quad (2.4)$$

Thus, we have

$$q(gx_n, gx_{n+1}) \leq k^n (q(gx_0, gx_1) + q(gy_0, gy_1)), \quad (2.5)$$

$$q(gy_n, gy_{n+1}) \leq k^n (q(gx_0, gx_1) + q(gy_0, gy_1)). \quad (2.6)$$

Let $m, n \in \mathbf{N}^*$ with $m > n$. Then by (q_2) and (2.5), we have

$$\begin{aligned} q(gx_n, gx_m) &\leq \sum_{i=n}^{m-1} q(gx_i, gx_{i+1}) \\ &\leq \sum_{i=n}^{m-1} k^i (q(gx_0, gx_1) + q(gy_0, gy_1)) \\ &\leq \frac{k^n}{1-k} (q(gx_0, gx_1) + q(gy_0, gy_1)). \end{aligned} \quad (2.7)$$

Similarly, we have

$$q(gy_n, gy_m) \leq \frac{k^n}{1-k} (q(gx_0, gx_1) + q(gy_0, gy_1)). \quad (2.8)$$

From part (3) of Lemma 2.1, we conclude that (gx_n) and (gy_n) are Cauchy sequences in (X, d) . Since X is complete, there are $u, v \in X$ such that $gx_n \rightarrow u$ and $gy_n \rightarrow v$. Using

the continuity of g , we get $g(gx_n) \rightarrow gu$ and $g(gy_n) \rightarrow gv$. Also, by continuity of F and commutativity of F and g , we have

$$\begin{aligned} gu &= \lim_{n \rightarrow \infty} g(gx_{n+1}) = \lim_{n \rightarrow \infty} g(F(x_n, y_n)) = \lim_{n \rightarrow \infty} F(gx_n, gy_n) = F(u, v), \\ gv &= \lim_{n \rightarrow \infty} g(gy_{n+1}) = \lim_{n \rightarrow \infty} g(F(y_n, x_n)) = \lim_{n \rightarrow \infty} F(gy_n, gx_n) = F(v, u). \end{aligned} \quad (2.9)$$

Hence, (u, v) is a coupled coincidence point of F and g . Moreover, by (2.1) we have

$$\begin{aligned} q(gu, gu) + q(gv, gv) &= q(F(u, v), F(u, v)) + q(F(v, u), F(v, u)) \\ &\leq k(q(gu, gu) + q(gv, gv)). \end{aligned} \quad (2.10)$$

Since $k < 1$, we conclude that $q(gu, gu) + q(gv, gv) = \theta$, and hence $q(gu, gu) = \theta$ and $q(gv, gv) = \theta$. \square

The continuity of F in Theorem 2.2 can be dropped. For this, we present the following useful lemma which is a variant of Lemma 2.1, (1).

Lemma 2.3. *Let (X, d) be a cone metric space, and let q be a c -distance on X . Let (x_n) be a sequence in X . Suppose that (α_n) and (β_n) are sequences in P converging to θ . If $q(x_n, y) \leq \alpha_n$ and $q(x_n, z) \leq \beta_n$, then $y = z$.*

Proof. Let $c \gg \theta$ be arbitrary. Since $\alpha_n \rightarrow \theta$, so there exists $N_1 \in \mathbb{N}$ such that $\alpha_n \ll c/2$ for all $n \geq N_1$. Similarly, there exists $N_2 \in \mathbb{N}$ such that $\beta_n \ll c/2$ for all $n \geq N_2$. Thus, for all $N \geq \max\{N_1, N_2\}$, we have

$$q(x_n, y) \ll \frac{c}{2}, \quad q(x_n, z) \ll \frac{c}{2}. \quad (2.11)$$

Take $e = c/2$, so by (q4), we get that $d(y, z) \ll c$ for each $c \gg \theta$; hence $y = z$. \square

Theorem 2.4. *Let (X, \leq) be a partially ordered set and suppose that (X, d) is a cone metric space. Let q be a c -distance on X . Let $F : X \times X \rightarrow X$ and let $g : X \rightarrow X$ be two mappings such that*

$$q(F(x, y), F(x^*, y^*)) + q(F(y, x), F(y^*, x^*)) \leq k(q(gx, gx^*) + q(gy, gy^*)), \quad (2.12)$$

for some $k \in [0, 1)$ and for all $x, y, x^*, y^* \in X$ with $(gx \leq gx^*) \wedge (gy \geq gy^*)$ or $(gx \geq gx^*) \wedge (gy \leq gy^*)$. Assume that F and g satisfy the following conditions:

- (1) $F(X \times X) \subseteq gX$,
- (2) gX is a complete subspace of X ,
- (3) F has the mixed g -monotone property.

Suppose that X has the following properties:

- (i) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a nonincreasing sequence $x_n \rightarrow x$, then $x \leq x_n$ for all n .

Assume there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$. Then F and g have a coupled coincidence point, say $(u, v) \in X \times X$. Also, $q(gu, gu) = q(gv, gv) = \theta$.

Proof. As in the proof of Theorem 2.2, we can construct two Cauchy sequences (gx_n) and (gy_n) in the complete cone metric space (gX, d) . Then, there exist $u, v \in X$ such that $gx_n \rightarrow gu$ and $gy_n \rightarrow gv$. Similarly we have for all $m > n \geq 1$

$$\begin{aligned} q(gx_n, gx_m) &\leq \frac{k^n}{1-k} [q(gx_0, gx_1) + q(gy_0, gy_1)], \\ q(gy_n, gy_m) &\leq \frac{k^n}{1-k} [q(gx_0, gx_1) + q(gy_0, gy_1)]. \end{aligned} \quad (2.13)$$

By (q3), we get that

$$q(gx_n, gu) \leq \frac{k^n}{1-k} [q(gx_0, gx_1) + q(gy_0, gy_1)], \quad (2.14)$$

$$q(gy_n, gv) \leq \frac{k^n}{1-k} [q(gx_0, gx_1) + q(gy_0, gy_1)]. \quad (2.15)$$

By summation, we get that

$$q(gx_n, gu) + q(gy_n, gv) \leq 2 \frac{k^n}{1-k} [q(gx_0, gx_1) + q(gy_0, gy_1)]. \quad (2.16)$$

Since (gx_n) is nondecreasing and (gy_n) is nonincreasing, using the properties (i), (ii) of X , we have

$$gx_n \leq gu, \quad gv \leq gy_n, \quad \forall n \geq 0. \quad (2.17)$$

From this and (2.14), we have

$$\begin{aligned} q(gx_n, F(u, v)) + q(gy_n, F(v, u)) &= q(F(x_{n-1}, y_{n-1}), F(u, v)) + q(F(y_{n-1}, x_{n-1}), F(v, u)) \\ &\leq k(q(gx_{n-1}, gu) + q(gy_{n-1}, gv)). \end{aligned} \quad (2.18)$$

Therefore

$$q(gx_n, F(u, v)) + q(gy_n, F(v, u)) \leq k[q(gx_{n-1}, gu) + q(gy_{n-1}, gv)]. \quad (2.19)$$

By (2.16), we have

$$\begin{aligned} q(gx_n, F(u, v)) + q(gy_n, F(v, u)) &\leq k[q(gx_{n-1}, gu) + q(gy_{n-1}, gv)] \\ &\leq k \frac{2k^{n-1}}{1-k} [q(gx_0, gx_1) + q(gy_0, gy_1)] \\ &= \frac{2k^n}{1-k} [q(gx_0, gx_1) + q(gy_0, gy_1)]. \end{aligned} \quad (2.20)$$

This implies that

$$q(gx_n, F(u, v)) \leq \frac{2k^n}{1-k} [q(gx_0, gx_1) + q(gy_0, gy_1)], \quad (2.21)$$

$$q(gy_n, F(v, u)) \leq \frac{2k^n}{1-k} [q(gx_0, gx_1) + q(gy_0, gy_1)]. \quad (2.22)$$

By (2.14), (2.21) and Lemma 2.3, we obtain $gu = F(u, v)$. Similarly, by (2.15), (2.22), and Lemma 2.3, we obtain $gv = F(v, u)$. Also, adjusting as the proof of Theorem 2.2, we get that

$$q(gu, gu) = q(gv, gv) = \theta. \quad (2.23)$$

□

Corollary 2.5. *Let (X, \leq) be a partially ordered set and suppose that (X, d) is a cone metric space. Let q be a c -distance on X . Let $F : X \times X \rightarrow X$, and let $g : X \rightarrow X$ be two mappings such that*

$$q(F(x, y), F(x^*, y^*)) \leq aq(gx, gx^*) + bq(gy, gy^*), \quad (2.24)$$

for some $a, b \in [0, 1)$ with $a + b < 1$ and for all $x, y, x^*, y^* \in X$ with $(gx \leq gx^*) \wedge (gy \geq gy^*)$ or $(gx \geq gx^*) \wedge (gy \leq gy^*)$. Assume that F and g satisfy the following conditions:

- (1) F is continuous,
- (2) g is continuous and commutes with F ,
- (3) $F(X \times X) \subseteq gX$,
- (4) (X, d) is complete,
- (5) F has the mixed g -monotone property.

If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$, then F and g have a coupled coincidence point (u, v) . Moreover, one has $q(gu, gu) = \theta$ and $q(gv, gv) = \theta$.

Proof. Given $x, x^*, y, y^* \in X$ such that $(gx \leq gx^*) \wedge (gy \geq gy^*)$. By (2.24), we have

$$\begin{aligned} q(F(x, y), F(x^*, y^*)) &\leq aq(gx, gx^*) + bq(gy, gy^*), \\ q(F(y, x), F(y^*, x^*)) &\leq aq(gy, gy^*) + bq(gx, gx^*). \end{aligned} \quad (2.25)$$

Thus

$$q(F(x, y), F(x^*, y^*)) + q(F(y, x), F(y^*, x^*)) \leq (a + b)(aq(gx, gx^*) + q(gy, gy^*)). \quad (2.26)$$

Since $a + b < 1$, the result follows from Theorem 2.2. □

Corollary 2.6. *Let (X, \leq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c -distance on X . Let $F : X \times X \rightarrow X$ be a continuous mapping having the mixed monotone property such that*

$$q(F(x, y), F(x^*, y^*)) \leq aq(x, x^*) + bq(y, y^*), \quad (2.27)$$

for some $a, b \in [0, 1)$ with $a + b < 1$ and for all $x, y, x^*, y^* \in X$ with $(x \leq x^*) \wedge (y \geq y^*)$ or $(x \geq x^*) \wedge (y \leq y^*)$. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then F has a coupled fixed point (x, y) . Moreover, one has $q(x, x) = \theta$ and $q(y, y) = \theta$.

Proof. It follows from Corollary 2.5 by taking $g = I_X$ (the identity map). \square

Corollary 2.7. Let (X, \leq) be a partially ordered set and suppose that (X, d) is a cone metric space. Let q be a c -distance on X . Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that

$$q(F(x, y), F(x^*, y^*)) \leq aq((gx, gx^*) + bq(gy, gy^*)), \quad (2.28)$$

for some $a, b \in [0, 1)$ with $a + b < 1$ and for all $x, y, x^*, y^* \in X$ with $(gx \leq gx^*) \wedge (gy \geq gy^*)$ or $(gx \geq gx^*) \wedge (gy \leq gy^*)$. Assume that F and g satisfy the following conditions:

- (1) $F(X \times X) \subseteq gX$,
- (2) gX is a complete subspace of X ,
- (3) F has the mixed g -monotone property.

Suppose that X has the following properties:

- (i) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a nonincreasing sequence $x_n \rightarrow x$, then $x \leq x_n$ for all n .

Assume there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$. Then F and g have a coupled coincidence point.

Proof. It follows from Theorem 2.4 by similar arguments to those given in proof of Corollary 2.5. \square

Corollary 2.8. Let (X, \leq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c -distance on X . Let $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property such that

$$q(F(x, y), F(x^*, y^*)) \leq aq((x, x^*) + bq(y, y^*)), \quad (2.29)$$

for some $a, b \in [0, 1)$ with $a + b < 1$ and for all $x, y, x^*, y^* \in X$ with $(x \leq x^*) \wedge (y \geq y^*)$ or $(x \geq x^*) \wedge (y \leq y^*)$. Suppose that X has the following properties:

- (i) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a nonincreasing sequence $x_n \rightarrow x$, then $x \leq x_n$ for all n .

Assume there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$. Then F has a coupled fixed point.

Proof. It follows from Corollary 2.7 by taking $g = I_X$ (the identity map). \square

Corollary 2.9. Let (X, \leq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c -distance on X , and let $F : X \times X \rightarrow X$ be a continuous mapping having the mixed monotone property such that

$$q(F(x, y), F(x^*, y^*)) + q(F(y, x), F(y^*, x^*)) \leq k(q(x, x^*) + q(y, y^*)), \quad (2.30)$$

for some $k \in [0, 1)$ and for all $x, y, x^*, y^* \in X$ with $(x \leq x^*) \wedge (y \geq y^*)$ or $(x \geq x^*) \wedge (y \leq y^*)$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then F has a coupled fixed point (x, y) . Moreover, we have $q(x, x) = \theta$ and $q(y, y) = \theta$.

Proof. It follows from Theorem 2.2 by taking $g = I_X$. □

Corollary 2.10. Let (X, \leq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c -distance on X . Let $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property such that

$$q(F(x, y), F(x^*, y^*)) + q(F(y, x), F(y^*, x^*)) \leq k(q(x, x^*) + q(y, y^*)), \quad (2.31)$$

for some $k \in [0, 1)$ and for all $x, y, x^*, y^* \in X$ with $(x \leq x^*) \wedge (y \geq y^*)$ or $(x \geq x^*) \wedge (y \leq y^*)$. Suppose that X has the following properties:

- (i) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a nonincreasing sequence $x_n \rightarrow x$, then $x \leq x_n$ for all n .

Assume there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$. Then F has a coupled fixed point.

Proof. It follows from Theorem 2.4 by taking $g = I_X$. □

Example 2.11. Let $E = C_R^1([0, 1])$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0, t \in [0, 1]\}$. Let $X = [0, 1]$ with usual order \leq . Define $d : X \times X \rightarrow X$ by $d(x, y)(t) = |x - y|e^t$ for all $x, y \in X$. Then (X, d) is a partially ordered cone metric space. Define $q : X \times X \rightarrow E$ by $q(x, y)(t) = ye^t$ for all $x, y \in X$. Then q is a c -distance. Define $F : X \times X \rightarrow X$ by

$$F(x, y) = \begin{cases} \frac{x - y}{2}, & x \geq y, \\ 0, & x < y. \end{cases} \quad (2.32)$$

Then,

- (1) $q(F(x, y), F(x^*, y^*)) + q(F(y, x), F(y^*, x^*)) \leq 1/2(q(x, x^*) + q(y, y^*))$, for all $x \leq x^*$ and $y \geq y^*$,
- (2) there is no $k \in [0, 1)$ such that $q(F(x, y), F(x^*, y^*)) \leq (k/2)(q(x, x^*) + q(y, y^*))$ for all $x \leq x^*$ and $y \geq y^*$,
- (3) there is no $k \in [0, 1)$ such that $q(F(x, y), F(x^*, y^*)) \leq (k/4)(q(x, x^*) + q(y, y^*))$ for all $x \leq x^*$ and $y \geq y^*$.

Note that $0 \leq F(0, 0)$ and $0 \geq F(0, 0)$. Thus by Corollary 2.10, we have F which has a coupled fixed point. Here $(0, 0)$ is a coupled fixed point of F .

Proof. The proof of (2.1) is easy. To prove (2.3), suppose the contrary; that is, there is $k \in [0, 1)$ such that $q(F(x, y), F(x^*, y^*)) \leq k/2(q(x, x^*) + q(y, y^*))$ for all $x \leq x^*$ and $y \geq y^*$. Take $x = 0, y = 1, x^* = 1$ and $y^* = 0$. Then

$$q(F(0, 1), F(1, 0))(t) \leq \frac{k}{2}(q(0, 1) + q(1, 0))(t). \quad (2.33)$$

Thus

$$q\left(0, \frac{1}{2}\right)(t) = \frac{1}{2}e^t \leq \frac{k}{2}e^t. \quad (2.34)$$

Hence $k \geq 1$ is a contradiction. The proof of (2.5) is similar to proof of (2.3). \square

Remark 2.12. Note that Theorems 3.1 and 3.2 of [39] are not applicable to Example 2.11.

Remark 2.13. Theorem 3.1 of [39] is a special case of Corollary 2.6 and Corollary 2.9.

Remark 2.14. Theorem 3.3 of [39] is a special case of Corollary 2.8 and Corollary 2.10.

3. Application

Consider the integral equations

$$\begin{aligned} x(t) &= \int_0^T f(t, x(s), y(s)) ds, \quad t \in [0, T], \\ y(t) &= \int_0^T f(t, y(s), x(s)) ds, \quad t \in [0, T], \end{aligned} \quad (3.1)$$

where $T > 0$ and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. Let $X = C([0, T], \mathbb{R})$ denote the space of \mathbb{R} -valued continuous functions on $I = [0, T]$. The purpose of this section is to give an existence theorem for a solution (x, y) to (3.1) that belongs to X , by using the obtained result given by Corollary 2.10. Let $E = \mathbb{R}^2$, and let $P \subset E$ be the cone defined by

$$P = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}. \quad (3.2)$$

We endow X with the cone metric $d : X \times X \rightarrow E$ defined by

$$d(u, v) = \left(\sup_{t \in I} |u(t) - v(t)|, \sup_{t \in I} |u(t) - v(t)| \right), \quad \forall u, v \in X. \quad (3.3)$$

It is clear that (X, d) is a complete cone metric space. Let $q(x, y) = d(x, y)$ for all $x, y \in X$. Then, q is a c -distance.

Now, we endow X with the partial order \leq given by

$$u, v \in X, \quad u \leq v \iff u(x) \leq v(x), \quad \forall x \in I. \quad (3.4)$$

Also, the product space $X \times X$ can be equipped with the partial order (still denoted \leq) given as follows:

$$(x, y) \leq (u, v) \iff x \leq u, y \geq v. \quad (3.5)$$

It is easy that (i) and (ii) given in Corollary 2.10 are satisfied.

Now, we consider the following assumptions:

- (a) $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous,
- (b) for all $t \in [0, T]$, the function $f(t, \cdot, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ has the mixed monotone property,
- (c) for all $t \in [0, T]$, for all $p, q, p', q' \in \mathbb{R}$ with $p \leq q$ and $p' \geq q'$, we have

$$f(t, q, q') - f(t, p, p') \leq \frac{1}{T} \varphi \left(\frac{q - p + p' - q'}{2} \right), \quad (3.6)$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is continuous nondecreasing and satisfies the following condition: There exists $0 < k < 1$ such that

$$\varphi(r) \leq kr \quad \forall r \geq 0, \quad (3.7)$$

- (d) there exists $x_0, y_0 \in C([0, T], \mathbb{R})$ such that

$$x_0(t) \leq \int_0^T f(t, x_0(s), y_0(s)) ds, \quad \int_0^T f(t, y_0(s), x_0(s)) ds \leq y_0(t), \quad \forall t \in [0, T]. \quad (3.8)$$

We have the following result.

Theorem 3.1. *Suppose that (a)–(d) hold. Then, (3.1) has at least one solution $(x^*, y^*) \in C([0, T], \mathbb{R}) \times C([0, T], \mathbb{R})$.*

Proof. Define the mapping $A : C([0, T], \mathbb{R}) \times C([0, T], \mathbb{R}) \rightarrow C([0, T], \mathbb{R})$ by

$$A(x, y)(t) = \int_0^T f(t, x(s), y(s)) ds, \quad x, y \in C([0, T], \mathbb{R}), \quad t \in [0, T]. \quad (3.9)$$

We have to prove that A has at least one coupled fixed point $(x^*, y^*) \in C([0, T], \mathbb{R}) \times C([0, T], \mathbb{R})$. □

From (b), it is clear that A has the mixed monotone property.

Now, let $x, y, u, v \in C([0, T], \mathbb{R})$ such that $(x \leq u$ and $y \geq v)$ or $(x \geq u$ and $y \leq v)$. Using (c), for all $t \in [0, T]$, we have

$$\begin{aligned} |A(u, v)(t) - A(x, y)(t)| &\leq \int_0^T [f(t, u(s), v(s)) - f(t, x(s), y(s))] ds \\ &\leq \frac{1}{T} \int_0^T \varphi \left(\frac{u(s) - x(s) + y(s) - v(s)}{2} \right) ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{T} \int_0^T \varphi \left(\frac{\sup_{z \in [0, T]} |u(z) - x(z)| + \sup_{z \in [0, T]} |y(z) - v(z)|}{2} \right) ds \\
&= \varphi \left(\frac{\sup_{z \in [0, T]} |u(z) - x(z)| + \sup_{z \in [0, T]} |y(z) - v(z)|}{2} \right) \\
&\leq k \left[\frac{\sup_{z \in [0, T]} |u(z) - x(z)| + \sup_{z \in [0, T]} |y(z) - v(z)|}{2} \right],
\end{aligned} \tag{3.10}$$

which implies that

$$\sup_{z \in [0, T]} |A(u, v)(t) - A(x, y)(t)| \leq k \left[\frac{\sup_{z \in [0, T]} |u(z) - x(z)| + \sup_{z \in [0, T]} |y(z) - v(z)|}{2} \right]. \tag{3.11}$$

Similarly, one can get

$$\sup_{z \in [0, T]} |A(v, u)(t) - A(y, x)(t)| \leq k \left[\frac{\sup_{z \in [0, T]} |u(z) - x(z)| + \sup_{z \in [0, T]} |y(z) - v(z)|}{2} \right]. \tag{3.12}$$

We deduce

$$\begin{aligned}
&\sup_{z \in [0, T]} |A(u, v)(t) - A(x, y)(t)| + \sup_{z \in [0, T]} |A(v, u)(t) - A(y, x)(t)| \\
&\leq k \left[\sup_{z \in [0, T]} |u(z) - x(z)| + \sup_{z \in [0, T]} |y(z) - v(z)| \right].
\end{aligned} \tag{3.13}$$

Thus

$$d(A(u, v), A(x, y)) + d(A(v, u), A(y, x)) \leq k[d(x, u) + d(y, v)]. \tag{3.14}$$

Thus, we proved that condition (2.31) of Corollary 2.10 is satisfied. Moreover, from (d), we have $x_0 \leq A(x_0, y_0)$ and $A(y_0, x_0) \leq y_0$. Finally, applying our Corollary 2.10, we get the desired result.

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