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Research Article

Vectorial Ekeland Variational Principles and Inclusion Problems in Cone Quasi-Uniform Spaces

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Some new vectorial Ekeland variational principles in cone quasi-uniform spaces are proved. Some new equivalent principles, vectorial quasivariational inclusion principle, vectorial quasi-optimization principle, vectorial quasiequilibrium principle are obtained. Also, several other important principles in nonlinear analysis are extended to cone quasi-uniform spaces. The results of this paper extend, generalize, and improve the corresponding results for Ekeland's variational principles of the directed vectorial perturbation type and other generalizations of Ekeland's variational principles in the setting of F-type topological space and quasi-metric spaces in the literatures. Even in usual real metric spaces, some of our results are new.

1. Introduction

Ekeland's variational principle [1] is a forceful tool in nonlinear analysis, control theory, global analysis, and many others. In the last two decades, it has been further studied, extended, and applied to many fields in mathematics (see, e.g., [2–17, 19–21, 26, 28, 29] and the references therein). Especially, we want to emphasize that many generalizations of Ekeland's variational principle to vector-valued functions have been recently obtained in [2–5, 8–10, 14–17] and the references therein. For example, in [2], the author proved some vectorial Ekeland's variational principles for vector-valued functions defined on quasi-metric spaces. In [3, 5, 14], the authors proved some vectorial Ekeland's variational principles for vector-valued functions defined on metric spaces. In [8], the authors proved some vector Ekeland's variational principle in a F-type topological space. However, all of these results are essentially Ekeland's principle of the directed vectorial perturbation type. But, the study for the case of nondirected vectorial perturbation type has just been started in [15]. On the other hand, variational inclusion problem is a very important problem and it contains many important problems such as complementarity problems, minimax inequalities, equilibrium problems, saddle point problems, optimization theory, bilevel problems, mathematical programs with equilibrium constraints, variational inequalities, and fixed point problems (see, e.g., [18–21] and the references therein).

In this paper, we first prove some nondirected vectorial perturbation type Ekeland's variational principles in cone quasi-uniform spaces, then by using our new principles, we introduce some variational inclusion principles and prove the equivalence among these results and other equivalents of our principle.

The paper is organized as follows: in Section 2, some properties of cone, some new equivalent characterizations of the pseudo-nuclear cone, and the definition of cone quasi-uniform spaces are given. In Section 3, we prove some new Ekeland's variational principles for both non-directed vectorial perturbation type and directed vectorial perturbation type in the setting of general topological vector spaces for vector-valued functions defined on complete cone quasi-uniform spaces. Some of new equivalent principles, vectorial quasi-variational inclusion principle, vectorial quasi-optimization principle, and vectorial quasi-equilibrium principle are introduced and proved, which have a wide practical background in quasi-variational inclusion problems, quasi-optimization problems, and quasi-equilibrium problems (see, e.g., [21]). In addition to these new equivalent principles, generalized Caristi-Kirk type coincidence point theorem for multivalued maps defined on cone quasi-uniform spaces, nonconvex maximal element theorem for the family of multivalued maps defined on cone quasi-uniform spaces, generalized vectorial Takahashi nonconvex minimization theorem, and Oettli-Théra type theorem defined on cone quasi-uniform spaces are also presented. The results of this paper not only give some new Ekeland's principles of the non-directed vectorial perturbation type and some new equivalent principles, but also extend and generalize or improve many corresponding results for Ekeland's variational principles of the directed vectorial perturbation type and other generalizations of Ekeland's variational principles in the setting of F-type topological space and quasi-metric spaces in the literatures [2, 3, 5, 8, 14, 15]. Even in usual real metric spaces, some of our results are new.

2. Preliminaries

Let $E(\tau)$ be a topological vector space and $V_\tau(\theta)$ (resp., $V_\sigma(\theta)$) denote the θ-neighborhood base with respect to the topology τ (resp., to the weak topology σ). A subset $K \subseteq E$ is called a closed convex cone if K is closed, $K + K \subseteq K$ and $\lambda K \subseteq K$ for all $\lambda \in [0, +\infty)$. A quasiorder (i.e., a reflexive and transitive relation) on E can be defined by K, that is, $x \leq y$ if and only if $y - x \in K$. We write $x < y$ whenever $x \leq y$ and $y \neq x$. The quasi-order $\leq$ is called a partial order if it is antisymmetric. Let E^* be the topological dual space of E (i.e., E^* is the set of all continuous linear functions on E) and let K^* be the dual cone of K, that is, $K^* = \{f \in E^* : f(x) \geq 0, \text{ for all } x \in K\}$.

We recall that a subset $B \subset E$ is said to be *K-saturated* if $B = [B]$, where $[B]$ is defined by

$$[B] = (B + K) \cap (B - K) = \cup\{[x, y] : x \in B, y \in B\}, \tag{2.1}$$

where $[x, y] = \{z \in E : x \leq z \leq y\} = (x + K) \cap (y - K)$. The cone K is said to be *normal* if there is a θ-neighborhood base for τ consisting of K-saturated sets.

The following Lemma 2.1 is an elementary result (refer to [22–24]) characterizing the concept of normality that will be used in our proof.

Lemma 2.1. *Let $E(\tau)$ be a topological vector space and $K \subseteq E$ be a cone. Then the following propositions are equivalent.*

(1) *K is a normal cone.*

(2) *There exists a θ-neighborhood base $V_\tau(\theta)$ for τ consisting of sets V for which $\theta \leq x \leq y \in V$ implies $x \in V$.*

(3) *For any two nets $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ in E, if $\theta \leq x_i \leq y_i$ for all $i \in I$ and $\{y_i\}_{i\in I}$ converges to zero for τ, then $\{x_i\}_{i\in I}$ converges to zero for τ.*

(4) *For any τ-neighborhood V of zero, there exists a τ-neighborhood W of zero such that $\theta \leq x \leq y \in W$ implies $x \in V$.*

The following notion of a pseudo-nuclear cone is a generalization of nuclear cone which has many applications in optimizations, fixed point theory, and the best approximation theory, and so forth.

Definition 2.2 (see [25]). Let $E(\tau)$ be a topological vector space (not necessarily locally convex) and $K \subseteq E$ be a closed convex cone (not necessary pointed). If $f \in E^*$, then we denote $f_\pi = \{x \in E : f(x) \leq 1\}$. A set of the kind $f_\pi \cap K$ is called a θ-pseudoslice of K, where $f \in E^*$, the dual space of E. We say that K is pseudo-nuclear if each θ-neighborhood in E contains a θ-pseudoslice of K.

To discuss the properties of pseudo-nuclear cone, we need the following existence theorem of a quasi-norms family which can determine the topology of $E(\tau)$.

Lemma 2.3 (see [26]). *Let $E(\tau)$ be a topological vector space and $\widetilde{U}(\tau)$ be a balanced θ-neighborhood base for the topology τ. Then there exist a directed set I and a quasi-norms family $\{\|\cdot\|_\alpha : \alpha \in I\}$ such that*

(a) *for any $k \in R$, $\alpha \in I$ and $x \in E$, $\|kx\|_\alpha = |k|\|x\|_\alpha$;*

(b) *for any $\alpha \in I$, there exists $\mu \in I$ such that $\alpha \leq \mu$ and $\|x + y\|_\alpha \leq \|x\|_\mu + \|y\|_\mu$ for any $x, y \in E$;*

(c) *for any $\alpha, \mu \in I$, if $\alpha \leq \mu$, then $\|x\|_\alpha \leq \|x\|_\mu$ for any $x \in E$;*

(d) *$\{\|\cdot\|_\alpha : \alpha \in I\}$ and $\widetilde{U}(\tau)$ determine the same topology τ;*

(e) *if τ is a T_0 topology (i.e., for any $x \neq \theta$, there exists a neighborhood of zero $U \in \widetilde{U}(\tau)$ such that $x \notin U$), then*

$$x = \theta \iff \|x\|_\alpha = 0, \quad \forall \alpha \in I. \tag{2.2}$$

The following lemma gives several equivalent conditions of pseudo-nuclearity.

Lemma 2.4. *Let $E(\tau)$ be a topological vector space and $K \subseteq E$ a cone. Then the following assertions are equivalent.*

(1) *K is a pseudo-nuclear cone.*

(2) *For all $V \in V_\tau(\theta)$, there exists $W \in V_\sigma(\theta)$ such that $\theta \le x \le y \in W$ implies $x \in V$.*

(3) *For all $V \in V_\tau(\theta)$, there exists $f \in K^*$ such that $\{x \in K : f(x) \le 1\} \subset V$.*

(4) *If $\{\|\cdot\|_\alpha : \alpha \in I\}$ is the family of quasi-norms which determines the topology of $E(\tau)$, then, for any $\alpha \in I$, there exists $f_\alpha \in K^*$ such that*

$$\|x\|_\alpha \le f_\alpha(x), \qquad \forall x \in K. \tag{2.3}$$

Proof. It follows from Theorem 3.8 in [24] that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) Since each neighborhood of zero with respect to the weak topology σ is also a neighborhood of zero with respect to the topology τ, for any $V \in V_\sigma(\theta)$, we have $V \in V_\tau(\theta)$. From (2), we know that there exists $W \in V_\sigma(\theta)$ such that $\theta \le x \le y \in W$ implies $x \in V$. This and (4) of Lemma 2.1 imply that K is a normal cone with respect to the weak topology σ. Since the weak topology σ is defined by semi-norms family $\{|f| : f \in E^*\}$, $E(\sigma)$ is a locally convex topological vector space. Therefore, $E^* = K^* - K^*$ with respect to the weak topology σ. From (2), it follows that, for any $V \in V_\tau(\theta)$, there exists $W \in V_\sigma(\theta)$ such that $\theta \le x \le y \in W$ implies $x \in V$. Thus there exist $f_1, \dots, f_m$ and $\varepsilon > 0$ such that $\{x \in E : |f_i(x)| < \varepsilon,\ i = 1, \dots, m\} \subset W$. Also, there exist $g_i, h_i \in K^*$ such that $f_i = g_i - h_i$ for each $i = 1, \dots, m$. Let $f = (2/\varepsilon)\sum_{i=1}^m (g_i + h_i)$. Then $f \in K^*$ and

$$\begin{aligned} \{x \in K : f(x) \le 1\} &= \left\{x \in K : \sum_{i=1}^m (g_i + h_i)(x) \le \frac{\varepsilon}{2}\right\} \\ &\subset \{x \in E : |f_i(x)| < \varepsilon,\ i = 1, \dots, m\} \\ &\subset W. \end{aligned} \tag{2.4}$$

Thus $\{x \in K : f(x) \le 1\} \subset V$, that is, (3) holds.

(3) $\Rightarrow$ (4) Suppose that $\{\|\cdot\|_\alpha : \alpha \in I\}$ is the family of *quasi-norms* which determines the topology of $E(\tau)$. Then, for any $\alpha \in I$, $\{x \in E : \|x\|_\alpha < 1\} \in V_\tau(\theta)$. It follows from (3) that, for any $\alpha \in I$, there exists $f_\alpha \in K^*$ such that $\{x \in K : f_\alpha(x) \le 1\} \subset \{x \in E : \|x\|_\alpha < 1\}$. If $x \in K$ and $f_\alpha(x) > 0$, then $\|x/f_\alpha(x)\|_\alpha < 1$, that is, $\|x\|_\alpha < f_\alpha(x)$. If $x \in K$ and $f_\alpha(x) = 0$, then, for any $l \in R^+$, $f_\alpha(lx) = 0$, which implies that $\|lx\|_\alpha = l\|x\|_\alpha < 1$. Since l is arbitrary, we know that $\|x\|_\alpha = 0$ and so $\|x\|_\alpha \le f_\alpha(x)$ for all $x \in K$, that is, (4) is true.

(4) $\Rightarrow$ (1) By Lemma 2.3(d), $\{\|\cdot\|_\alpha : \alpha \in I\}$ and $\widetilde{U}(\tau)$ determine the same topology τ. Then, for any $V \in V_\tau(\theta)$, there exist $\alpha_1, \alpha_2, \dots, \alpha_m \in I$ and positive numbers $\delta_1, \delta_2, \dots, \delta_m$ such that $\bigcap_{i=1}^m \{x \in E : \|x\|_{\alpha_i} < \delta_i\} \subset V$. By (4), there exists $f_{\alpha_i} \in K^*$ $(i = 1, 2, \dots, m)$ such that

$\|x\|_{\alpha_i} \leq f_{\alpha_i}(x)$ for all $x \in K$. Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\}$ and $f(x) = (2/\delta)\sum_{i=1}^{m} f_{\alpha_i}(x)$. Then $f \in E^*$ and

$$\begin{aligned}\{x \in E : f(x) \leq 1\} \cap K &= \left\{x \in K : \sum_{i=1}^{m} f_{\alpha_i}(x) \leq \frac{\delta}{2}\right\} \\ &\subset \bigcap_{i=1}^{m}\{x \in K : |f_{\alpha_i}(x)| < \delta\} \\ &\subset \bigcap_{i=1}^{m}\{x \in E : \|x\|_{\alpha_i} < \delta_i\} \\ &\subset V.\end{aligned} \tag{2.5}$$

This shows that (1) holds. This completes the proof. □

To discuss Ekeland's principles of the nondirected vectorial perturbation type, by enlightening of the work in [27], we define the following cone quasi-uniform space, which is an extension of the cone uniform space in [27]. About the discussion and applications of cone uniform space, one can refer to [15, 27–29].

Definition 2.5. Let X be a nonempty set and $(D, \prec)$ be a directed set. Let E be a topological vector space and $K \subseteq E$ be a cone. Let the family

$$\mathcal{P} = \{d_\lambda : X \times X \longrightarrow E : \lambda \in D\} \tag{2.6}$$

satisfy the following conditions:

(d1) for any $\lambda \in D$, $d_\lambda(x, y) \in K$ for all $x, y \in X$;

(d2) for any $\lambda \in D$, $d_\lambda(x, y) = \theta$ if and only if $x = y$;

(d3) for any $\lambda \in D$, there exists $\mu \in D$ with $\lambda \prec \mu$ such that

$$d_\lambda(x, y) \leq d_\mu(x, z) + d_\mu(z, y), \quad \forall x, y, z \in X. \tag{2.7}$$

Then $\mathcal{P}$ is called a *family of cone* quasi-metrics on X and (X, P) is called a *cone* quasi-uniform *space*.

Definition 2.6. Let $\{x_n\}$ be a sequence in a cone quasi-uniform space (X, P).

(1) The sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ if, for any $\lambda \in D$ and θ-neighborhood U in E, there exists a positive integer N such that $d_\lambda(x, x_n) \in U$ for any $n \geq N$, that is, $\lim_{n\to\infty} d_\lambda(x, x_n) = \theta$, which is denoted by $\lim_{n\to\infty} x_n = x$.

(2) The sequence $\{x_n\}$ is called a Cauchy sequence in X if, for any $\lambda \in D$ and θ-neighborhood U in E, there exists a positive integer N such that $d_\lambda(x_n, x_m) \in U$ for any $m > n \geq N$, that is, $\lim_{m,n\to\infty} d_\lambda(x_n, x_m) = \theta$.

(3) If every Cauchy sequence is convergent in (X, P), then (X, P) is called a sequentially complete cone quasi-uniform space.

(4) For any net $\{x_\alpha\}_{\alpha\in I}$ in a cone quasi-uniform space (X, P), the convergence of the net and the Cauchy net can be defined similarly. If every Cauchy net is convergent in (X, P), then (X, P) is called a complete cone quasi-uniform space.

Remark 2.7. If we take $E = R$, then the cone quasi-uniform space is a generalization of F-type topological spaces [30]. If $P = \{d : X \times X \to E\}$, then (X, P) reduces to a cone quasi-metric space (X, d). The notions on convergence and completeness in a cone quasi-uniform space are complicated because of the lack of symmetry of the cone quasi-metric. Similarly, we can define the other kinds of convergence and completeness in a cone quasi-uniform space. For more details, the reader can refer to same discussions in the case of quasi-metric spaces [31].

Recall that a cone K has a *base* [8] or is *well based* [11] if there exists a convex subset B with $\theta \notin \overline{B}$ such that $K = \cup_{r>0} rB$.

Lemma 2.8 (see [8]). *Let $E(\tau)$ be a topological vector space and $K \subset E$ be a closed convex cone. Then the cone K has a base if and only if*

$$(K^*)^\circ = \{f \in E^* : f(x) > 0,\ \forall x \in K \setminus \{\theta\}\} \neq \emptyset. \tag{2.8}$$

Definition 2.9 (see [16]). Let $(\mathrm{X}, \leq)$ be a quasiordered set.

(1) X is said to be *totally ordered upperseparable* (resp., *totally ordered lower-separable*) if, for any totally ordered nonempty subset M of X, there exists an increasing sequence (resp., decreasing sequence) $\{x_n\} \subset M$ such that, for any $x \in M$, there exists $x_{n_0} \in \{x_n\}$ satisfying $x \leq x_{n_0}$ (resp., $x_{n_0} \leq x$).

(2) X is said to be *totally ordered separable* if X is both totally ordered upperseparable and totally ordered lower-separable.

Lemma 2.10 (see [16]). *Let $(\mathrm{X}, \leq)$ be a partially ordered set such that every increasing sequence has an upper bound, let $(Y, \leq_Y)$ be a totally ordered upper-separable (resp., totally ordered lower-separable) partially ordered set, and let $\Phi : X \to Y$ be an increasing (resp., decreasing) mapping. Then, for any $x \in \mathrm{X}$, there exists $v \geq x$ such that $\Phi(y) = \Phi(v)$ for all $y \geq v$. Moreover, if Φ is strictly monotone, then v is a maximal element.*

3. Vectorial Ekeland's Variational Principle

Let $E(\tau)$ be a topological vector space, let K be a closed convex cone in E, let (X, P) be a complete cone quasi-uniform space, and let $\varphi : E \to (0, +\infty)$ be a nondecreasing function, that is, $x \leq y$ implies that $\varphi(x) \leq \varphi(y)$. For a function $F : X \times X \to E$, we first give the following conditions:

(H1) $F(x, x) = \theta$ for all $x \in X$;

(H2) $F(x, y) + F(y, z) \in F(x, z) + K$ for all $x, y, z \in X$;

(H3) for any fixed $x, x_0 \in X$, the mapping $F(x, \cdot) : X \to E$ is bound from below and

$$\{y \in X : \varphi(F(x_0, x))F(x, y) + d_\lambda(x, xy) \leq \theta,\ \forall \lambda \in D\}. \tag{3.1}$$

Is a closed subset in X.

Lemma 3.1. *Let $E(\tau)$ be a topological vector space with a T_0 topology τ and $K \subset E$ be a closed convex pseudo-nuclear cone or a closed point convex cone (i.e., $K \cap (-K) = \{\theta\}$). Let (X, P) be a complete cone quasi-uniform space and $F : X \times X \to E$ be a function satisfying (H1) and (H2). For any fixed $x_0 \in X$, one defines a relation $\preccurlyeq$ on X as follows:*

$$x \preccurlyeq x' \Longleftrightarrow \varphi(F(x_0, x))F(x, x') + d_\lambda(x, x') \leq \theta, \quad \forall \lambda \in D. \tag{3.2}$$

Then $(X, \preccurlyeq)$ is a partially ordered set.

Proof. It is clear that $x \preccurlyeq x$. If $x \preccurlyeq y$ and $y \preccurlyeq x$, then we have

$$\begin{aligned} &\varphi(F(x_0, x))F(x, y) + d_\lambda(x, y) \leq \theta, \quad \forall \lambda \in D, \\ &\varphi(F(x_0, y))F(y, x) + d_\lambda(y, x) \leq \theta, \quad \forall \lambda \in D. \end{aligned} \tag{3.3}$$

This shows that $F(x, y) \leq \theta$ and $F(y, x) \leq \theta$. It follows from the condition (H2) that

$$\begin{aligned} F(x_0, x) &\leq F(x_0, y) + F(y, x) \leq F(x_0, y), \\ F(x_0, y) &\leq F(x_0, x) + F(x, y) \leq F(x_0, x). \end{aligned} \tag{3.4}$$

It follows from Lemma 2.3 that there exists a family $\{\|\cdot\|_\alpha : \alpha \in I\}$ of *quasi-norms* which defines the topology τ of E. If K is pseudo-nuclear, by Lemma 2.4, it follows that, for any $\alpha \in I$, there exists $f_\alpha \in K^*$ with the property:

$$\|x\|_\alpha \leq f_\alpha(x), \quad \forall x \in K. \tag{3.5}$$

For any $\alpha \in I$, we know that f_α is an increasing function on E. Then it follows from (3.4) that $\varphi(F(x_0, x)) = \varphi(F(x_0, y))$, $f_\alpha(F(x_0, x)) = f_\alpha(F(x_0, y))$ and then $f_\alpha(F(y, x)) = f_\alpha(F(x, y)) = 0$. Thus, by (3.3), it follows that $f_\alpha(d_\lambda(x, y)) = f_\alpha(d_\lambda(y, x)) = 0$, for any $\alpha \in I$ and $\lambda \in D$. It follows from (3.5) that $d_\lambda(x, y) = \theta$ for all $\lambda \in D$, that is, $x = y$. If K is a closed convex cone, then the ordering relation $\leq$ is a partial ordering. It follows from (3.3) and (3.4) that $F(x_0, x) = F(x_0, y)$ and $F(y, x) = F(x, y) = \theta$. Again by (3.3), it follows that $d_\lambda(x, y) = \theta$ for all $\lambda \in D$, that is, $x = y$. If $x \preccurlyeq y$ and $y \preccurlyeq z$, then we have

$$\begin{aligned} &\varphi(F(x_0, x))F(x, y) + d_\lambda(x, y) \leq \theta, \quad \forall \lambda \in D, \\ &\varphi(F(x_0, y))F(y, z) + d_\lambda(y, z) \leq \theta, \quad \forall \lambda \in D, \end{aligned} \tag{3.6}$$

which imply that $F(x, y) \leq \theta$ and $F(y, z) \leq \theta$ and so

$$F(x_0, y) \leq F(x_0, x) + F(x, y) \leq F(x_0, x). \tag{3.7}$$

By (d3) of Definition 2.5, it follows that, for any $\lambda \in D$, there exists $\mu \in D$ with $\lambda \prec \mu$ such that $d_\lambda(x,z) \le d_\mu(x,y) + d_\mu(y,z)$ and so

$$\begin{aligned}
&\varphi(F(x_0,x))F(x,z) + d_\lambda(x,z) \\
&\quad \le \varphi(F(x_0,x))\left[F(x,y) + F(y,z)\right] + d_\mu(x,y) + d_\mu(y,z) \\
&\quad \le \varphi(F(x_0,x))F(x,y) + \varphi(F(x_0,y))F(y,z) + d_\mu(x,y) + d_\mu(y,z) \\
&\quad \le \theta.
\end{aligned} \tag{3.8}$$

This shows that $x \preccurlyeq z$. Therefore, $\preccurlyeq$ is a partial order on X. This completes the proof. □

First, we give a general Ekeland's principle of the nondirected vectorial perturbation type as follows:

Theorem 3.2. *Let $E(\tau)$ be a topological vector space with a T_0 topology τ and $K \subset E$ be a closed convex cone. Let (X,P) be a complete cone quasi-uniform space and let $F : X \times X \to E$ be a function satisfying the conditions $(H1)$, $(H2)$, and $(H3)$. If the cone K is pseudo-nuclear, then, for any $x_0 \in X$, there exists $v \in X$ such that*

(1) *$\varphi(\theta)F(x_0,v) + d_\lambda(x_0,v) \in -K$ for all $\lambda \in D$;*

(2) *for all $x \in X$ with $x \neq v$, there exists $\lambda \in D$ such that*

$$\varphi(F(x_0,v))F(v,x) + d_\lambda(v,x) \notin -K. \tag{3.9}$$

Proof. For any fixed $x_0 \in X$, if $\preceq$ is defined by (3.2), then Lemma 3.1 shows that $(X,\preceq)$ is a partially ordered set.

First, we prove that any increasing net $\{x_\gamma\}_{\gamma\in\Lambda}$ is a Cauchy net. It follows from Lemma 2.3 that there exists a family $\{\|\cdot\|_\alpha : \alpha \in I\}$ of quasi-norms which defines the topology τ of E. Since K is pseudo-nuclear, by Lemma 2.4, it follows that, for any $\alpha \in I$, there exists $f_\alpha \in K^*$ with the property:

$$\|x\|_\alpha \le f_\alpha(x), \quad \forall x \in K. \tag{3.10}$$

From the proof of Lemma 3.1, we can show that $\{F(x_0,x_\gamma)\}_{\gamma\in\Lambda}$ is a decreasing net and, for any $\nu \ge \gamma$,

$$d_\lambda(x_\gamma,x_\nu) + \varphi(F(x_0,x_\gamma))F(x_\gamma,x_\nu) \le \theta, \quad \forall \lambda \in D, \tag{3.11}$$

and so

$$d_\lambda(x_\gamma,x_\nu) \le -\varphi(F(x_0,x_\gamma))F(x_\gamma,x_\nu), \quad \forall \lambda \in D. \tag{3.12}$$

It follows from $F(x_0,x_\nu) \le F(x_0,x_\gamma) + F(x_\gamma,x_\nu)$ that

$$-F(x_\gamma,x_\nu) \le F(x_0,x_\gamma) - F(x_0,x_\nu). \tag{3.13}$$

Thus we have

$$d_\lambda(x_\gamma, x_\nu) \le \varphi(F(x_0, x_\gamma))\left[F(x_0, x_\gamma) - F(x_0, x_\nu)\right], \quad \forall \lambda \in D, \tag{3.14}$$

and so, for any $\alpha \in I$,

$$f_\alpha(d_\lambda(x_\gamma, x_\nu)) \le \varphi(F(x_0, x_\gamma))\left[f_\alpha(F(x_0, x_\gamma)) - f_\alpha(F(x_0, x_\nu))\right], \quad \forall \lambda \in D. \tag{3.15}$$

It follows from the condition (H3) that $\{F(x_0, x_\gamma)\}$ is bounded from below. This and (3.15) imply that

$$\lim_{\gamma,\nu} f_\alpha(d_\lambda(x_\gamma, x_\nu)) = 0, \quad \forall \lambda \in D. \tag{3.16}$$

It follows from (3.10) that, for any $\alpha \in I$,

$$\lim_{\gamma,\nu} \left\| d_\lambda(x_\gamma, x_\nu) \right\|_\alpha = 0, \quad \forall \lambda \in D. \tag{3.17}$$

This shows that $\{x_\gamma\}_{\gamma\in\Lambda}$ is a Cauchy net in X. The completeness implies that $\{x_\gamma\}_{\gamma\in\Lambda}$ converges to some $v \in X$. It follows from $x_\gamma \preccurlyeq x_\nu$ for any $\nu \ge \gamma$ that

$$d_\lambda(x_\gamma, x_\nu) + \varphi(F(x_0, x_\gamma))F(x_\gamma, x_\nu) \le \theta, \quad \forall \lambda \in D. \tag{3.18}$$

By (H3), we get that

$$d_\lambda(x_\gamma, v) + \varphi(F(x_0, x_\gamma))F(x_\gamma, v) \le \theta, \quad \forall \lambda \in D. \tag{3.19}$$

Thus $x_\gamma \preccurlyeq v$, that is, v is an upper bound of the net $\{x_\gamma\}_{\gamma\in\Lambda}$. Assume that $A \subset \{x \in X : x \succcurlyeq x_0\}$ is a totally ordered set. Then A is also a directed set. If we represent A by $\{x_x\}_{x\in A}$, where $x_x = x$, then $\{x_x\}_{x\in A}$ is an increasing net. By the results just proved above, we can know that A has an upper bound. By the well-known Zorn's lemma, the set $\{x \in X : x \succcurlyeq x_0\}$ has a maximal element $v \succcurlyeq x_0$, that is,

$$\varphi(F(x_0, x_0))F(x_0, v) + d_\lambda(x_0, v) \le \theta, \quad \forall \lambda \in D. \tag{3.20}$$

Therefore, it follows from $F(x_0, x_0) = \theta$ that the conclusion (1) holds. For any $x \in X$ with $x \ne v$, if $\varphi(F(x_0, v))F(v, x) + d_\lambda(v, x) \le \theta$ for all $\lambda \in D$, then $v \preccurlyeq x$. Since v is a maximal element, we have that $v = x$, which is a contradiction. Therefore, conclusion (2) holds. This completes the proof. □

If the cone K has a base, we have the following result.

Theorem 3.3. *Let $E(\tau)$ be a topological vector space with a T_0 topology τ and let $K \subset E$ be a closed convex cone. Let (X, P) be a sequentially complete cone quasi-uniform space and let $F : X \times X \to E$*

be a mapping satisfying the conditions $(H1)$, $(H2)$, *and* $(H3)$. *If the cone* K *has a base, then, for any* $x_0 \in X$, *there exists* $v \in X$ *such that*

(1) $\varphi(\theta)F(x_0, v) + d_\lambda(x_0, v) \in -K$ *for all* $\lambda \in D$;

(2) *for all* $x \in X$ *with* $x \neq v$, *there exists* $\lambda \in D$ *such that*

$$\varphi(F(x_0, v))F(v, x) + d_\lambda(v, x) \notin -K. \tag{3.21}$$

Proof. Since cone K has a base, it follows from Lemma 2.3 that $(K^*)^\circ \neq \emptyset$. Let $\xi \in (K^*)^\circ$. For any fixed $x_0 \in X$, assume that $\preceq$ is defined by (3.2). Similarly, as in the proofs of Lemma 3.1 and Theorem 3.2, we can prove that $(X, \preceq)$ is a partially ordered set and any increasing sequence $\{x_n\}_{n\in N}$ has an upper bound. The only difference is that $f_\alpha \in K^*$ is replaced by $\xi \in (K^*)^\circ$. Assume that $x \preccurlyeq y$ and $x \neq y$. Then we have

$$d_\lambda(x, y) \leq \varphi(F(x_0, x))\left[F(x_0, x) - F(x_0, y)\right], \quad \forall \lambda \in D. \tag{3.22}$$

$x \neq y$ implies that there exists $\lambda_0 \in D$ such that $d_{\lambda_0}(x, y) \neq \theta$. Then we have

$$0 < \xi(d_{\lambda_0}(x, y)) \leq \varphi(F(x_0, x))\left[\xi(F(x_0, x)) - \xi(F(x_0, y))\right], \tag{3.23}$$

that is, $\xi(F(x_0, x)) < \xi(F(x_0, y))$. Let $\eta(x) = \xi(F(x_0, x))$. Then $\eta : X \to R$ is strictly increasing. Assume that $A \subset X$ is a totally ordered set. Then $\eta(A)$ is also a totally set and, for any $x, y \in A$, $x \preccurlyeq y$ if and only if $\eta(x) \leq \eta(y)$. It follows from Theorem 2.3 in [16] that the set R of real numbers is totally ordered separable and so there exist an increasing sequence $\{y_n\} \subset \eta(A)$ and a decreasing sequence $\{z_n\} \subset \eta(A)$ such that, for any $y \in \eta(A)$, there exist positive integers n_1 and n_2 satisfying

$$z_{n_1} \leq y \leq y_{n_2}. \tag{3.24}$$

Let $\{u_n\}$ and $\{v_n\}$ be two sequences in A such that $\eta(u_n) = y_n$ and $\eta(v_n) = z_n$. Since $\eta : X \to R$ is strictly increasing, we know that $\{u_n\}$ is increasing and $\{v_n\}$ is decreasing. For any $x \in A$, by (3.24), we have

$$\eta(v_{n_1}) = z_{n_1} \leq \eta(x) \leq y_{n_2} = \eta(u_{n_2}) \tag{3.25}$$

and so

$$v_{n_1} \leq x \leq u_{n_2}. \tag{3.26}$$

Thus $(X, \preceq)$ is totally ordered separable. By using Lemma 2.10, we can get that there exists a maximal element $v \succcurlyeq x_0$, that is,

$$\varphi(F(x_0, x_0))F(x_0, v) + d_\lambda(x_0, v) \leq \theta, \quad \forall \lambda \in D. \tag{3.27}$$

Then, in the same way as in the proof of Theorem 3.2, we can prove that conclusions (1) and (2) holds. This completes the proof. □

Remark 3.4. From the proof of Theorems 3.2 and 3.3 we can see that if cone K is a pointed convex cone and replace the pseudo-nuclearity of K with the regularity of K (i.e., any monotone and bounded net in K is convergent), then the conclusions of Theorems 3.2 and 3.3 still hold.

Clearly, our nondirected vectorial perturbation type Ekeland's principle contains the directed vectorial perturbation type Ekeland's principle as a special case. From Theorem 3.2, we have the following Theorem 3.5, which improves the corresponding results in [2, 3, 5, 8, 14].

Theorem 3.5. *Let $(X, \{d_\lambda\}_{\lambda\in D})$ be a complete F-type topological space, let $E(\tau)$ be a Hausdorff topological vector space, and let $K \subset E$ be a closed convex cone. Let $F : X \times X \to E$ be a function satisfying the conditions $(H1)$, $(H2)$, and $(H3)$. Let $e_\lambda \in K \setminus \{\theta\}$ for all $\lambda \in D$. If K is pseudo-nuclear, or $E(\tau)$ is a locally convex space and $K \cap (-K) = \{\theta\}$, then, for any $x_0 \in X$, there exists $v \in X$ such that*

(1) *$\varphi(\theta)F(x_0, v) + d_\lambda(x_0, v)e_\lambda \in -K$ for all $\lambda \in D$;*

(2) *for all $x \in X$ with $x \neq v$, there exists $\lambda \in D$ such that*

$$\varphi(F(x_0, v))F(v, x) + d_\lambda(v, x)e_\lambda \notin -K. \tag{3.28}$$

Proof. Let $\widetilde{d}_\lambda : X \times X \to \widetilde{E}$ be a mapping defined by $\widetilde{d}_\lambda(x, y) = d_\lambda(x, y)e_\lambda$ and

$$\widetilde{P} = \left\{\widetilde{d}_\lambda : X \times X \longrightarrow \widetilde{E} : \lambda \in D\right\}. \tag{3.29}$$

It is clearly that, for any net $\{x_\gamma\}_{\gamma\in\Lambda}$ and $x \in X$,

$$\begin{gathered}\lim_{\gamma}\widetilde{d}_\lambda(x_\gamma, x) = \theta \Longleftrightarrow \lim_{\gamma} d_\lambda(x_\gamma, x) = 0,\\ \lim_{\gamma,\nu}\widetilde{d}_\lambda(x_\gamma, x_\nu) = \theta \Longleftrightarrow \lim_{\gamma,\nu} d_\lambda(x_\gamma, x_\nu) = 0.\end{gathered} \tag{3.30}$$

Since $(X, \{d_\lambda\}_{\lambda\in D})$ is a complete F-type topological space, we have that $(X, \widetilde{P})$ is a complete cone quasi-uniform space. If K is pseudo-nuclear, then the conclusion of Theorem 3.5 can be obtained by using Theorem 3.2 for $(X, \widetilde{P})$. If $E(\tau)$ is a locally convex space and $K \cap (-K) = \{\theta\}$, then, for each $e_\lambda \in K \setminus \{\theta\}$, we have $-e_\lambda \notin K$. By using well-known separation theorem of convex sets, there exist a continuous linear function $f_\lambda \in E^*$ and a number c, such that $f_\lambda(-e_\lambda) < c$, and $f_\lambda(x) > c$ for any $x \in K$. This shows that $0 = f_\lambda(\theta) > c$ and $f_\lambda(-e_\lambda) < 0$. If $x \in K$, then $nx \in K$ for any $n \in N$. So we have that $f_\lambda(x) > c/n$. Then by letting $n \to \infty$, we get that $f_\lambda(x) \geq 0$ for any $x \in K$. Thus, for each $e_\lambda \in K \setminus \{\theta\}$, there exists a continuous function $f_\lambda \in K^*$, such that $f_\lambda(e_\lambda) > 0$. For any fixed $x_0 \in X$, if $\preceq$ is defined by

$$x \preccurlyeq x' \Longleftrightarrow \varphi(F(x_0, x))F(x, x') + \widetilde{d}_\lambda(x, x') \leq \theta, \quad \forall \lambda \in D. \tag{3.31}$$

By Lemma 3.1, $(X, \preceq)$ is a partially ordered set. Similarly as in the proofs of Theorem 3.2, we can prove that any increasing net $\{x_\gamma\}_{\gamma\in\Lambda}$ has an upper bound. The only difference is that

$f_\alpha \in K^*$ is replaced by $f_\lambda \in K^*$. Then, in the same way as in the proof of Theorem 3.2, we can prove that the conclusion of Theorem 3.5 holds. This completes the proof. □

In the following Theorem 3.6, we introduce some new principles, which are equivalent with our Ekeland principle.

Theorem 3.6. *Let $E(\tau)$ be a topological vector space with a T_0 topology τ, K be a closed convex and pseudo-nuclear cone in E and (X, P) be a complete cone quasi-uniform space. Let $F : X \times X \to E$ be a function satisfying the conditions $(H1)$–$(H3)$. Then, for any $x_0 \in X$, the following conclusions hold and they are equivalent.*

(1) *(Vectorial Ekeland type variational principle in cone quasi-uniform spaces) There exists*

$$v \in S(x_0) = \{x \in X : \varphi(\theta)F(x_0, x) + d_\lambda(x_0, x) \in -K,\ \forall \lambda \in D\} \tag{3.32}$$

such that, for all $x \in X$ with $x \neq v$, there exists $\lambda \in D$ satisfying the following:

$$\varphi(F(x_0, v))F(v, x) + d_\lambda(v, x) \notin -K. \tag{3.33}$$

(2) *(Vectorial quasivariational inclusion principle in cone quasi-uniform spaces) Let Z be a vector space and $P \subset Z$ be a nonempty convex subset. Let $G : X \times X \to 2^Z \setminus \emptyset$ and $H : X \to 2^X \setminus \emptyset$ be both multivalued mappings with nonempty values. Suppose that*

$$d_\lambda(x, y) + \varphi(F(x_0, x))F(x, y) \leq \theta, \quad \forall \lambda \in D, \tag{3.34}$$

holds, if $y \in H(x)$ and $\theta \in G(x, y)$. Then there exists $v \in S(x_0)$ such that $\theta \notin G(v, y)$ for any $y \in H(v) \setminus \{v\}$.

(3) *(Vectorial quasi-optimization principle in cone quasi-uniform spaces) Let Z be a vector space and $P \subset Z$ be a nonempty convex subset. Let $B : X \times X \to 2^Z \setminus \emptyset$ and $H : X \to 2^X \setminus \emptyset$ be both multivalued mappings with nonempty values. Suppose that* (3.34) *holds if $y \in H(x)$ and $B(x, y) \not\subseteq P$. Then there exists $v \in S(x_0)$ such that $B(v, y) \subset P$ for any $y \in H(v) \setminus \{v\}$.*

(4) *(Vectorial quasi-optimization principle in cone quasi-uniform spaces) Let Z be a vector space and $P \subset Z$ be a nonempty convex subset. Let $B : X \times X \to 2^Z \setminus \emptyset$ and $H : X \to 2^X \setminus \emptyset$ be both multivalued maps with nonempty values. Suppose that* (3.34) *holds if $y \in H(x)$ and $B(x, y) \cap [B(x, x) - P] \neq \emptyset$. Then there exists $v \in S(x_0)$ such that $B(v, y) \cap [B(v, v) - P] = \emptyset$ for any $y \in H(v) \setminus \{v\}$.*

(5) *(Generalized Caristi-Kirk type coincidence point in cone quasi-uniform spaces) Let I be an index set. For each $i \in I$, let $T_i : X \to 2^X$ be a multivalued mapping, let M be a nonempty subset of X, and let $g : M \to X$ be a surjective mapping. Suppose further that, if, for each $x \in M$ with $g(x) \notin \bigcap_{l \in I} T_l(x)$, there exist $l_0 \in I$ and $y \in T_{l_0}(x) \setminus \{g(x)\}$ such that*

$$d_\lambda(g(x), y) + \varphi(F(x_0, g(x)))F(g(x), y) \leq \theta, \quad \forall \lambda \in D. \tag{3.35}$$

Then there exists a coincidence point $u \in M$ *of* g *and* $\{T_i\}_{i\in I}$*, that is,* $g(u) \in \bigcap_{l\in I} T_l(u)$ *such that* $g(u) \in S(x_0)$.

(6) *(Nonconvex maximal element for a family of multivalued maps in cone quasi-uniform spaces) Let* I *be any index set and for each* $i \in I$, $T_i : X \to 2^X$ *be a multivalued mapping. Assume that, for each* $(x,i) \in S(x_0) \times I$ *with* $T_i(x) \neq \emptyset$*, there exists* $y = y(x,i) \in X$ *with* $y \neq x$ *such that*

$$d_\lambda(x,y) + \varphi(F(x_0,x))F(x,y) \leq \theta, \quad \forall \lambda \in D. \tag{3.36}$$

Then there exists $v \in S(x_0)$ *such that* $T_i(v) = \emptyset$ *for each* $i \in I$.

(7) *(Generalized Takahashi nonconvex minimization Theorem in cone quasi-uniform spaces) Suppose that, for each* $x \in S(x_0)$ *with*

$$\{u \in X : F(x,u) \ngeq \theta\} \neq \emptyset, \tag{3.37}$$

there exists $y = y(x) \in X$ *with* $y \neq x$ *such that* (3.34) *holds. Then there exists* $v \in S(x_0)$ *such that* $F(v,y) \geq \theta$ *for all* $y \in X$.

(8) *(Oettli-Théra type theorem in cone quasi-uniform spaces) Let* $M \subset X$ *and suppose that, for any* $x \in S(x_0) \setminus M$*, there exists* $y \neq x$ *such that* (3.34) *holds. Then there exists* $v \in S(x_0) \cap M$.

Proof. It follows from Theorem 3.2 that (1) holds.

(1) $\Rightarrow$ (2) By (1), there exists $v \in S(x_0)$ such that, for all $x \in X$ with $x \neq v$, there exists $\lambda \in D$ satisfying

$$\varphi(F(x_0,v))F(v,x) + d_\lambda(v,x) \notin -K. \tag{3.38}$$

If (2) does not hold for v, then there exists $y \in H(v) \setminus \{v\}$ with $\theta \in G(v,y)$ such that

$$\varphi(F(x_0,v))F(v,y) + d_\lambda(v,y) \leq \theta, \tag{3.39}$$

which contradicts (3.38). Thus $\theta \notin G(v,y)$ for any $y \in H(v) \setminus \{v\}$, that is, (2) holds.

(2) $\Rightarrow$ (3) Let $G(x,y) = B(x,y) - (Z \setminus P)$. Then $\theta \in G(x,y)$ implies that

$$B(x,y) \cap (Z \setminus P) \neq \emptyset. \tag{3.40}$$

This shows that $B(x,y) \nsubseteq P$ and so, if $y \in H(x)$ and $\theta \in G(x,y)$, then the conditions of (3) implies that (3.34) holds. It follows from (2) that there exists $v \in S(x_0)$ such that $\theta \notin G(v,y)$

for any $y \in H(v) \setminus \{v\}$. This implies that $B(v,y) \cap (Z \setminus P) = \emptyset$ for any $y \in H(v) \setminus \{v\}$. Thus $B(v,y) \subset P$ for any $y \in H(v) \setminus \{v\}$, that is, (3) holds.

(3) $\Rightarrow$ (4) Let

$$Q(x,y) = \begin{cases} -P, & \text{if } B(x,y) \cap (B(x,x) - P) \neq \emptyset, \\ P, & \text{if } B(x,y) \cap (B(x,x) - P) = \emptyset. \end{cases} \tag{3.41}$$

If $y \in H(x)$ and $Q(x,y) \not\subseteq P$, then $B(x,y) \cap (B(x,x) - P) \neq \emptyset$. By the condition of (4), we have that (3.34) holds. This shows that the condition of (3) holds for $Q(x,y)$. By the conclusion of (3), there exists $v \in S(x_0)$ such that $Q(v,y) \subset P$ for any $y \in H(v) \setminus \{v\}$. Thus $B(v,y) \cap (B(v,v) - P) = \emptyset$, that is, (4) holds.

(4) $\Rightarrow$ (1) Let

$$Z = \{z = \{z_\lambda\}_{\lambda \in D} : z_\lambda \in E, \ \forall \lambda \in D\}, \qquad P = \{\{z_\lambda\}_{\lambda \in D} \in Z : z_\lambda \in K, \ \forall \lambda \in D\}. \tag{3.42}$$

If, for any $x, y \in Z$ and any scalar α, we define

$$x + y = \{x_\lambda + y_\lambda\}_{\lambda \in D}, \qquad \alpha \cdot x = \{\alpha x_\lambda\}_{\lambda \in D}, \tag{3.43}$$

then Z is a vector space and P is a cone in Z. Let $B : X \times X \to 2^Z \setminus \emptyset$ be a multivalued mapping defined by

$$B(x,y) = \left\{\{d_\lambda(x,y) + \varphi(F(x_0,x))F(x,y)\}_{\lambda \in D}\right\} \tag{3.44}$$

and $H : X \to 2^X \setminus \emptyset$ be a multivalued mapping defined by $H(x) = X$ for any $x \in X$. If

$$B(x,y) \cap [B(x,x) - P] \neq \emptyset \tag{3.45}$$

(note that $B(x,x) = 0$), then $B(x,y) \cap (-P) \neq \emptyset$. This shows that (3.34) holds. It follows from (4) that there exists $v \in S(x_0)$ such that $B(v,y) \cap [B(v,v) - P] = \emptyset$ for any $y \in H(v) \setminus \{v\}$. Thus, for all $y \in X$ with $y \neq v$, there exists $\lambda \in D$ satisfying the following:

$$\varphi(F(x_0,v))F(v,y) + d_\lambda(v,y) \notin -K. \tag{3.46}$$

That is, (1) holds.

(1) $\Rightarrow$ (5) From (1), there exists $v \in S(x_0)$ such that, for all $x \in X$ with $x \neq v$, there exists $\lambda \in D$ satisfying the following:

$$\varphi(F(x_0,v))F(v,x) + d_\lambda(v,x) \notin -K. \tag{3.47}$$

Since g is a surjective mapping, there exists $u \in D$ such that $g(u) = v$. We claim that $g(u) \in \bigcap_{l\in I} T_l(u)$. If $g(u) \notin \bigcap_{l\in I} T_l(u)$, by the hypotheses of (V), there exist $l_0 \in I$ and $y \in T_{l_0}(u) \setminus \{g(u)\}$ such that

$$d_\lambda(g(x), y) + \varphi(F(x_0, g(x)))F(g(x), y) \le \theta, \quad \forall\lambda \in D, \tag{3.48}$$

which contradicts (3.47). Thus $g(u) \in \bigcap_{l\in I} T_l(u)$. $v \in S(x_0)$ implies that $g(u) = v \in S(x_0)$.

(5) $\Rightarrow$ (6) If the conclusion of (6) dose not hold, then, for any $x \in S(x_0)$, there exists $i \in I$ such that $T_i(x) \neq \emptyset$. By the hypotheses of (6), there exists $y = y(x,i) \in X$ with $y \neq x$ such that (3.34) holds. Let $D = X$, $g = I_d$: (the identity mapping of X) and

$$H_i(x) = \begin{cases} (T_i(x) \cup \{y(x,i)\}) \setminus \{x\}, & \text{if } x \in S(x_0) \text{ and } T_i(x) \neq \emptyset, \\ \emptyset, & \text{if } x \in S(x_0) \text{ and } T_i(x) = \emptyset, \\ \{x\}, & \text{if } x \notin S(x_0). \end{cases} \tag{3.49}$$

Then the conditions of (5) are satisfied for H_i, $D = X$ and $g = I_d$. Thus, from (5), it follows that there exists $v \in S(x_0)$ such that $v \in \bigcap_{l\in I} H_l(v)$, which contradicts the definition of H_i. Therefore, there exists $v \in S(x_0)$ such that $T_i(v) = \emptyset$ for any $i \in I$.

(6) $\Rightarrow$ (7) Let

$$T_y(x) = \begin{cases} \{y\} \cap \{u \in X : F(x,u) \ngeq \theta\}, & \text{if } x \in S(x_0), \\ \emptyset, & \text{if } x \notin S(x_0). \end{cases} \tag{3.50}$$

From this, we know that, if $T_y(x) \neq \emptyset$, then $F(x,y) \ngeq \theta$. By the hypotheses of (7), there exists $z = z(x) \in X$ with $z \neq x$ such that

$$d_\lambda(x,z) + \varphi(F(x_0,x))F(x,z) \le \theta, \quad \forall\lambda \in D. \tag{3.51}$$

By using (6), there exists $v \in S(x_0)$ such that $T_y(v) = \emptyset$ for any $y \in X$, that is, $F(v,y) \ge \theta$ for any $y \in X$.

(7) $\Rightarrow$ (8) Suppose that (7) and the hypothesis of (8) hold. Suppose that, for any $x \in S(x_0)$, $x \notin M$. By the hypothesis of (8), there exists $v \neq x$ such that

$$d_\lambda(x,v) + \varphi(F(x_0,x))F(x,v) \le \theta, \quad \forall\lambda \in D. \tag{3.52}$$

Equation (3.52) shows that the conditions of (7) naturally hold. Then there exists $v \in S(x_0)$ such that $F(v,y) \ge \theta$ for any $y \in X$, which contradicts (3.52). Therefore, there exists $v \in S(x_0) \cap M$.

(8) $\Rightarrow$ (1) Let

$$\begin{gathered} \Gamma(u) = \{x \in X : u \neq x, d_\lambda(u,x) + \varphi(F(x_0,u))F(u,x) \le \theta, \ \forall\lambda \in D\}, \\ M = \{u \in X : \Gamma(u) = \emptyset\}. \end{gathered} \tag{3.53}$$

If $u \notin M$, then $\Gamma(u) \neq \emptyset$, that is, there exists $x \in \Gamma(u)$ such that

$$d_\lambda(u,x) + \varphi(F(x_0,u))F(u,x) \leq \theta, \quad \forall\lambda \in D. \tag{3.54}$$

This shows that the conditions of (8) hold and so there exists $v \in S(x_0) \cap M$.$v \in M$ shows that, for all $x \in X$ with $x \neq v$, there exists $\lambda \in D$ satisfying the following:

$$\varphi(F(x_0,v))F(v,x) + d_\lambda(v,x) \notin K. \tag{3.55}$$

This completes the proof. □

In the following, we provide some examples to illustrate our results.

Example 3.7. Let $E = L^p[0,1]$, $K = \{x \in L^p[0,1] : x(t) \geq 0, \text{ a.e. } t \in [0,1]\}$, where $p \geq 1$. Then K is a nuclear and regular cone if $p = 1$ and a regular cone, but not a nuclear cone in $L^p[0,1]$ if $p > 1$. However, for any $p \geq 1$, K is not a solid cone (see, e.g., [32]). Let $X = L^p[0,1]$ and define a mapping $d : X \times X \to L^p[0,1]$ by $d(x,y) = |x-y|$. Then d is a cone quasi-metric on X. Since $L^p[0,1]$ is complete, (X,d) is also a complete quasi-metric space. Define a mapping $F : X \times X \to L^p[0,1]$ by $F(x,y) = \sin y - \sin x$ for any $x, y \in X$. Then, for each $x \in X$, $F(x,\cdot) : X \to E$ is bound from below and

$$\{y \in X : F(x,y) + |x-y| \leq 0\} \tag{3.56}$$

is a closed subset in X. It follows from Theorem 3.2 and Remark 3.4 that, for any $x_0 \in X$, there exists $v \in X$ such that

(1) $\sin v(t) - \sin x_0(t) + |x_0(t) - v(t)| \leq 0$ for a.e. $t \in [0,1]$;

(2) for all $x \in X$ with $x \neq v$, there exists a measuable set $A \subset [0,1]$ with a positive measure such that

$$\sin x(t) - \sin v(t) + |x(t) - v(t)| > 0, \quad \text{a.e. } t \in A. \tag{3.57}$$

In fact, $v = x_0$ meets the demands.

Example 3.8. Let $E = l_1$ endowed with the topology defined by the semi-norms $\{p_n\}_{n\in N}$, where $p_n((x_k)_{k\in N}) = \sum_{k=1}^n |x_k|$. The cone

$$K = \{x = (x_k)_{k\in N} \in l_1 : x_k \geq 0, \ \forall k \in N\} \tag{3.58}$$

is a nuclear cone, but not well based (see [33]). Let $X = [0,1]$, define a mapping $d : X \times X \to l_1$ by

$$d(x,y) = \begin{cases} \left(\dfrac{\sqrt[k]{|x-y|}}{k^2}\right)_{k\in N}, & \text{if } x < y, \\ \left(\dfrac{1}{k^2}\right)_{k\in N}, & \text{if } x > y, \end{cases} \tag{3.59}$$

and a mapping $F : X \times X \to l_1$ by

$$F(x,y) = \left(\frac{1}{k}\sin\frac{1}{y+k} - \frac{1}{k}\sin\frac{1}{x+k}\right)_{k\in N}. \tag{3.60}$$

Then (X,d) is a complete cone quasi-metric space, for each $x \in X$, $F(x,\cdot) : X \to E$ is bound from below and

$$\{y \in X : F(x,y) + d(x,y) \leq 0\} \tag{3.61}$$

is a closed subset in X. It follows from Theorem 3.2 and Remark 3.4 that, for any $x_0 \in X$, there exists $v \in X$ such that

$$\left(\frac{1}{k}\sin\frac{1}{v+k} - \frac{1}{k}\sin\frac{1}{x_0+k}\right)_{k\in N} + d(x_0,v) \in -K \tag{3.62}$$

and, for all $x \in X$ with $x \neq v$,

$$\left(\frac{1}{k}\sin\frac{1}{v+k} - \frac{1}{k}\sin\frac{1}{x+k}\right)_{k\in N} + d(v,x) \notin -K. \tag{3.63}$$

Remark 3.9. In above examples, the cone quasi-metric is not directed metric, the cone is not solid or not well based, thus, Example 3.7 and Example 3.8 show that our results are different from the results in [2–5, 8–10, 14–17].

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