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Research Article

Implicit Mann Type Iteration Method Involving Strictly Hemicontractive Mappings in Banach Spaces

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We proved that the modified implicit Mann iteration process can be applied to approximate the fixed point of strictly hemicontractive mappings in smooth Banach spaces.

1. Introduction

Let K be a nonempty subset of an arbitrary Banach space X and let X^* be its dual space. The symbols $D(T)$ and $F(T)$ stand for the domain and the set of fixed points of T (for a single-valued mapping $T : X \to X$, $x \in X$ is called a *fixed point* of T iff $Tx = x$). We denote by J the *normalized duality mapping* from X to 2^{X^*} defined by

$$J(x) = \left\{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \left\| f^* \right\|^2 \right\}, \quad x \in X, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. In a smooth Banach space, J is singlevalued (we denoted by j).

Remark 1.1. (1) X is called uniformly smooth if X^* is uniformly convex.

(2) In a uniformly smooth Banach space, J is uniformly continuous on bounded subsets of X.

Let $T : D(T) \subset X \to X$ be a mapping.

Definition 1.2. The mapping T is called *Lipshitz* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \tag{1.2}$$

for all $x, y \in D(T)$. If $L = 1$, then T is called *nonexpansive* and if $0 \leq L < 1$, then T is called *contractive*.

Definition 1.3 (see [1, 2]). (1) The mapping T is said to be *pseudocontractive* if

$$\|x - y\| \leq \|x - y + r((I - T)x - (I - T)y)\|, \tag{1.3}$$

for all $x, y \in D(T)$ and $r > 0$.

(2) The mapping T is said to be *strongly pseudocontractive* if there exists a constant $t > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|, \tag{1.4}$$

for all $x, y \in D(T)$ and $r > 0$.

(3) The mapping T is said to be *local strongly pseudocontractive* if for each $x \in D(T)$ there exists a constant $t > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|, \tag{1.5}$$

for all $y \in D(T)$ and $r > 0$.

(4) The mapping T is said to be *strictly hemicontractive* if $F(T) \neq \emptyset$ and if there exists a constant $t > 1$ such that

$$\|x - q\| \leq \|(1 + r)(x - q) - rt(Tx - q)\|, \tag{1.6}$$

for all $x \in D(T)$, $q \in F(T)$ and $r > 0$.

Clearly, each strongly pseudocontractive mapping is local strongly pseudocontractive.

Chidume [1] established that the Mann iteration sequence converges strongly to the unique fixed point of T in case T is a Lipschitz strongly pseudocontractive mapping from a bounded closed convex subset of L_p (or l_p) into itself. Schu [3] generalized the result in [1] to both uniformly continuous strongly pseudocontractive mappings and real smooth Banach spaces. Park [4] extended the result in [1] to both strongly pseudocontractive mappings and certain smooth Banach spaces. Rhoades [5] proved that the Mann and Ishikawa iteration methods may exhibit different behaviors for different classes of nonlinear mappings. Afterwards, several generalizations have been made in various directions (see, e.g., [6–13]).

In 2001, Xu and Ori [14] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \ldots, N\}$) with $\{\alpha_n\}$ a real sequence in $(0, 1)$ and an initial point $x_0 \in K$:

$$\begin{aligned} x_1 &= (1-\alpha_1)x_0 + \alpha_1 T_1 x_1, \\ x_2 &= (1-\alpha_2)x_1 + \alpha_2 T_2 x_2, \\ &\vdots \\ x_N &= (1-\alpha_N)x_{N-1} + \alpha_N T_N x_N, \\ x_{N+1} &= (1-\alpha_{N+1})x_N + \alpha_{N+1} T_{N+1} x_{N+1}, \\ &\vdots \end{aligned} \tag{1.7}$$

which can be written in the following compact form:

$$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_n x_n, \quad n \geq 1, \tag{1.8}$$

where $T_n = T_{n(\bmod N)}$ (here the mod N function takes values in I). Xu and Ori [14] proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [11], Osilike proved the following results.

Theorem 1.4. *Let X be a real Banach space and let K be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be N strictly pseudocontractive mappings from K to K with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence satisfying the following conditions:*

(i) $0 < \alpha_n < 1$,

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$,

(iii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (1.8). *Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if* $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Remark 1.5. One can easily see that for $\alpha_n = 1 - 1/n^{1/2}$, $\sum_{n=1}^{\infty}(1-\alpha_n)^2 = \infty$. Hence the results of Osilike [11] are needed to be improved.

Let K be a nonempty closed bounded convex subset of an arbitrary smooth Banach space X and let $T : K \to K$ be a continuous strictly hemicontractive mapping. We proved that the implicit Mann type iteration method converges strongly to a unique fixed point of T.

The results presented in this paper extend and improve the corresponding results particularly in [1, 3, 4, 7, 8, 10, 11, 13, 15].

2. Preliminaries

We need the following results.

Lemma 2.1 (see [4]). *Let X be a smooth Banach space. Suppose that one of the following holds:*

(a) *J is uniformly continuous on any bounded subsets of X,*

(b) $\langle x-y, j(x)-j(y)\rangle \leq \|x-y\|^2$ *for all x, y in X,*

(c) *for any bounded subset D of X, there is a function $c : [0,\infty) \to [0,\infty)$ such that*

$$\operatorname{Re}\langle x-y, j(x)-j(y)\rangle \leq c(\|x-y\|), \tag{2.1}$$

for all $x, y \in D$, where c satisfies $\lim_{t\to 0^+}(c(t)/t) = 0$.

Then for any $\epsilon > 0$ and any bounded subset K, there exists $\delta > 0$ such that

$$\|sx+(1-s)y\|^2 \leq (1-2s)\|y\|^2 + 2s\operatorname{Re}\langle x, j(y)\rangle + 2s\epsilon, \tag{2.2}$$

for all $x, y \in K$ and $s \in [0,\delta]$.

Remark 2.2. (1) If X is uniformly smooth, then (a) in Lemma 2.1 holds.

(2) If X is a Hilbert space, then (b) in Lemma 2.1 holds.

Lemma 2.3 (see [8]). *Let $T : D(T) \subset X \to X$ be a mapping with $F(T) \neq \emptyset$. Then T is strictly hemicontractive if and only if there exists a constant $t > 1$ such that for all $x \in D(T)$ and $q \in F(T)$, there exists $j(x-q) \in J(x-q)$ satisfying*

$$\operatorname{Re}\ \langle x-Tx, j(x-q)\rangle \geq \left(1-\frac{1}{t}\right)\|x-q\|^2. \tag{2.3}$$

Lemma 2.4 (see [10]). *Let X be an arbitrary normed linear space and let $T : D(T) \subset X \to X$ be a mapping.*

(a) *If T is a local strongly pseudocontractive mapping and $F(T) \neq \emptyset$, then $F(T)$ is a singleton and T is strictly hemicontractive.*

(b) *If T is strictly hemicontractive, then $F(T)$ is a singleton.*

Lemma 2.5 (see [10]). *Let $\{\theta_n\}$,$\{\sigma_n\}$, and $\{\omega_n\}$ be nonnegative real sequences and let $\epsilon' > 0$ be a constant satisfying*

$$\sigma_{n+1} \leq (1-\theta_n)\sigma_n + \epsilon'\theta_n + \omega_n, \quad n \geq 1, \tag{2.4}$$

where $\sum_{n=1}^{\infty}\theta_n = \infty$, $\theta_n \leq 1$ for all $n \geq 1$ and $\sum_{n=1}^{\infty}\omega_n < \infty$. Then $\limsup_{n\to\infty}\sigma_n \leq \epsilon'$.

3. Main Results

We now prove our main results.

Lemma 3.1. *Let X be a smooth Banach space. Suppose that one of the following holds:*

(a) *J is uniformly continuous on any bounded subsets of X,*

(b) $\langle x-y, j(x)-j(y)\rangle \leq \|x-y\|^2$ *for all x,y in X,*

(c) *for any bounded subset D of X, there is a function* $c:[0,\infty)\rightarrow[0,\infty)$ *such that*

$$\operatorname{Re}\langle x-y, j(x)-j(y)\rangle \leq c(\|x-y\|) \tag{3.1}$$

for all $x,y\in D$, *where c satisfies* $\lim_{t\to 0^+} c((t)/t)=0$.

Then for any $\epsilon>0$ *and any bounded subset K, there exists* $\delta>0$ *such that*

$$\begin{aligned}\|\alpha x+\beta y+\gamma z\|^2 \leq (1-2\alpha)\|x\|^2 &+ 2\frac{\alpha\beta}{1-\alpha}\operatorname{Re}\langle y, j(x)\rangle\\ &+2\frac{\alpha\gamma}{1-\alpha}\operatorname{Re}\langle z, j(x)\rangle+2\epsilon\alpha\end{aligned} \tag{3.2}$$

for all $x,y,z\in K$ *and* $\alpha,\beta,\gamma\in[0,\delta];\alpha+\beta+\gamma=1$.

Proof. For $\alpha,\beta,\gamma\in[0,\delta];\alpha+\beta+\gamma=1$, by using (2.2), consider

$$\begin{aligned}\|\alpha x+\beta y+\gamma z\|^2 &= \left\|\alpha x+(1-\alpha)\left(\frac{\beta}{1-\alpha}y+\frac{\gamma}{1-\alpha}z\right)\right\|^2\\ &\leq (1-2\alpha)\|x\|^2+2\epsilon\alpha+2\alpha\operatorname{Re}\left\langle\frac{\beta}{1-\alpha}y+\frac{\gamma}{1-\alpha}z, j(x)\right\rangle\\ &= (1-2\alpha)\|x\|^2+2\epsilon\alpha+2\frac{\alpha\beta}{1-\alpha}\operatorname{Re}\langle y, j(x)\rangle+2\frac{\alpha\gamma}{1-\alpha}\operatorname{Re}\langle z, j(x)\rangle.\end{aligned} \tag{3.3}$$

This completes the proof. □

Theorem 3.2. *Let X be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let K be a nonempty closed bounded convex subset of X and let* $T:K\rightarrow K$ *be a continuous strictly hemicontractive mapping. Let* $\{\alpha_n\}$,$\{\beta_n\}$ *and* $\{\gamma_n\}$ *be real sequences in* $[0,1]$ *satisfying conditions*

(iv) $\alpha_n+\beta_n+\gamma_n=1$, *for all* $n\geq 1$,

(v) $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(vi) $\sum_{n=1}^{\infty}\beta_n<\infty$ *and* $\sum_{n=1}^{\infty}\gamma_n<\infty$.

For a sequence $\{v_n\}$ *in K, suppose that* $\{x_n\}$ *is the sequence generated from an arbitrary* $x_0\in K$ *by*

$$x_n=\alpha_n x_{n-1}+\beta_n T x_n+\gamma_n T v_n, \quad n\geq 1, \tag{3.4}$$

satisfying $\sum_{n=1}^{\infty}\|v_n-x_n\|<\infty$.

Then the sequence $\{x_n\}$ *converges strongly to a unique fixed point q of T.*

Proof. By [2, Corollary 1], T has a unique fixed point q in K. It follows from Lemma 2.4 that $F(T)$ is a singleton. That is, $F(T)=\{q\}$ for some $q\in K$.

Set $M = 1 + \operatorname{diam} K$. It is easy to verify that

$$M = \sup_{n\geq 1}\|x_n - q\| + \sup_{n\geq 1}\|Tx_n - q\| + \sup_{n\geq 1}\|Tv_n - q\|. \tag{3.5}$$

Also

$$\begin{aligned}\|v_n - q\|^2 &\leq \|v_n - x_n\|^2 + \|x_n - q\|^2 + 2\|v_n - x_n\|\|x_n - q\| \\ &\leq \|v_n - x_n\|^2 + \|x_n - q\|^2 + 2M\|v_n - x_n\|.\end{aligned} \tag{3.6}$$

Consider

$$\begin{aligned}\|x_n - q\|^2 &= \|\alpha_n x_{n-1} + \beta_n Tx_n + \gamma_n Tv_n - q\|^2 \\ &= \|\alpha_n(x_{n-1} - q) + \beta_n(Tx_n - q) + \gamma_n(Tv_n - q)\|^2 \\ &\leq \alpha_n\|x_{n-1} - q\|^2 \\ &\quad + \beta_n\|Tx_n - q\|^2 + \gamma_n\|Tv_n - q\|^2 \\ &\leq \|x_{n-1} - q\|^2 + M^2(\beta_n + \gamma_n),\end{aligned} \tag{3.7}$$

where the first inequality holds by the convexity of $\|\cdot\|^2$.

Now we put $k = 1/t$, where t satisfies (2.3). Using (3.4) and Lemma 3.1, we infer that

$$\begin{aligned}\|x_n - q\|^2 &= \|\alpha_n x_{n-1} + \beta_n Tx_n + \gamma_n Tv_n - q\|^2 \\ &= \|\alpha_n(x_{n-1} - q) + \beta_n(Tx_n - q) + \gamma_n(Tv_n - q)\|^2 \\ &\leq (1 - 2\alpha_n)\|x_{n-1} - q\|^2 + 2\frac{\alpha_n\beta_n}{1-\alpha_n}\operatorname{Re}\langle Tx_n - q, j(x_{n-1} - q)\rangle \\ &\quad + 2\frac{\alpha_n\gamma_n}{1-\alpha_n}\operatorname{Re}\langle Tv_n - q, j(x_{n-1} - q)\rangle + 2\epsilon\alpha_n \\ &= (1 - 2\alpha_n)\|x_{n-1} - q\|^2 + 2\frac{\alpha_n\beta_n}{1-\alpha_n}\operatorname{Re}\langle Tx_n - q, j(x_n - q)\rangle \\ &\quad + 2\frac{\alpha_n\beta_n}{1-\alpha_n}\operatorname{Re}\langle Tx_n - q, j(x_{n-1} - q) - j(x_n - q)\rangle \\ &\quad + 2\frac{\alpha_n\gamma_n}{1-\alpha_n}\operatorname{Re}\langle Tv_n - q, j(v_n - q)\rangle \\ &\quad + 2\frac{\alpha_n\gamma_n}{1-\alpha_n}\operatorname{Re}\langle Tv_n - q, j(x_{n-1} - q) - j(v_n - q)\rangle + 2\epsilon\alpha_n \\ &\leq (1 - 2\alpha_n)\|x_{n-1} - q\|^2 + 2\frac{\alpha_n\beta_n}{1-\alpha_n}k\|x_n - q\|^2 \\ &\quad + 2\frac{\alpha_n\beta_n}{1-\alpha_n}\|Tx_n - q\|\|j(x_{n-1} - q) - j(x_n - q)\| \\ &\quad + 2\frac{\alpha_n\gamma_n}{1-\alpha_n}k\|v_n - q\|^2 \\ &\quad + 2\frac{\alpha_n\gamma_n}{1-\alpha_n}\|Tv_n - q\|\|j(x_{n-1} - q) - j(v_n - q)\| + 2\epsilon\alpha_n\end{aligned}$$

$$\begin{aligned}
&\leq (1-2\alpha_n)\left\|x_{n-1}-q\right\|^2 + 2\frac{\alpha_n\beta_n}{1-\alpha_n}k\left\|x_n-q\right\|^2 + 2M\frac{\alpha_n\beta_n}{1-\alpha_n}\delta_n \\
&\quad + 2\frac{\alpha_n\gamma_n}{1-\alpha_n}k\left\|v_n-q\right\|^2 + 2M\frac{\alpha_n\gamma_n}{1-\alpha_n}\eta_n + 2\epsilon\alpha_n \\
&\leq (1-2\alpha_n)\left\|x_{n-1}-q\right\|^2 + 2\frac{\alpha_n\beta_n}{1-\alpha_n}k\left\|x_n-q\right\|^2 \\
&\quad + 2\frac{\alpha_n\gamma_n}{1-\alpha_n}k\left\|v_n-q\right\|^2 + 2M\alpha_n \max\{\delta_n, \eta_n\} + 2\epsilon\alpha_n,
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\delta_n &= \left\|j(x_{n-1}-q) - j(x_n-q)\right\|, \\
\eta_n &= \left\|j(x_{n-1}-q) - j(v_n-q)\right\|.
\end{aligned} \tag{3.9}$$

Also, we have

$$\begin{aligned}
\|x_{n-1}-x_n\| &= \left\|x_{n-1} - \alpha_n x_{n-1} - \beta_n T x_n - \gamma_n T v_n\right\| \\
&= \left\|\beta_n(x_{n-1}-Tx_n) + \gamma_n(x_{n-1}-Tv_n)\right\| \\
&\leq \beta_n\|x_{n-1}-Tx_n\| + \gamma_n\|x_{n-1}-Tv_n\| \\
&\leq 2M(\beta_n+\gamma_n) \\
&< \infty
\end{aligned} \tag{3.10}$$

implies

$$\|x_{n-1}-x_n\| \longrightarrow 0, \tag{3.11}$$

as $n \to \infty$, and consequently

$$\|x_{n-1}-v_n\| \leq \|x_{n-1}-x_n\| + \|x_n-v_n\| \longrightarrow 0 \tag{3.12}$$

as $n \to \infty$. Since J is uniformly continuous on any bounded subsets of X, we have

$$\delta_n,\ \eta_n \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.13}$$

For any given $\epsilon > 0$ and the bounded subset K, there exists a $\delta > 0$ satisfying (2.2). Note that (3.13) and (vi) ensure that there exists an N such that

$$\beta_n, \gamma_n < \min\left\{\delta, \frac{\epsilon}{8M^2k}\right\}, \quad \delta_n, \eta_n \leq \frac{\epsilon}{4M}, \qquad n \geq N. \tag{3.14}$$

Now substituting (3.6) in (3.8) to obtain

$$\begin{aligned}\|x_n - q\|^2 \le{}& (1-2\alpha_n)\|x_{n-1}-q\|^2 + 2k\alpha_n\|x_n-q\|^2 \\ &+ 2M\alpha_n \max\{\delta_n, \eta_n\} + 2\epsilon\alpha_n \\ &+ 2\frac{\alpha_n\gamma_n}{1-\alpha_n}k\left(\|v_n-x_n\|^2 + 2M\|v_n-x_n\|\right),\end{aligned} \tag{3.15}$$

by using (3.7), implies

$$\begin{aligned}\|x_n - q\|^2 \le{}& (1-2(1-k)\alpha_n)\|x_{n-1}-q\|^2 + 2\epsilon\alpha_n \\ &+ 2M^2k\alpha_n(\beta_n+\gamma_n) + 2M\alpha_n\max\{\delta_n,\eta_n\} \\ &+ 2\frac{\alpha_n\gamma_n}{1-\alpha_n}k\left(\|v_n-x_n\|^2 + 2M\|v_n-x_n\|\right) \\ \le{}& (1-2(1-k)\alpha_n)\|x_{n-1}-q\|^2 + 3\epsilon\alpha_n \\ &+ 2\frac{\alpha_n\gamma_n}{1-\alpha_n}k\left(\|v_n-x_n\|^2 + 2M\|v_n-x_n\|\right)\end{aligned} \tag{3.16}$$

for all $n \ge N$.

Put

$$\begin{gathered}\sigma_n = \|x_{n-1}-q\|^2, \quad \theta_n = 2(1-k)\alpha_n, \quad \epsilon' = \frac{3\epsilon}{2(1-k)}, \\ \omega_n = 2\frac{\alpha_n\gamma_n}{1-\alpha_n}k\left(\|v_n-x_n\|^2 + 2M\|v_n-x_n\|\right),\end{gathered} \tag{3.17}$$

and we have from (3.16)

$$\sigma_{n+1} \le (1-\theta_n)\sigma_n + \epsilon'\theta_n + \omega_n, \quad n \ge 1. \tag{3.18}$$

For $k < 1/2$, set $\delta = 1/2(1-k) < 1$. Because $\alpha_n \le \delta$, we imply $1-\alpha_n \ge 1-\delta$ and $2(1-k)\alpha_n \le 1$. Now observe that $\sum_{n=1}^{\infty}\theta_n = \infty, \theta_n \le 1$ for all $n \ge 1$ and $\sum_{n=1}^{\infty}\omega_n < \infty$. It follows from Lemma 2.5 that

$$\limsup_{n\to\infty}\|x_n - q\|^2 \le \epsilon'. \tag{3.19}$$

Letting $\epsilon' \to 0^+$, we obtain that $\limsup_{n\to\infty}\|x_n-q\|^2 = 0$, which implies that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 3.3. *Let X be a smooth Banach space satisfying any one of the Axioms* (*a*)–(*c*) *of Lemma 3.1. Let K be a nonempty closed bounded convex subset of X and let $T: K \to K$ be a Lipschitz strictly hemicontractive mapping. Let $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in* $[0,1]$ *satisfying the conditions (iv)–(vi).*

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (3.4)*. Then the sequence $\{x_n\}$ converges strongly to a unique fixed point q of T.*

Corollary 3.4. *Let X be a smooth Banach space satisfying any one of the Axioms* (a)–(c) *of Lemma 3.1. Let K be a nonempty closed bounded convex subset of X and let $T : K \to K$ be a continuous strictly hemicontractive mapping. Suppose that $\{\alpha_n\}$ be a real sequence in* $[0,1]$ *satisfying the conditions (v) and* $\lim_{n\to\infty}\alpha_n = 0$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (1.8). *Then the sequence $\{x_n\}$ converges strongly to a unique fixed point q of T.*

Corollary 3.5. *Let X be a smooth Banach space satisfying any one of the Axioms* (a)–(c) *of Lemma 3.1. Let K be a nonempty closed bounded convex subset of X and let $T : K \to K$ be a Lipschitz strictly hemicontractive mapping. Suppose that $\{\alpha_n\}$ be a real sequence in* $[0,1]$ *satisfying the conditions (v) and* $lim_{n\to\infty}\alpha_n = 0$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (1.8). *Then the sequence $\{x_n\}$ converges strongly to a unique fixed point q of T.*

Remark 3.6. Similar results can be found for the iteration processes involved error terms; we omit the details.

Remark 3.7. Theorem 3.2 and Corollary 3.3 extend and improve Theorem 1.4 in the following directions.

We do not need the assumption $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$ as in Theorem 1.4.

4. Applications for Multistep Implicit Iterations

Let K be a nonempty closed convex subset of a smooth Banach space X and let $T, T_1, T_2, \ldots, T_p : K \to K (p \geq 2)$ be a family of $p+1$ mappings.

Algorithm 4.1. For a given $\mathrm{x}_0 \in \mathrm{K}$, compute the sequence $\{\mathrm{x}_n\}$ by the implicit iteration process of arbitrary fixed order $\mathrm{p} \geq 2$:

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + \beta_n T x_n + \gamma_n T_1 y_n^1, \\ y_n^i &= \beta_n^i x_{n-1} + \left(1 - \beta_n^i\right) T_{i+1} y_n^{i+1}, \quad i = 1, 2, \ldots, p-2, \\ y_n^{p-1} &= \beta_n^{p-1} x_{n-1} + \left(1 - \beta_n^{p-1}\right) T_p x_n, \quad n \geq 1, \end{aligned} \tag{4.1}$$

which is called the *multistep implicit iteration process,* where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\beta_n^i\}, i = 1, 2, \ldots, p-1$ are real sequences in $[0,1]$ and $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$.

For $p = 3$, we obtain the following three-step implicit iteration process.

Algorithm 4.2. For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + \beta_n T x_n + \gamma_n T_1 y_n^1, \\ y_n^1 &= \beta_n^1 x_{n-1} + \left(1 - \beta_n^1\right) T_2 y_n^2, \\ y_n^2 &= \beta_n^2 x_{n-1} + \left(1 - \beta_n^2\right) T_3 x_n, \quad n \geq 1, \end{aligned} \tag{4.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\beta_n^1\}$ and $\{\beta_n^2\}$ are real sequences in $[0,1]$ satisfying some certain conditions.

For $p = 2$, we obtain the following two-step implicit iteration process.

Algorithm 4.3. For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + \beta_n T x_n + \gamma_n T_1 y_n^1, \\ y_n^1 &= \beta_n^1 x_{n-1} + \left(1-\beta_n^1\right) T_2 x_n, \quad n \geq 1, \end{aligned} \tag{4.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\beta_n^1\}$ are real sequences in $[0,1]$ satisfying some certain conditions.

If $T_1 = T$, $T_2 = I$ and $\beta_n^1 = 0$ in (4.3), we obtain the following implicit Mann iteration process.

Algorithm 4.4. For any given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T x_n, \quad n \geq 1, \tag{4.4}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying some certain conditions.

Theorem 4.5. *Let X be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let K be a nonempty closed bounded convex subset of X and let $T, T_1, T_2, \ldots, T_p : K \to K (p \geq 2)$ be $p+1$ mappings. Let T, T_1 be continuous strictly hemicontractive mappings. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\beta_n^i\}$, $i = 1, 2, \ldots, p-1$ be real sequences in $[0,1]$ satisfying the conditions (iv)–(vi) and $\sum_{n=1}^{\infty}(1-\beta_n^1) < \infty$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (4.1). Then $\{x_n\}$ converges strongly to the common fixed point of $\bigcap_{i=1}^{p} F(T_i) \cap F(T) \neq \emptyset$.*

Proof. By applying Theorem 3.2 under assumption that T and T_1 are continuous strictly hemicontractive mappings, we obtain Theorem 4.5 which proves strong convergence of the iteration process defined by (4.1). Consider by taking $T_1 = T$ and $v_n = y_n^1$,

$$\begin{aligned} \|v_n - x_n\| &= \left\|y_n^1 - x_n\right\| \\ &= \left\|\beta_n^1 x_{n-1} + \left(1-\beta_n^1\right) T_2 y_n^2 - x_n\right\| \\ &= \left\|\beta_n^1 (x_{n-1} - x_n) + \left(1-\beta_n^1\right)\left(T_2 y_n^2 - x_n\right)\right\| \\ &\leq \beta_n^1 \|x_{n-1} - x_n\| + \left(1-\beta_n^1\right)\left\|T_2 y_n^2 - x_n\right\| \\ &\leq \beta_n^1 \|x_{n-1} - x_n\| + M'\left(1-\beta_n^1\right). \end{aligned} \tag{4.5}$$

From (4.5) and the condition $\sum_{n=1}^{\infty}(1-\beta_n^1) < \infty$, we obtain

$$\sum_{n=1}^{\infty} \|v_n - x_n\| < \infty. \tag{4.6}$$

This completes the proof. □

Corollary 4.6. *Let X be a smooth Banach space satisfying any one of the Axioms (a)–(c) of Lemma 3.1. Let K be a nonempty closed bounded convex subset of X and let $T, T_1, T_2, \ldots, T_p : K \to K(p \geq 2)$ be $p+1$ mappings. Let T, T_1 be Lipschitz strictly hemicontractive mappings. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\beta_n^i\}$, $i = 1, 2, \ldots, p-1$ be real sequences in $[0,1]$ satisfying the conditions (iv)-(vi) and $\sum_{n=1}^{\infty}(1-\beta_n^1) < \infty$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (4.1). Then $\{x_n\}$ converges strongly to the common fixed point of $\bigcap_{i=1}^{p} F(T_i) \cap F(T) \neq \emptyset$.*

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