Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 292740, 16 pages
doi:10.1155/2012/292740

Research Article

A Generalization of Itô's Formula and the Stability of Stochastic Volterra Integral Equations

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Received 17 May 2012; Accepted 12 July 2012

Academic Editor: Yansheng Liu

It is well known that Itô's formula is an essential tool in stochastic analysis. But it cannot be used for general stochastic Volterra integral equations (SVIEs). In this paper, we first introduce the concept of quasi-Itô process which is a generalization of well-known Itô process. And then we extend Itô's formula to a more general form applicable to some kinds of SVIEs. Furthermore, the stability in probability for some SVIEs is analyzed by the generalized Itô's formula. Our work shows that the generalized Itô's formula is powerful and flexible to use in many relevant fields.

1. Introduction

Nowadays, more and more people have realized that stochastic differential equation (SDE) is an important subject which provides more realistic models in many areas of science and applications, such as in biomathematics, filtering problems, physics, stochastic control, and mathematical finance. It is known that Itô SDEs of the form

$$dX(t) = f(X(t), t)dt + g(X(t), t)dW(t), \tag{1.1}$$

have been used and applied broadly, and their fundamental theories have been well developed [1–3].

In [1–4] and many other references, we see that Itô's formula plays a key role in the study of stochastic analysis. It is applied in the studying of stochastic control, stochastic neural network, backward SDEs, and numerical solutions of SDEs. Itô's formula can be seen as a stochastic version of chain rule in calculus. It is very useful in evaluating Itô integral, in investigating the existence and uniqueness, the stability and the oscillation of solutions to

SDEs, and so does in many other aspects of stochastic calculus [5–15]. Hence, we can imagine that if there was no Itô's formula, many known results might be very difficult to get.

Here, and throughout this paper, what we mentioned is all in a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which an m-dimensional Brownian motion $W(\cdot)$ is defined with $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ being its natural filtration augmented by the $\mathbb{P}$-null sets in $\mathcal{F}$. The mathematical expectation with respect to the given probability measure $\mathbb{P}$ is denoted by $\mathbb{E}(\cdot)$. For convenience, we state Itô's formula in [1] as follows.

Definition 1.1. A d-dimensional Itô process is an $\mathbb{R}^d$-valued continuous adapted process $x(t) = (x_1(t), \ldots, x_d(t))^T$ on $t \geq 0$ of the form

$$x(t) = x(0) + \int_0^t f(s)ds + \int_0^t g(s)dW(s), \tag{1.2}$$

where $f = (f_1, \ldots, f_d)^T \in \mathcal{L}^1(\mathbb{R}^+; \mathbb{R}^d)$ and $g = (g_{ij})_{d\times m} \in \mathcal{L}^2(\mathbb{R}^+; \mathbb{R}^{d\times m})$. We will say that $x(t)$ has a stochastic differential $dx(t)$ on $t \geq 0$ given by

$$dx(t) = f(t)dt + g(t)dW(t). \tag{1.3}$$

Two natural questions are whether the compound function $V(x(t), t)$ is an Itô process, if $x(t)$ is an Itô process and $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R})$, And if it is, what is its stochastic differential? This leads to the following very famous Itô's formula.

Theorem 1.2. *Let $x(t)$ be a d-dimensional Itô process on $t \geq 0$ with the stochastic differential* (1.3)*, and $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R})$. Then $V(x(t), t)$ is an Itô process with the stochastic differential given by*

$$\begin{aligned} dV(x(t), t) = {} & \left[V_t(x(t), t) + V_x(x(t), t)f(t) + \frac{1}{2}\operatorname{trace}\left(g^T(t)V_{xx}(x(t), t)g(t)\right)\right]dt \\ & + V_x(x(t), t)g(t)dW(t) \ \text{a.s.} \end{aligned} \tag{1.4}$$

From Theorem 1.2, it is easy to see that the differential form of the Itô process is more convenient to apply than its integral form. To describe the realistic world better, it is natural to extend SDE (1.1) to a more general case as the following stochastic Volterra integral equation (SVIE):

$$x(t) = \varphi(t, \omega) + \int_{t_0}^t f(x(s), t, s, \omega)ds + \int_{t_0}^t g(x(s), t, s, \omega)dW(s), \tag{1.5}$$

where $\varphi(t, \omega)$ is a continuous stochastic process. It is easy to see that SDE (1.1) is a special case of SVIE (1.5). Many scholars have given some results for SVIE (1.5) (see [16, 17]). However, it is noted that the solutions decided by (1.5) are not Itô processes; hence they do not satisfy the conditions in Theorem 1.2. So the Itô's formula cannot be used for these SVIEs. It is one of the reasons that many basic theories of SVIEs have not been accomplished.

Motivated by the previous discussions, in this paper, we extend Itô's formula to a more general form applicable to SVIEs, by employing the technique in stochastic analysis.

Based on the generalized Itô's formula and Lyapunov method, the stochastic stability to some kinds of SVIEs is investigated. Consequently, some sufficient conditions, which ensure the global stochastic asymptotic stability of the trivial solution, are established. By constructing an appropriate Lyapunov function, a condition ensuring global stochastic asymptotic stability of a linear SVIE is given. Our work shows that the generalized Itô's formula is powerful and flexible to use. Obviously, it can also be used in many other relevant fields.

2. Quasi-ItÔ Process and Generalized Itô's Formula

In this section, we begin with introducing the concept of quasi-Itô process. Set $\Delta = \{(t,s) : 0 \le s \le t < \infty\}$. Let $C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+;\mathbb{R})$ stand for the family of all real-valued functions $V(x,t)$ defined on $\mathbb{R}^d \times \mathbb{R}^+$ such that they are continuously twice differentiable at x and once at t. For any $p \in [1,\infty)$, we define

$$\mathcal{L}^p\left(\mathbb{R}^+;\mathbb{R}^{d\times m}\right) = \Big\{\varphi : \mathbb{R}^+ \times \Omega \longrightarrow \mathbb{R}^{d\times m} \mid \varphi(\cdot) \text{ is measurable } \mathcal{F}_t\text{-adapted and } \int_0^T |\varphi(t)|^p dt < \infty \text{ a.s. for every } T > 0\Big\},$$
$$\mathcal{L}\left(\mathbb{R}^+, \mathcal{L}^p\left((0,t);\mathbb{R}^{d\times m}\right)\right) = \left\{\varphi : \Delta \times \Omega \longrightarrow \mathbb{R}^{m\times d} \mid \varphi(t,\cdot) \in \mathcal{L}^p\left((0,t);\mathbb{R}^{d\times m}\right)\right\}. \tag{2.1}$$

Definition 2.1. A d-dimensional quasi-Itô process is an $\mathbb{R}^d$-valued continuous adapted process $x(t) = (x_1(t),\ldots,x_d(t))^T$ on $t \ge 0$ of the form

$$x(t) = \varphi(t) + \int_0^t f(t,s)ds + \int_0^t g(t,s)dW(s), \tag{2.2}$$

where $f = (f_1,\ldots,f_d)^T \in \mathcal{L}(\mathbb{R}^+,\mathcal{L}^1((0,t);\mathbb{R}^d))$, $g = (g_{ij})_{d\times m} \in \mathcal{L}(\mathbb{R}^+,\mathcal{L}^2((0,t);\mathbb{R}^{d\times m}))$, for all $s \ge 0$, $f(\cdot,s)$ and $g(\cdot,s)$ are continuous, and $\varphi(t)$ is an $\mathcal{F}_t$-adapted continuous stochastic process.

We will say that $x(t)$ has quasistochastic differential $dx(s)$ or $Dx(s)$ for $t \ge 0$ given by

$$dx(s) = d\varphi(s) + f(t,s)ds + g(t,s)dW(s), \tag{2.3}$$

or

$$Dx(s) = f(t,s)ds + g(t,s)dW(s), \tag{2.4}$$

in which $Dx(s) = dx(s) - d\varphi(s)$.

Remark 2.2. Definition 2.1 is well defined under the condition that $\int_0^t g(t,s)dW(s)$ is continuous for t. The proof of the continuity of $\int_0^t g(t,s)dW(s)$ is similar to Theorem 1.5.13 in [1]. Here we do not verify it. But in the proof we will need two approximation theorems as follows.

Lemma 2.3 (see [2, page 116]). *Letting $g : \mathbb{R}^d \times \Omega \to \mathbb{R}$ be $\mathfrak{B}(\mathbb{R}^d) \times \mathcal{F}$-measurable, then g could be approximated pointwise by bounded functions of the form*

$$\sum_{k=1}^{m} \phi_k(x)\varphi_k(\omega). \tag{2.5}$$

Lemma 2.4. *Let $\phi(t,s) : \mathbb{R}^2 \to \mathbb{R}$ be $\mathfrak{B}(\mathbb{R}^2)$-measurable, and for all $s \geq 0$, $g(\cdot,s)$ is continuous; then $\phi(t,s)$ is approximated by functions on the form*

$$\sum_{k=1}^{m} g_k^1(t) g_k^2(s), \tag{2.6}$$

where $g_k^1(t)$ is continuous and $g_k^2(s)$ is $\mathfrak{B}(\mathbb{R}^1)$-measurable for every $k \in \{1,2,\ldots,m\}$.

Similar to Theorem 1.2, it again raises the following question. If $x(t)$ is a quasi-Itô process and $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+;\mathbb{R})$, then whether the compound function $V(x(t),t)$ is a quasi-Itô process. And if it is, then what is its quasistochastic differential? We now have the result which is a well generalization of Itô's formula.

Theorem 2.5. *Let $x(t)$ be a d-dimensional quasi-Itô process on $t \geq 0$ with the quasistochastic differential*

$$dx(s) = d\varphi(s) + f(t,s)ds + g(t,s)dW(s), \tag{2.7}$$

or

$$Dx(s) = f(t,s)ds + g(t,s)dW(s), \tag{2.8}$$

with $Dx(s) = dx(s) - d\varphi(s)$. Here f, g are defined as Definition 2.1, $\varphi(t)$ is a continuous stochastic process, and for every $t \in [0,\infty)$, $\varphi(t)$ is $\mathcal{F}_0$-measurable. Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+;\mathbb{R})$ and

$$h(t,s) = \varphi(t) + \int_0^s f(t,r)dr + \int_0^s g(t,r)dW(r), \quad t \in [0,+\infty). \tag{2.9}$$

Then $V(x(t),t)$ is a quasi-Itô process with the quasistochastic differential given by

$$\begin{aligned} dV&(x(s),s) \\ &= dV(\varphi(s),0) \\ &\quad + \left[V_t(h(t,s),s) + V_x(h(t,s),s)f(t,s) + \frac{1}{2}\,\text{trace}\left(g^T(t,s)V_{xx}(h(t,s),s)g(t,s)\right)\right]ds \\ &\quad + V_x(h(t,s),s)g(t,s)dW(s) \quad \text{a.s.,} \end{aligned} \tag{2.10}$$

or

$$DV(x(s),s) = \left[V_t(h(t,s),s) + V_x(h(t,s),s)f(t,s) + \frac{1}{2}\,\text{trace}\left(g^T(t,s)V_{xx}(h(t,s),s)g(t,s)\right)\right]ds + V_x(h(t,s),s)g(t,s)dW(s)\ \text{a.s.} \tag{2.11}$$

with $DV(x(s),s) = dV(x(s),s) - dV(\varphi(s),0)$.

Proof. Setting $t^* \in [0,\infty)$ is arbitrary and

$$y(t) = \varphi(t^*) + \int_0^t f(t^*,s)ds + \int_0^t g(t^*,s)dW(s)\text{a.s.} \tag{2.12}$$

By Itô's formula, we can derive that for any $t \geq 0$,

$$\begin{aligned} V(y(t),t) &= V(\varphi(t^*),0) \\ &\quad + \int_0^t \left[V_t(y(s),s) + V_x(y(s),s)f(t^*,s) + \frac{1}{2}\,\text{trace}\left(g^T(t^*,s)V_{xx}(y(s),s)g(t^*,s)\right)\right]ds \\ &\quad + \int_0^t V_x(y(s),s)g(t^*,s)dW(s)\ \text{a.s.} \end{aligned} \tag{2.13}$$

So $V(y(t^*),t^*) = V(x(t^*),t^*)$. Setting $t = t^*$, then we have

$$\begin{aligned} V(x(t^*),t^*) &= V(\varphi(t^*),0) \\ &\quad + \int_0^{t^*} \left[V_t(y(s),s)+V_x(y(s),s)f(t^*,s)+\frac{1}{2}\,\text{trace}\left(g^T(t^*,s)V_{xx}(y(s),s)g(t^*,s)\right)\right]ds \\ &\quad + \int_0^{t^*} V_x(y(s),s)g(t^*,s)dW(s) \\ &= V(\varphi(t^*),0) \\ &\quad + \int_0^{t^*} \Big[V_t(h(t^*,s),s) + V_x(h(t^*,s),s)f(t^*,s) \\ &\qquad + \frac{1}{2}\,\text{trace}\left(g^T(t^*,s)V_{xx}(h(t^*,s),s)g(t^*,s)\right)\Big]ds \\ &\quad + \int_0^{t^*} V_x(h(t^*,s),s)g(t^*,s)dW(s). \end{aligned} \tag{2.14}$$

Since t^* is arbitrary, (2.10) must be required. The proof is complete. □

Remark 2.6. When $\varphi(t), f(t,s)$ and $g(t,s)$, are independent of t, that is, when $\varphi(t) = x(0), f(t,s) = f(s)$ and $g(t,s) = g(s)$, then it is easy to check that

$$h(t,s) = x(0) + \int_0^s f(r)dr + \int_0^s g(r)dW(r) = x(s), \tag{2.15}$$

and the generalized Itô's formula becomes classical Itô's formula.

Example 2.7. Suppose that

$$x(t) = m(t) + \int_0^t tW(s)ds + \int_0^t e^{-\cos t - s/2 + W(s)} dW(s), \tag{2.16}$$

where $m(t)$ is a continuous function.

We find $y(t) = x^2(t)$. Here we have

$$\begin{aligned} h(t,s) &= m(t) + \int_0^s tW(r)dr + \int_0^s e^{-\cos t - r/2 + W(r)} dW(r) \\ &= m(t) + t\overline{W}(s) + e^{-\cos t - s/2 + W(s)} - e^{-\cos t}, \end{aligned} \tag{2.17}$$

where $\overline{W}(s) \triangleq \int_0^s W(r)dr$. Let $V(x,t) = x^2$. Then $V_t = 0, V_x = 2x, V_{xx} = 2$. So by (2.11) we obtain

$$\begin{aligned} Dy(t) = {} & \left[\left(2m(t) + 2t\overline{W}(s) + 2e^{-\cos t - s/2 + W(s)} - 2e^{-\cos t}\right)tW(s) + e^{-2\cos t - s + 2W(s)}\right]ds \\ & + 2\left(m(t) + t\overline{W}(s) + e^{-\cos t - s/2 + W(s)} - e^{-\cos t}\right)e^{-\cos t - s/2 + W(s)} dW(s). \end{aligned} \tag{2.18}$$

Therefore,

$$\begin{aligned} x^2(t) = {} & \int_0^t \left[\left(2m(t) + 2t\overline{W}(s) + 2e^{-\cos t - s/2 + W(s)} - 2e^{-\cos t}\right)tW(s) + e^{-2\cos t - s + 2W(s)}\right]ds \\ & + \int_0^t \left[2\left(m(t) + t\overline{W}(s) + e^{-\cos t - s/2 + W(s)} - e^{-\cos t}\right)e^{-\cos t - s/2 + W(s)}\right]dW(s) + e^{2t}. \end{aligned} \tag{2.19}$$

Sometimes function $\varphi(t,\omega)$ is required to be $\mathcal{F}_t$-adapted instead of $\mathcal{F}_0$-measurable. We suppose that $\mathcal{C}$ is the set of all absolutely continuous $\mathcal{F}_t$-adapted processes. That is, if $\phi \in \mathcal{C}$, then $\phi(t,\omega)$ is absolutely continuous for almost all $\omega \in \Omega$ and $\phi(t,\omega)$ is $\mathcal{F}_t$-measurable for any $t \geq 0$.

Theorem 2.8. *Let $x(t)$ be a d-dimensional quasi-Itô process on $t \geq 0$ with the quasistochastic differential*

$$dx(s) = d\varphi(s) + f(t,s)ds + g(t,s)dW(s), \tag{2.20}$$

or

$$Dx(s) = f(t,s)ds + g(t,s)dW(s), \tag{2.21}$$

with $Dx(s) = dx(s) - d\varphi(s)$. *Here* f, g *are defined as Definition 2.1 and* $\varphi \in \mathcal{C}$. *Let* $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R})$ *and*

$$\overline{h}(t,s) = \int_0^s (\varphi'(r) + f(t,r))dr + \int_0^s g(t,r)dW(r) \quad , t \in [0,+\infty). \tag{2.22}$$

Then $V(x(t),t)$ *is a quasi-Itô process with the quasistochastic differential given by*

$$\begin{aligned} dV(x(s),s) = &\Big[V_t\big(\overline{h}(t,s),s\big) + V_x\big(\overline{h}(t,s),s\big)(\varphi'(s) + f(t,s)) \\ &+ \frac{1}{2}\operatorname{trace}\Big(g^T(t,s)V_{xx}\big(\overline{h}(t,s),s\big)g(t,s)\Big)\Big]ds \\ &+ V_x\big(\overline{h}(t,s),s\big)g(t,s)dW(s) \text{ a.s.} \end{aligned} \tag{2.23}$$

Proof. It is easy to see that

$$dx(s) = (\varphi'(s) + f(t,s))ds + g(t,s)dW(s) \triangleq \overline{f}(t,s)\mathrm{d}s + g(t,s)dW(s). \tag{2.24}$$

Let

$$\overline{h}(t,s) = \int_0^s \overline{f}(t,r)dr + \int_0^s g(t,r)dW(r), \quad t \in [0,+\infty). \tag{2.25}$$

From Theorem 2.5 we have

$$\begin{aligned} dV(x(s),s) = &\Big[V_t\big(\overline{h}(t,s),s\big) + V_x\big(\overline{h}(t,s),s\big)(\varphi'(s) + f(t,s)) \\ &+ \frac{1}{2}\operatorname{trace}\Big(g^T(t,s)V_{xx}\big(\overline{h}(t,s),s\big)g(t,s)\Big)\Big]ds \\ &+ V_x\big(\overline{h}(t,s),s\big)g(t,s)dW(s) \text{ a.s.} \end{aligned} \tag{2.26}$$

The proof is complete. □

Example 2.9. Let

$$x(t) = e^{\sin \overline{W}(t)} + \int_0^t tW(s)ds + \int_0^t e^{-\cos t - s/2 + W(s)}dW(s). \tag{2.27}$$

Find $y(t) = x^2(t)$. Here we have

$$\begin{aligned}\overline{h}(t,s) &= \int_0^s \left(tW(r) + W(r)e^{\sin \overline{W}(r)} \cos \overline{W}(r)\right)dr + \int_0^s e^{-\cos t - r/2 + W(r)} dW(r) \\ &= t\overline{W}(s) + e^{\sin \overline{W}(s)} + e^{-\cos t - s/2 + W(s)} - e^{-\cos t}.\end{aligned} \tag{2.28}$$

So by Theorem 2.8, we obtain

$$\begin{aligned}x^2(t) = &\int_0^t \Big[2\left(t\overline{W}(s) + e^{\sin \overline{W}(s)} + e^{-\cos t - s/2 + W(s)} - e^{-\cos t}\right) \\ &\quad \times \left(tW(s) + W(r)e^{\sin \overline{W}(r)} \cos \overline{W}(r)\right) + e^{-2\cos t - s + 2W(s)}\Big] ds \\ &+ \int_0^t \Big[\left(2t\overline{W}(s) + 2e^{\sin \overline{W}(s)} + 2e^{-\cos t - s/2 + W(s)} - 2e^{-\cos t}\right) e^{-\cos t - s/2 + W(s)}\Big] dW(s).\end{aligned} \tag{2.29}$$

3. Stability in Probability of SVIEs

In this section, we use the generalized Itô's formula to investigate the stability for the d-dimensional SVIE:

$$x(t) = \varphi(t,\omega)x_0 + \int_{t_0}^t f(x(s),t,s,\omega)ds + \int_{t_0}^t g(x(s),t,s,\omega)dW(s). \tag{3.1}$$

Assuming further that $\varphi(t,\omega)$ is $\mathcal{F}_{t_0}$-measurable, $\varphi(t_0,\omega) = 1$ and

$$f(0,t,s) = 0, \quad g(0,t,s) = 0, \quad (t,s) \in \Delta \text{ a.s.} \tag{3.2}$$

Hence, (3.1) has solution $x(t) \equiv 0$ corresponding to initial value $x(t_0) = 0$. This solution is called trivial solution. For any $\eta \in C([t_0,\infty);\mathbb{R}^d)$, define

$$\begin{aligned}h(\eta,t,s) &= \eta(s) + \varphi(t,\omega)x_0 - \varphi(s,\omega)x_0, \\ \mathcal{L}V(\eta,t,s) &= V_t(h(\eta,t,s),s) + V_x(h(\eta,t,s),s)f(\eta(s),s,\omega) \\ &\quad + \frac{1}{2}\operatorname{trace}\left[g^T(\eta(s),s,\omega)V_{xx}(h(\eta,t,s))g(\eta(s),s,\omega)\right].\end{aligned} \tag{3.3}$$

Then by Theorem 2.5,

$$V(x(t),t) = V(\varphi(t,\omega)x_0,t_0) + \int_{t_0}^t \mathcal{L}V(x,t,s)ds + \int_{t_0}^t V_x(h(x,t,s),s)g(x(s),s,\omega)dW(s) \text{ a.s.} \tag{3.4}$$

Let $\mathcal{K}$ denote the family of all continuous nondecreasing functions $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(0) = 0$ and $\mu(x) > 0$ if $x > 0$. For $h > 0$, let $S_h = \{x \in \mathbb{R}^d : |x| < h\}$. A continuous function $V(x,t)$ defined on $S_h \times [t_0, \infty)$ is said to be positive definite if $V(0,t) = 0$, and, for some $\mu \in \mathcal{K}$,

$$V(x,t) \geq \mu(|x|) \quad \forall (x,t) \in S_h \times [t_0, \infty). \tag{3.5}$$

A function $V(x,t)$ is said to be decrescent if $V(x,t) \leq \mu(|x|), (x,t) \in S_h \times [t_0, \infty)$ for some $\mu \in \mathcal{K}$. A function $V(x,t)$ defined on $\mathbb{R}^d \times [t_0, \infty)$ is said to be radially unbounded if $\liminf_{|x| \to \infty, t \geq t_0} V(x,t) = \infty$.

Definition 3.1. (1) The trivial solution of (3.1) is said to be stochastically stable if for every pair $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r, t_0) > 0$ such that

$$\mathbb{P}(|x(t; t_0, x_0)| < r,\ t \geq t_0) \geq 1 - \varepsilon, \tag{3.6}$$

whenever $|x_0| < \delta$.

(2) The trivial solution of (3.1) is said to be stochastically asymptotically stable if it is stochastically stable, and, moreover, for every $\varepsilon \in (0,1)$, there exists a $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ such that

$$\mathbb{P}\left(\lim_{t \to \infty} x(t; t_0, x_0) = 0\right) \geq 1 - \varepsilon, \tag{3.7}$$

whenever $|x_0| < \delta_0$.

(3) The trivial solution of (3.1) is said to be globally stochastically asymptotically stable if it is stochastically stable, and, moreover, for all $x_0 \in \mathbb{R}^d$

$$\mathbb{P}\left(\lim_{t \to \infty} x(t; t_0, x_0) = 0\right) = 1. \tag{3.8}$$

Lemma 3.2. *If there exists an $\varepsilon > 0$, such that $|\varphi(t,\omega)| > \varepsilon, t \in [t_0, \infty)$ a.s. Then for any $x_0 \in \mathbb{R}^d$ and $x_0 \neq 0$, one has*

$$\mathbb{P}(|x(t; t_0, x_0)| \neq 0, t \geq t_0) = 1. \tag{3.9}$$

Similar to the proof of Lemma 3.2 in [2, pp. 120], it is easy to get the lemma. Here we do not recount it.

Theorem 3.3. *Suppose that there exists a $K > 0$, such that $|\varphi(t,\omega)| \leq K$ a.s. If there exists a positive definite function $V(x,t) \in C^{2,1}(S_h \times [t_0, \infty); \mathbb{R}^+)$, such that for any $(x,t,s) \in C([t_0, \infty); \mathbb{R}^d) \times \Delta$, there is*

$$\mathcal{L}V(x,t,s) \leq 0. \tag{3.10}$$

Then the trivial solution to (3.1) is stochastically stable.

Proof. From the definition of the positive definite function, we know that $V(0,t) = 0$, and there exists nonnegative nondecreasing function $\mu(x)$, such that $V(x,t) \geq \mu(|x|)$ for any $(x,t) \in S_h \times [t_0,\infty)$. Choose any $\varepsilon \in (0,1)$, $r > 0$. Without loss of generality, we assume that $r < h$. Since $V(x,t)$ is continuous and $V(0,t_0) = 0$, we could find $\delta = \delta(\varepsilon,r,t_0) > 0$, such that for any $t \in [t_0,\infty), \omega \in \Omega$ there is

$$\frac{1}{\varepsilon}\sup_{x\in S_\delta} V\big(\varphi(t,\omega)x,t_0\big) < \mu(r). \tag{3.11}$$

Define $x(t) = x(t;t_0,x_0)$. Choose any $x_0 \in S_\delta$. Let τ be the first time of $x(t)$ going out the ball S_r, that is, $\tau = \inf\{t \geq t_0 : x(t) \overline{\in} S_r\}$. By Theorem 2.5, for any $t \geq t_0$, there is

$$\begin{aligned} V(x(\tau\wedge t),\tau\wedge t) = V\big(\varphi(\tau\wedge t,\omega)x_0,t_0\big) &+ \int_{t_0}^{\tau\wedge t} \mathcal{L}V(x,\tau\wedge t,s)ds \\ &+ \int_{t_0}^{\tau\wedge t} V_x(h(x,\tau\wedge t,s),s)g(x(s),s,\omega)dW(r). \end{aligned} \tag{3.12}$$

Taking the expectation for both sides, and using $\mathcal{L}V(x,t,s) \leq 0$, we get

$$\mathbb{E}V(x(\tau\wedge t),\tau\wedge t) \leq \mathbb{E}V\big(\varphi(\tau\wedge t,\omega)x_0,t_0\big). \tag{3.13}$$

On the other hand, we have

$$\begin{aligned} \mathbb{E}V(x(\tau\wedge t),\tau\wedge t) &= \mathbb{E}\big[\big(I_{\{\tau\leq t\}} + I_{\{\tau>t\}}\big)V(x(\tau\wedge t),\tau\wedge t)\big] \\ &\geq \mathbb{E}\big[I_{\{\tau\leq t\}}V(x(\tau\wedge t),\tau\wedge t)\big] \\ &\geq \mu(r)\mathbb{E}\big[I_{\{\tau\leq t\}}\big] \\ &= \mu(r)\mathbb{P}(\tau\leq t), \end{aligned} \tag{3.14}$$

So combining (3.11) and (3.14), it follows that

$$\mathbb{P}(\tau\leq t) \leq \frac{\mathbb{E}V\big(\varphi(\tau\wedge t,\omega)x_0,t_0\big)}{\mu(r)} < \varepsilon. \tag{3.15}$$

Letting $t \to \infty$, we obtain

$$\mathbb{P}(\tau\leq\infty) < \varepsilon, \tag{3.16}$$

that is,

$$\mathbb{P}(|x(t;t_0,x_0)| < r,\ t\geq t_0) > 1-\varepsilon. \tag{3.17}$$

The proof is complete. □

Theorem 3.4. *Suppose that the conditions in Lemma 3.2 hold. If there exists a positive definite function* $V(x,t) \in C^{2,1}(S_h \times [t_0,\infty);\mathbb{R}^+)$*, which has infinitesimal upper bounded and*

$$\begin{aligned} &\mu_1(|x|) \leq V(x,t) \leq \mu_2(|x|), \quad (x,t) \in S_h \times [t_0,\infty), \\ &\mathcal{L}V(x,t,s) \leq -\mu_3(|x(t)|), \quad (x,t,s) \in C\Big([t_0,\infty);\mathbb{R}^d\Big) \times \Delta, \end{aligned} \tag{3.18}$$

in which $\mu_2(x)$ *is concave function. Then the trivial solution to* (3.1) *is stochastically asymptotically stable.*

Proof. It is clear that the conditions in Theorem 3.3 are satisfied. Hence, the solution to (3.1) is stochastically stable. So it is only necessary to show that for any $\varepsilon \in (0,1)$, there exists a $\delta = \delta_0(\varepsilon,t_0) > 0$, such that for any $|x_0| < \delta_0$,

$$\mathbb{P}\Big(\lim_{t\to\infty} x(t;t_0,x_0) = 0\Big) \geq 1-\varepsilon \tag{3.19}$$

holds. Fixing $\varepsilon \in (0,1)$, in view of Theorem 3.3, there exists a $\delta_0 = \delta_0(\varepsilon,t_0) > 0$, such that $|x_0| < \delta_0$ holds provided only that

$$\mathbb{P}\left\{|x(t;t_0,x_0)| < \frac{h}{2},\ t \geq t_0\right\} \geq 1-\frac{\varepsilon}{4}. \tag{3.20}$$

Fix $x_0 \in S_{\delta_0}$, and denote $x(t) = x(t;t_0,x_0)$. Choose any $0 < \beta < |x_0|$ and $0 < \alpha < \beta$. Define stopping time

$$\tau_\alpha = \inf\{t \geq t_0 : |x(t)| \leq \alpha\}, \qquad \tau_h = \inf\left\{t \geq t_0 : |x(t)| \geq \frac{h}{2}\right\}. \tag{3.21}$$

From Theorem 2.5, for any $t \geq t_0$, there is

$$\begin{aligned} 0 &\leq \mathbb{E}V(x(\tau_\alpha \wedge \tau_h \wedge t), \tau_\alpha \wedge \tau_h \wedge t) \\ &= \mathbb{E}V\big(\varphi(\tau_\alpha \wedge \tau_h \wedge t,\omega)x_0,t_0\big) + \mathbb{E}\int_{t_0}^{\tau_\alpha\wedge\tau_h\wedge t} \mathcal{L}V(x,\tau_\alpha \wedge \tau_h \wedge t,s)ds \\ &\leq \mathbb{E}V\big(\varphi(\tau_\alpha \wedge \tau_h \wedge t,\omega)x_0,t_0\big) - \mu_3(\alpha)\mathbb{E}(\tau_\alpha \wedge \tau_h \wedge t - t_0). \end{aligned} \tag{3.22}$$

Therefore

$$\mathbb{E}(\tau_\alpha \wedge \tau_h \wedge t - t_0) \leq \frac{\mathbb{E}V\big(\varphi(\tau_\alpha \wedge \tau_h \wedge t,\omega)x_0,t_0\big)}{\mu_3(\alpha)}. \tag{3.23}$$

So

$$\mathbb{P}(t \leq \tau_\alpha \wedge \tau_h) \leq \frac{\mathbb{E}V\big(\varphi(\tau_\alpha \wedge \tau_h \wedge t,\omega)x_0,t_0\big)}{\mu_3(\alpha)(t-t_0)}. \tag{3.24}$$

Letting $t \to \infty$, it yields

$$\mathbb{P}(\tau_\alpha \wedge \tau_h < \infty) = 1. \tag{3.25}$$

Clearly, from (3.20) it follows that $\mathbb{P}(\tau_h < \infty) \le \varepsilon/4$. Therefore

$$1 = \mathbb{P}(\tau_\alpha \wedge \tau_h < \infty) \le \mathbb{P}(\tau_\alpha < \infty) + \mathbb{P}(\tau_h < \infty) \le \mathbb{P}(\tau_\alpha < \infty) + \frac{\varepsilon}{4}. \tag{3.26}$$

So

$$\mathbb{P}(\tau_\alpha < \infty) \ge 1 - \frac{\varepsilon}{4}. \tag{3.27}$$

Choose sufficiently large θ_α, such that

$$\mathbb{P}(\tau_\alpha < \theta_\alpha) \ge 1 - \frac{\varepsilon}{2}. \tag{3.28}$$

Then

$$\mathbb{P}(\tau_\alpha < \tau_h \wedge \theta_\alpha) \ge \mathbb{P}(\{\tau_\alpha < \theta_\alpha\} \cap \{\tau_h = \infty\}) \ge \mathbb{P}(\tau_\alpha < \theta_\alpha) - \mathbb{P}(\tau_h = \infty) \ge 1 - \frac{3\varepsilon}{4}. \tag{3.29}$$

Again define two stopping times as

$$\sigma = \begin{cases} \tau_\alpha, & \tau_\alpha < \tau_h \wedge \ \theta_\alpha, \\ \infty, & \text{otherwise}, \end{cases} \quad \text{and } \tau_\beta = \inf\{t > \sigma : |x(t)| \ge \beta\}. \tag{3.30}$$

By reason that

$$\begin{aligned} x(t) &= \varphi(t,\omega)x_0 + \int_{t_0}^{t} f(x(s),t,s,\omega)ds + \int_{t_0}^{t} g(x(s),t,s,\omega)dW(s) \\ &= \varphi_3(t,r,x_0,\omega) + \int_{r}^{t} f(x(s),t,s,\omega)ds + \int_{r}^{t} g(x(s),t,s,\omega)dW(s), \quad t \ge r, \end{aligned} \tag{3.31}$$

in which

$$\varphi_3(t,r,x_0,\omega) = \varphi(t,\omega)x_0 + \int_{t_0}^{r} f(x(s),t,s,\omega)ds + \int_{t_0}^{r} g(x(s),t,s,\omega)dW(s). \tag{3.32}$$

From Theorem 2.5, it follows that for any $t \ge \theta_\alpha$, there is

$$\mathbb{E}V\big(x(\tau_\beta \wedge t), \tau_\beta \wedge t\big) \le \mathbb{E}V\big(\varphi_3(\tau_\beta \wedge t, \sigma \wedge t, x_0, \omega), \sigma \wedge t\big). \tag{3.33}$$

Note that if $\omega \in \{\tau_\alpha \geq \tau_h \wedge \theta_\alpha\}$, then

$$V(x(\tau_\beta \wedge t), \tau_\beta \wedge t) = V(\varphi_3(\tau_\beta \wedge t, \sigma \wedge t, x_0, \omega), \sigma \wedge t). \tag{3.34}$$

Consequently,

$$\mathbb{E}\left(I_{\{\tau_\alpha<\tau_h\wedge\theta_\alpha\}}V(x(\tau_\beta \wedge t), \tau_\beta \wedge t)\right) \leq \mathbb{E}\left(I_{\{\tau_\alpha<\tau_h\wedge\theta_\alpha\}}V(\varphi_3(\tau_\beta \wedge t, \sigma \wedge t, x_0, \omega), \sigma \wedge t)\right). \tag{3.35}$$

From the total probability formula, it yields that

$$\begin{aligned}\mathbb{E}\left(I_{\{\tau_\alpha<\tau_h\wedge\theta_\alpha\}}V(x(\tau_\beta \wedge t), \tau_\beta \wedge t)\right) &\geq \mathbb{E}\left(I_{\{\tau_\alpha<\tau_h\wedge\theta_\alpha\}}V(x(\tau_\beta \wedge t), \tau_\beta \wedge t) \mid \tau_\beta \leq t\right)\mathbb{P}(\tau_\beta \leq t) \\ &\geq \mu_1(\beta)\mathbb{P}(\tau_\beta \leq t).\end{aligned} \tag{3.36}$$

Since $\mu_2(\cdot)$ is a concave function, and by (3.18), we have

$$\begin{aligned}\mathbb{E}\left(I_{\{\tau_\alpha<\tau_h\wedge\theta_\alpha\}}V(\varphi_3(\tau_\beta \wedge t, \sigma \wedge t, x_0, \omega), \sigma \wedge t)\right) &\leq \mathbb{E}\left(I_{\{\tau_\alpha<\tau_h\wedge\theta_\alpha\}}\mu_2\left(\left|\varphi_3(\tau_\beta \wedge t, \sigma \wedge t, x_0, \omega)\right|\right)\right) \\ &\leq \mathbb{E}\mu_2\left(\left|\varphi_3(\tau_\beta \wedge t, \tau_\alpha, x_0, \omega)\right|\right) \\ &\leq \mu_2\left(\mathbb{E}\left|\varphi_3(\tau_\beta \wedge t, \tau_\alpha, x_0, \omega)\right|\right).\end{aligned} \tag{3.37}$$

From (3.31) it follows that

$$\mathbb{E}\left|\varphi_3(\tau_\beta \wedge t, \tau_\alpha, x_0, \omega)\right| - \mathbb{E}|x(\tau_\alpha)| \leq \mathbb{E}\left|\varphi(\tau_\beta \wedge t, \omega)x_0 - \varphi(\tau_\alpha, \omega)x_0\right|. \tag{3.38}$$

From Lemma 3.2, it is known that $\lim_{\alpha\to 0}\tau_\alpha = \infty$. Hence $\lim_{\alpha\to 0}\tau_\beta = \infty$. Letting $\alpha \to 0$ and taking the limit for (3.38), there is

$$\lim_{\alpha\to 0}\left[\mathbb{E}\left|\varphi_3(\tau_\beta \wedge t, \tau_\alpha, x_0, \omega)\right| - \mathbb{E}|x(\tau_\alpha)|\right] = 0. \tag{3.39}$$

Thus we could choose sufficiently small $\varepsilon_1 > 0, \alpha_1 > 0$, such that $\mu_2(\alpha_1 + \varepsilon_1)/\mu_1(\beta) < \varepsilon/4$ and

$$\mathbb{E}\left|\varphi_3\left(\tau_{\beta_{\alpha_1}} \wedge t, \tau_{\alpha_1}, x_0, \omega\right)\right| - \mathbb{E}|x(\tau_{\alpha_1})| \leq \varepsilon_1 \tag{3.40}$$

hold. That is,

$$\mathbb{E}\left|\varphi_3\left(\tau_{\beta_{\alpha_1}} \wedge t, \tau_{\alpha_1}, x_0, \omega\right)\right| \leq \mathbb{E}|x(\tau_{\alpha_1})| + \varepsilon_1 = \alpha_1 + \varepsilon_1. \tag{3.41}$$

Combining (3.35), (3.36), (3.37), and (3.41), it follows that for sufficiently large t there is

$$\mathbb{P}(\tau_\beta \leq t) \leq \frac{\mu_2(\alpha_1 + \varepsilon_1)}{\mu_1(\beta)} < \frac{\varepsilon}{4}. \tag{3.42}$$

Letting $t \to \infty$, it yields that $\mathbb{P}(\tau_\beta \le \infty) < \varepsilon/4$. In view of (3.29), it deduces that

$$\mathbb{P}(\{\tau_\beta = \infty\} \cap \{\sigma < \infty\}) \ge \mathbb{P}(\tau_\alpha < \tau_\alpha \wedge \theta) - \mathbb{P}(\tau_\beta < \infty) \ge 1 - \varepsilon, \tag{3.43}$$

which shows that

$$\mathbb{P}\left(\omega : \limsup_{t \to \infty} |x(t)| \le \beta\right) \ge 1 - \varepsilon. \tag{3.44}$$

By the arbitrariness of β, we have

$$\mathbb{P}\left(\omega : \limsup_{t \to \infty} |x(t)| = 0\right) \ge 1 - \varepsilon. \tag{3.45}$$

The proof is complete. □

Theorem 3.5. *Suppose that the conditions in Theorem 3.4 are satisfied and $V(x,t)$ is radially unbounded. Then the trivial solution to* (3.1) *is globally stochastically asymptotically stable.*

Proof. In view of Theorem 3.4, it is known that the trivial solution to (3.1) is stochastically asymptotically stable. Therefore it is only necessary to explain that for any $x_0 \in \mathbb{R}^d$, there is

$$\mathbb{P}\left(\lim_{t \to \infty} x(t; t_0, x_0) = 0\right) = 1. \tag{3.46}$$

Choose any $x_0 \in \mathbb{R}^d$ and $\varepsilon \in (0,1)$. Denote $x(t) = x(t; t_0, x_0)$. From that $V(x,t)$ is radially unbounded and that $\varphi(t,\omega)$ is bounded, we could find a sufficiently large $h > |x_0|$, such that

$$\inf_{t \ge t_0, |x| \ge h} \left\{ \frac{\mathbb{E}V(\varphi(t,\omega)x_0, t_0)}{\mathbb{E}V(x,t)} \right\} \le \frac{\varepsilon}{4}. \tag{3.47}$$

Define stopping time $\tau_h = \{t > t_0 : |x(t)| \ge h\}$. Then from Theorem 2.5 and conditional property formula, we could prove that for any $t > t_0$, there is

$$\mathbb{E}V(x(\tau_h), \tau_h)\mathbb{P}(\tau_h \le t) \le \mathbb{E}V(x(\tau_h \wedge t), \tau_h \wedge t) \le \mathbb{E}V(\varphi(\tau_h \wedge t, \omega)x_0, t_0). \tag{3.48}$$

Therefore

$$\mathbb{P}(\tau_h < t) \le \frac{\mathbb{E}V(\varphi(\tau_h \wedge t, \omega)x_0, t_0)}{\mathbb{E}V(x(\tau_h), \tau_h)} \le \frac{\varepsilon}{4}. \tag{3.49}$$

Letting $t \to \infty$, it yields that $\mathbb{P}(\tau_h < \infty) \le \varepsilon/4$, that is:

$$\mathbb{P}(|x(t; t_0, x_0)| \le h, t \ge t_0) \ge 1 - \frac{\varepsilon}{4}. \tag{3.50}$$

In the following, applying the method in Theorem 3.4, we obatin that

$$\mathbb{P}\left(\lim_{t\to\infty} x(t;t_0,x_0)=0\right)\geq 1-\varepsilon. \tag{3.51}$$

Hence, by the arbitrariness of ε, (3.46) holds. It completes the proof. □

To illustrate the theorem developed in this section, an example now is discussed.

Example 3.6. Consider a scale linear SVIE:

$$x(t)=\varphi(t,\omega)x_0+\int_0^t a(t)b(s)x(s)ds+\int_0^t a(t)c(s)x(s)dW(s),\quad t\in[0,\infty), \tag{3.52}$$

in which $a(t)b(t)<0, t\geq 0$, $\varphi(0,\omega)=1$.

Letting $y(t)=a^{-1}(t)x(t)$, $\varphi_1(t,\omega)=a^{-1}(t)\varphi(t,\omega)$, then (3.52) is changed into

$$y(t)=\varphi_1(t,\omega)x_0+\int_0^t a(s)b(s)y(s)ds+\int_0^t a(s)c(s)y(s)dW(s)\quad ,t\in[0,\infty). \tag{3.53}$$

By Lemma 2.3, it follows that

$$h(t,s)=y(s)+\varphi_1(t,\omega)x_0-\varphi_1(s,\omega)x_0. \tag{3.54}$$

Setting $V(x,t)=x^2$, then

$$\mathcal{L}V(y(s),t,s)=2a(s)b(s)y^2(s)+\left(\varphi_1(t,\omega)-\varphi_1(s,\omega)\right)a(s)b(s)x_0y(s)+|a(s)c(s)|^2\left|y(s)\right|^2. \tag{3.55}$$

If $a^{-1}(t)\varphi(t,\omega)$ is increasing and bounded almost surely and $2|b(s)|\geq|a(s)||c(s)|^2$, then $\mathcal{L}V(y(s),t,s)<0$. From Theorem 2.5, the solution to (3.52) is stochastically stable.

Setting $V(x,t)=x^{1/2}$, then

$$\begin{aligned}\mathcal{L}V(y(s),t,s)&=\frac{1}{2}\left[y(s)+\varphi_1(t,\omega)x_0-\varphi_1(s,\omega)x_0\right]^{-1/2}a(s)b(s)y(s)\\&\quad-\frac{1}{8}\left[y(s)+\varphi_1(t,\omega)x_0-\varphi_1(s,\omega)x_0\right]^{-3/2}a^2(s)b^2(s)y^2(s).\end{aligned} \tag{3.56}$$

If $a^{-1}(t)\varphi(t,\omega)$ is increasing and bounded almost surely, and $a^{-1}(t)=-b(t)$, then $\mathcal{L}V(y(s),t,s)\leq -y(s)^{1/2}/2$. In view of Theorem 3.3, for any $x_0>0$, there is

$$\mathbb{P}\left(\lim_{t\to\infty} x(t;t_0,x_0)=0\right)=1. \tag{3.57}$$

Remark 3.7. The generalized Itô's formula provides a powerful tool to deal with SVIEs. But we also remind of its complexity, which will bring some difficulties when the almost sure exponential stability and the moment exponential stability for SVIEs are discussed. In this point, we shall go on to discuss in another papers.

Acknowledgments

This work was supported by the NNSF of China (nos.11126219, 11171081, and 11171056), the NNSF of Shandong Province (no. ZR2010AQ021), the Natural Scientific Research Innovation Foundation in Harbin Institute of Technology (no. HIT.NSRIF. 2011104), and NCET-08-0755.

References

[1] X. Mao, *Stochastic Differential Equations and Their Applications*, Horwood Publishing Series in Mathematics & Applications, Horwood Publishing Limited, Chichester, UK, 1997.
[2] O. Bernt, *Stochastic Differential Equations*, Springer, 6th edition, 2006.
[3] T. C. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York, NY, USA, 1988.
[4] A. Friedman, *Stochastic Differential Equations and Applications*, vol. 1, Academic Press, 1975.
[5] H. Huang, D. Ho, and J. Lam, "Stochastic stability analysis of fuzzy Hopfield neural networks with timevarying delays," *IEEE Transactions on Circuits and Systems II-Express Briefs*, vol. 52, pp. 251–255, 2005.
[6] C. Li, L. Chen, and K. Aihara, "Stochastic synchronization of genetic oscillator networks," *BMC Systems Biology*, vol. 1, article 6, 2007.
[7] W. Zhang and L. Wang, "Robust stochastic stability analysis for uncertain neutral-type delayed neural networks driven by Wiener process," *Journal of Applied Mathematics*, Article ID 829594, 12 pages, 2012.
[8] W. Li, H. Su, and K. Wang, "Global stability analysis for stochastic coupled systems on networks," *Automatica*, vol. 47, no. 1, pp. 215–220, 2011.
[9] W. Li, H. Su, D. Wei, and K. Wang, "Global stability of coupled nonlinear systems with Markovian switching," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, pp. 2609–2616, 2012.
[10] M. Xue, S. Zhou, and S. Hu, "Stability of nonlinear neutral stochastic functional differential equations," *Journal of Applied Mathematics*, vol. 2010, Article ID 425762, 26 pages, 2010.
[11] X. Li, A. Gray, D. Jiang, and X. Mao, "Sufficient and necessary conditions of stochastic permanence and extinction for stochastic logistic populations under regime switching," *Journal of Mathematical Analysis and Applications*, vol. 376, no. 1, pp. 11–28, 2011.
[12] X. Li, D. Jiang, and X. Mao, "Population dynamical behavior of Lotka-Volterra system under regime switching," *Journal of Computational and Applied Mathematics*, vol. 232, no. 2, pp. 427–448, 2009.
[13] M. Liu and K. Wang, "Persistence and extinction of a stochastic single-specie model under regime switching in a polluted environment," *Journal of Theoretical Biology*, vol. 264, pp. 934–944, 2010.
[14] M. Liu and K. Wang, "Persistence and extinction in stochastic non-autonomous logistic systems," *Journal of Mathematical Analysis and Applications*, vol. 375, no. 2, pp. 443–457, 2011.
[15] E. Pardoux and P. Protter, "Stochastic Volterra equations with anticipating coefficients," *The Annals of Probability*, vol. 18, no. 4, pp. 1635–1655, 1990.
[16] M. A. Berger and V. J. Mizel, "Volterra equations with Itô integrals. I," *Journal of Integral Equations*, vol. 2, no. 3, pp. 187–245, 1980.
[17] E. Pardoux and P. Protter, "Stochastic Volterra equations with anticipating coefficients," *The Annals of Probability*, vol. 18, no. 4, pp. 1635–1655, 1990.