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Research Article

The System of Mixed Equilibrium Problems for Quasi-Nonexpansive Mappings in Hilbert Spaces

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We first introduce the iterative procedure to approximate a common element of the fixed-point set of two quasinonexpansive mappings and the solution set of the system of mixed equilibrium problem (SMEP) in a real Hilbert space. Next, we prove the weak convergence for the given iterative scheme under certain assumptions. Finally, we apply our results to approximate a common element of the set of common fixed points of asymptotic nonspreading mapping and asymptotic *TJ* mapping and the solution set of SMEP in a real Hilbert space.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, let C be a nonempty closed convex subset of H, and let T be a mapping of C into H, then $T : C \to H$ is said to be nonexpansive if $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in C$. A mapping $T : C \to H$ is said to be quasinonexpansive if $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T) := \{x \in C : Tx = x\}$. It is well known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is a closed and convex set [1]. A mapping $T : C \to C$ is said to be firmly nonexpansive [2] if

$$\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle, \tag{1.1}$$

for all $x, y \in C$, and it is an important example of nonexpansive mappings in a Hilbert space.

Let $\varphi : C \to \mathbb{R}$ be a real-valued function, and let $F : C \times C \to \mathbb{R}$ be an equilibrium bifunction, that is, $F(u, u) = 0$ for each $u \in C$. The mixed equilibrium problem is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) \ge 0 \quad \forall y \in C. \tag{1.2}$$

Denote the set of solution of (1.2) by $\mathrm{MEP}(F,\varphi)$. In particular, if $\varphi = 0$, this problem reduces to the equilibrium problem, which is to find $x \in C$ such that

$$F(x,y) \geq 0 \quad \forall y \in C. \tag{1.3}$$

The set of solution of (1.3) is denoted by $\mathrm{EP}(F)$.

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, Min-Max problems, the Nash equilibrium problems in noncooperative games, and others; see, for example, Blum and Oettli [3] and Moudafi [4]. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.3).

Let $F_1, F_2 : C \times C \to \mathbb{R}$ be two monotone bifunctions and $\mu > 0$ is constant. In 2009, Moudafi [5] introduced an alternating algorithm for approximating a solution of the system of equilibrium problems, finding $(x', y') \in C \times C$ such that

$$(\mathrm{SEP})\begin{cases} F_1(x',z) + \dfrac{1}{\mu}\langle y' - x', x' - z\rangle \geq 0, & \forall z \in C, \\ F_2(y',z) + \dfrac{1}{\mu}\langle x' - y', y' - z\rangle \geq 0, & \forall z \in C. \end{cases} \tag{1.4}$$

For such mappings F_1 and F_2 and two given positive constants $\lambda, \mu > 0$, Plubtieng and Sombut [6] considered the following system of mixed equilibrium problem, finding $(x', y') \in C \times C$ such that

$$(\mathrm{SMEP})\begin{cases} F_1(x',z) + \varphi(z) - \varphi(x') + \dfrac{1}{\lambda}\langle y' - x', x' - z\rangle \geq 0, & \forall z \in C, \\ F_2(y',z) + \varphi(z) - \varphi(y') + \dfrac{1}{\mu}\langle x' - y', y' - z\rangle \geq 0, & \forall z \in C. \end{cases} \tag{1.5}$$

In particular, if $\lambda = \mu$ and $\varphi \equiv 0$, then problem (SMEP) reduces to (SEP). Furthermore, Plubtieng and Sombut [6] introduced the following iterative procedure to approximate a common element of the fixed-point set of a quasi-nonexpansive mapping T and the solution set of (SMEP) in a Hilbert space H. Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be given by

$$\begin{gathered} x_1 \in C \text{ chosen arbitrary}, \\ u_n \in C, \quad F_2(u_n,z) + \varphi(z) - \varphi(u_n) + \frac{1}{\mu}\langle z - u_n, u_n - x_n\rangle \geq 0, \quad \forall z \in C, \\ y_n \in C, \quad F_1(y_n,z) + \varphi(z) - \varphi(y_n) + \frac{1}{\lambda}\langle z - y_n, y_n - u_n\rangle \geq 0, \quad \forall z \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{1.6}$$

where $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,1)$ and satisfying appropriate conditions. The weak convergence theorems are obtained in a real Hilbert space.

On the other hand, in 1953, Mann [7] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N}, \tag{1.7}$$

where the initial point x_0 is taken in C arbitrarily, and $\{\alpha_n\}$ is a sequence in $[0,1]$.

For two nonexpansive mappings T_1, T_2 of C into itself, Moudafi [4] studied weak convergence theorems in the following iterative process:

$$\begin{gathered} x_0 \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\left(\beta_n T_1 x_n + \left(1 - \beta_n\right) T_2 x_n\right), \end{gathered} \tag{1.8}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0,1]$ and $F(T_1) \cap F(T_2) \neq \emptyset$. Recently, Iemoto and Takahashi [8] also considered this iterative procedure for T_1 is a nonexpansive mapping and $T_2 : C \to C$ is a nonspreading mapping. Very recently, Kim [9] studied the weak and strong convergence for the Moudafi's iterative scheme (1.8) of two quasi-nonexpansive mappings.

In this paper, inspired and motivated by Plubtieng and Sombut [6], Moudafi [4], Iemoto and Takahashi [8], and Kim [9], we first introduce the iterative procedure to approximate a common element of the common fixed point set of two quasi-nonexpansive mappings and the solution set of SMEP in a real Hilbert space. Next, we prove the weak convergence theorem for the given iterative scheme under certain assumptions. Finally, we apply our results to approximate a common element of the set of common fixed point of asymptotic nonspreading mapping and asymptotic TJ mapping and the solution set of SMEP in a real Hilbert space.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers, and let $\mathbb{R}$ be the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively, and let C be a closed convex subset of H. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

From [10], for each $x, y \in H$ and $\lambda \in [0,1]$, we have

$$\left\|\lambda x + (1-\lambda) y\right\|^2 = \lambda \|x\|^2 + (1-\lambda)\left\|y\right\|^2 - \lambda(1-\lambda)\left\|x - y\right\|^2. \tag{2.1}$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \left\|x - y\right\| \quad \forall y \in C. \tag{2.2}$$

P_C is called the metric projection of H onto C. It is well know that P_C is a nonexpansive mapping of H onto C and satisfies

$$\left\|P_C x - P_C y\right\|^2 \le \left\langle P_C x - P_C y, x - y\right\rangle \quad \forall x, y \in H. \tag{2.3}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$,

$$\langle x - P_C x, P_C y - y \rangle \geq 0,$$
$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C y\|^2 \quad \forall x \in H,\ y \in C. \tag{2.4}$$

Further, for all $x \in H$ and $y \in C$, $y = P_C x$ if and only if $\langle x - y, y - z \rangle \geq 0$, for all $z \in C$.

Lemma 2.1 (see [11]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let P be the metric projection of H onto C, and let $\{x_n\}_{n\in\mathbb{N}}$ be in H. If*

$$\|x_{n+1} - u\| \leq \|x_n - u\|, \tag{2.5}$$

for all $u \in C$ and $n \in \mathbb{N}$. Then, $\{P_C x_n\}$ converges strongly to an element of C.

Theorem 2.2 (Opial's theorem, [10]). *Let H be a real Hilbert space, and suppose that $x_n \rightharpoonup x$, then*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \tag{2.6}$$

for all $y \in H$ with $x \neq y$.

All Hilbert space and l^p $(1 < p < \infty)$ satisfy Opial's condition, while L^p with $1 < p \neq 2 < \infty$ do not.

For solving the mixed equilibrium problem for an equilibrium bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$,

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$,

(A3) for each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous,

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and semicontinuous.

The following lemma appears implicitly in [3, 12].

Lemma 2.3 (see [3]). *Let C be a nonempty closed convex subset of H, and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $r > 0$ and $x \in H$, then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \quad \forall y \in C. \tag{2.7}$$

The following lemma was also given in [12].

Lemma 2.4 (see [12]). *Let C be a nonempty closed closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$–$(A4)$, then, for any $r > 0$ and $x \in H$, define a mapping $T_r x : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0,\ \forall\ y \in C \right\}, \tag{2.8}$$

for all $z \in H, r \in \mathbb{R}$. *Then the following hold:*

(i) T_r *is single valued,*

(ii) T_r *is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \quad \forall x, y \in H; \tag{2.9}$$

(iii) $F(T_r) = \mathrm{EP}(F)$,

(iv) $\mathrm{EP}(F)$ *is closed and convex.*

We note that Lemma 2.4 is equivalent to the following lemma.

Lemma 2.5 (see [6]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $F : C \times C \to \mathbb{R}$ *be an equilibrium bifunction satisfying (A1)–(A4) and let* $\varphi : C \to \mathbb{R}$ *be a lower semicontinuous and convex functional. For each* $r > 0$ *and* $x \in H$, *define a mapping*

$$S_r(x) = \left\{ y \in C : F(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y \rangle \geq 0, \forall z \in C \right\}, \quad \forall x \in H. \tag{2.10}$$

Then, the following results hold:

(i) *for each* $x \in C$, $S_r(x) \neq \emptyset$,

(ii) S_r *is single valued,*

(iii) S_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y \rangle, \tag{2.11}$$

(iv) $F(S_r) = \mathrm{MEP}(F, \varphi)$,

(v) $\mathrm{MEP}(F, \varphi)$ *is closed and convex.*

3. Main Results

In this section, we prove the weak convergence for approximating a common element of the common fixed point set of two quasi-nonexpansive mappings and the solution set of the system of mixed equilibrium problems in a Hilbert space.

To begin with, let us state and proof the following characterizations of the solution set of GMEP.

Lemma 3.1. *Let* C *be a closed convex subset of a real Hilbert space* H. *Let* F_1 *and* F_2 *be two mappings from* $C \times C \to \mathbb{R}$ *satisfying (A1)–(A4), and let* $S_{1,\lambda}$ *and* $S_{2,\mu}$ *be defined as in Lemma 2.5 associated to* F_1 *and* F_2, *respectively. For given* $x', y' \in C$, (x', y') *is a solution of problem* (1.5) *if and only if* x' *is a fixed point of the mapping* $G : C \to C$ *defined by*

$$G(x) = S_{1,\lambda}(S_{2,\mu} x), \quad \forall x \in C, \tag{3.1}$$

where $y' = S_{2,\mu} x'$.

Proof. For given $x', y' \in C$, we observe the following equivalency:

$$
\begin{aligned}
(x', y') \text{ is a solution of problem (1.5)} &\Longleftrightarrow \begin{cases} F_1(x', z) + \varphi(z) - \varphi(x') + \dfrac{1}{\lambda}\langle y' - x', x' - z\rangle \ge 0 \\ \hfill \forall z \in C, \\ F_2(y', z) + \varphi(z) - \varphi(y') + \dfrac{1}{\mu}\langle x' - y', y' - z\rangle \ge 0 \\ \hfill \forall z \in C, \end{cases} \\
&\Longleftrightarrow x' = S_{1,\lambda} y' \qquad y' = S_{2,\mu} x', \\
&\Longleftrightarrow x' = S_{1,\lambda}(S_{2,\mu} x'), \\
&\Longleftrightarrow x' = Gx'.
\end{aligned}
\tag{3.2}
$$

This completes the proof. □

We note from Lemma 3.1 that the mapping G is nonexpansive. Moreover, if C is a closed bounded convex subset of H, then the solution of problem (1.5) always exists. Throughout this paper, we denote the set of fixed points of G by Ω.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4). Let $r, \lambda > 0$ and $S_{1,r}$ and $S_{2,\lambda}$ be defined as in Lemma 2.5 associated to F_1 and F_2, respectively. Let $T_i : C \to C$, $i = 1, 2$, be two quasi-nonexpansive mappings such that $I - T_i$ are demiclosed at zero, that is, if $\{w_n\} \subset C$, $w_n \rightharpoonup w$, and $(I - T_i)w_n \to 0$, then $w \in F(T_i)$, with $F(T_1) \cap F(T_2) \cap \Omega \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$, be given by*

$$
\begin{gathered}
x_1 \in C \text{ chosen arbitrary}, \\
u_n \in C, \quad F_2(u_n, z) + \varphi(z) - \varphi(u_n) + \frac{1}{\lambda}\langle z - u_n, u_n - x_n\rangle \ge 0, \quad \forall z \in C, \\
y_n \in C, \quad F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{r}\langle z - y_n, y_n - u_n\rangle \ge 0, \quad \forall z \in C, \\
x_{n+1} = a_n x_n + (1 - a_n)(b_n T_1 y_n + (1 - b_n) T_2 y_n), \quad \forall n \in \mathbb{N},
\end{gathered}
\tag{3.3}
$$

where $\{a_n\}$, $\{b_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, and satisfy

$$
\liminf_{n\to\infty} a_n(1 - a_n) > 0, \qquad \liminf_{n\to\infty} b_n(1 - b_n) > 0,
\tag{3.4}
$$

then $x_n \rightharpoonup \overline{x} := \lim_{n\to\infty} P_{F(T_1)\cap F(T_2)\cap\Omega} x_n$ and $(\overline{x}, \overline{y})$ is a solution of problem (1.5), where $\overline{y} = S_{2,\lambda}\overline{x}$.

Proof. Let $x^* \in F(T_1) \cap F(T_2) \cap \Omega$, then $x^* = T_1 x^* = T_2 x^*$ and $x^* = S_{1,r}(S_{2,\lambda} x^*)$.

Putting $y^* = S_{2,\lambda} x^*$, $y_n = S_{1,r} u_n$, and $u_n = S_{2,\lambda} x_n$, we have

$$
\|y_n - x^*\| = \|S_{1,r} u_n - S_{1,r} y^*\| \le \|u_n - y^*\| = \|S_{2,\lambda} x_n - S_{2,\lambda} x^*\| \le \|x_n - x^*\|.
\tag{3.5}
$$

Next, we prove that

$$\lim_{n\to\infty} \|x_n - x^*\| \text{ exists.} \tag{3.6}$$

Since T_1 and T_2 are quasi-nonexpansive, we obtain that

$$\begin{aligned}
\left\|b_n T_1 y_n + (1-b_n) T_2 y_n - x^*\right\|^2 &= \left\|b_n (T_1 y_n - x^*) + (1-b_n)(T_2 y_n - x^*)\right\|^2 \\
&= b_n \left\|T_1 y_n - x^*\right\|^2 + (1-b_n)\left\|T_2 y_n - x^*\right\|^2 \\
&\quad - b_n(1-b_n)\left\|T_1 y_n - T_2 y_n\right\|^2 \\
&\le b_n \left\|T_1 y_n - x^*\right\|^2 + (1-b_n)\left\|T_2 y_n - x^*\right\|^2 \\
&\le b_n \left\|y_n - x^*\right\|^2 + (1-b_n)\left\|y_n - x^*\right\|^2 \\
&= \left\|y_n - x^*\right\|^2 \\
&\le \|x_n - x^*\|^2,
\end{aligned} \tag{3.7}$$

which gives that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|a_n x_n + (1-a_n)(b_n T_1 y_n + (1-b_n) T_2 y_n) - x^*\right\|^2 \\
&= \left\|a_n (x_n - x^*) + (1-a_n)(b_n T_1 y_n + (1-b_n) T_2 y_n - x^*)\right\|^2 \\
&= a_n \|x_n - x^*\|^2 + (1-a_n)\left\|b_n T_1 y_n + (1-b_n) T_2 y_n - x^*\right\|^2 \\
&\quad - a_n(1-a_n)\left\|x_n - (b_n T_1 y_n + (1-b_n) T_2 y_n)\right\|^2 \\
&\le a_n \|x_n - x^*\|^2 + (1-a_n)\|x_n - x^*\|^2 \\
&\quad - a_n(1-a_n)\left\|x_n - (b_n T_1 y_n + (1-b_n) T_2 y_n)\right\|^2 \\
&= \|x_n - x^*\|^2 - a_n(1-a_n)\left\|x_n - (b_n T_1 y_n + (1-b_n) T_2 y_n)\right\|^2 \\
&\le \|x_n - x^*\|^2.
\end{aligned} \tag{3.8}$$

Hence, $\{\|x_n - x^*\|\}$ is a nonincreasing sequence, and hence, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. This implies that $\{x_n\}, \{y_n\}, \{u_n\}, \{T_1 y_n\}$, and $\{T_2 y_n\}$ are bounded. From (3.8), we have

$$a_n(1-a_n)\left\|x_n - (b_n T_1 y_n + (1-b_n) T_2 y_n)\right\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.9}$$

Since $\liminf_{n\to\infty} a_n(1-a_n) > 0$, this implies that

$$\lim_{n\to\infty} \left\|x_n - \left(b_n T_1 y_n + (1-b_n) T_2 y_n\right)\right\| = 0. \tag{3.10}$$

Furthermore, since $0 < a \leq a_n \leq b < 1$, we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \left\|a_n x_n + (1-a_n)\left(b_n T_1 y_n + (1-b_n)T_2 y_n\right) - x_n\right\| \\ &= \left\|(1-a_n)\left(b_n T_1 y_n + (1-b_n)T_2 y_n\right) - (1-a_n)x_n\right\| \\ &= (1-a_n)\left\|b_n T_1 y_n + (1-b_n)T_2 y_n - x_n\right\| \\ &\leq (1-a)\left\|b_n T_1 y_n + (1-b_n)T_2 y_n - x_n\right\|.\end{aligned} \tag{3.11}$$

From (3.10), we conclude that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

From (3.7), we have

$$\begin{aligned}b_n(1-b_n)\left\|T_1 y_n - T_2 y_n\right\|^2 &= b_n\left\|T_1 y_n - x^*\right\|^2 + (1-b_n)\left\|T_2 y_n - x^*\right\|^2 \\ &\quad - \left\|b_n T_1 y_n + (1-b_n)T_2 y_n - x^*\right\|^2 \\ &\leq b_n\left\|y_n - x^*\right\|^2 + (1-b_n)\left\|y_n - x^*\right\|^2 \\ &\quad - \left\|b_n T_1 y_n + (1-b_n)T_2 y_n - x^*\right\|^2 \\ &= \left\|y_n - x^*\right\|^2 - \left\|b_n T_1 y_n + (1-b_n)T_2 y_n - x^*\right\|^2 \\ &\leq \|x_n - x^*\|^2 - \left\|b_n T_1 y_n + (1-b_n)T_2 y_n - x^*\right\|^2 \\ &= \left(\|x_n - x^*\| - \left\|b_n T_1 y_n + (1-b_n)T_2 y_n - x^*\right\|\right) \\ &\quad \times \left(\|x_n - x^*\| + \left\|b_n T_1 y_n + (1-b_n)T_2 y_n - x^*\right\|\right) \\ &\leq M\left(\|x_n - x^*\| - \left\|b_n T_1 y_n + (1-b_n)T_2 y_n - x^*\right\|\right) \\ &\leq M\left(\left\|x_n - \left(b_n T_1 y_n + (1-b_n)T_2 y_n\right)\right\|\right),\end{aligned} \tag{3.13}$$

where M is a constant satisfying $M \geq \sup_{n\geq 1}[\|x_n - x^*\| + \|b_n T_1 y_n + (1-b_n)T_2 y_n - x^*\|]$. Again from (3.10), we conclude that

$$\lim_{n\to\infty} b_n(1-b_n)\left\|T_1 y_n - T_2 y_n\right\| = 0. \tag{3.14}$$

Using $\liminf_{n\to\infty} b_n(1-b_n) > 0$, we have

$$\lim_{n\to\infty} \left\|T_1 y_n - T_2 y_n\right\| = 0. \tag{3.15}$$

Now, we prove that

$$\lim_{n\to\infty} \left\|x_{n+1} - T_1 y_n\right\| = 0, \qquad \lim_{n\to\infty} \left\|x_{n+1} - T_2 y_n\right\| = 0. \tag{3.16}$$

We observe that

$$\begin{aligned}\|x_{n+1}-T_1y_n\| &= \|a_nx_n+(1-a_n)(b_nT_1y_n+(1-b_n)T_2y_n)-T_1y_n\|\\ &= \|a_nx_n+(1-a_n)(b_nT_1y_n+(1-b_n)T_2y_n)-(a_n+(1-a_n))T_1y_n)\|\\ &= \|a_n(x_n-T_1y_n)+(1-a_n)(b_nT_1y_n+(1-b_n)T_2y_n-T_1y_n)\|\\ &= \|a_n(x_n-T_1y_n)+(1-a_n)(b_nT_1y_n+T_2y_n-b_nT_2y_n-T_1y_n)\|\\ &= \|a_n(x_n-T_1y_n)+(1-a_n)((1-b_n)T_2y_n-(1-b_n)T_1y_n)\|\\ &= \|a_n(x_n-T_1y_n)+(1-a_n)(1-b_n)(T_2y_n-T_1y_n)\|\\ &\le a_n\|x_n-T_1y_n\|+(1-a_n)(1-b_n)\|T_2y_n-T_1y_n\|\\ &= a_n\|x_n-x_{n+1}+x_{n+1}-T_1y_n\|+(1-a_n)(1-b_n)\|T_2y_n-T_1y_n\|\\ &\le a_n\|x_n-x_{n+1}\|+a_n\|x_{n+1}-T_1y_n\|+(1-a_n)(1-b_n)\|T_2y_n-T_1y_n\|,\end{aligned}\tag{3.17}$$

which gives that

$$(1-a_n)\|x_{n+1}-T_1y_n\|\le a_n\|x_n-x_{n+1}\|+(1-a_n)(1-b_n)\|T_2y_n-T_1y_n\|.\tag{3.18}$$

Since $0<a\le a_n\le b<1$, we have

$$\begin{aligned}a_n(1-a_n)\|x_{n+1}-T_1y_n\| &\le (1-a_n)\|x_{n+1}-T_1y_n\|\\ &\le a_n\|x_n-x_{n+1}\|+(1-a_n)(1-b_n)\|T_2y_n-T_1y_n\|.\end{aligned}\tag{3.19}$$

Using (3.12) and (3.15), we conclude that

$$\lim_{n\to\infty}a_n(1-a_n)\|x_{n+1}-T_1y_n\|=0,\tag{3.20}$$

which gives that

$$\lim_{n\to\infty}\|x_{n+1}-T_1y_n\|=0,\tag{3.21}$$

since $\liminf_{n\to\infty}a_n(1-a_n)>0$. Similarly, we have

$$\begin{aligned}\|x_{n+1}-T_2y_n\| &= \|a_nx_n+(1-a_n)(b_nT_1y_n+(1-b_n)T_2y_n)-T_2y_n\|\\ &= \|a_n(x_n-T_2y_n)+(1-a_n)(b_nT_1y_n+(1-b_n)T_2y_n-T_2y_n)\|\\ &= \|a_n(x_n-T_2y_n)+(1-a_n)b_n(T_1y_n-T_2y_n)\|\\ &\le a_n\|x_n-T_2y_n\|+(1-a_n)b_n\|T_1y_n-T_2y_n\|\\ &= a_n\|x_n-x_{n+1}+x_{n+1}-T_2y_n\|+(1-a_n)b_n\|T_2y_n-T_1y_n\|\\ &\le a_n\|x_n-x_{n+1}\|+a_n\|x_{n+1}-T_2y_n\|+(1-a_n)b_n\|T_2y_n-T_1y_n\|,\end{aligned}\tag{3.22}$$

which implies that

$$(1-a_n)\left\|x_{n+1}-T_2y_n\right\| \le a_n\|x_n-x_{n+1}\| + (1-a_n)b_n\left\|T_2y_n-T_1y_n\right\|. \tag{3.23}$$

Thus, we have

$$\begin{aligned} a_n(1-a_n)\left\|x_{n+1}-T_2y_n\right\| &\le (1-a_n)\left\|x_{n+1}-T_2y_n\right\| \\ &\le a_n\|x_n-x_{n+1}\| + (1-a_n)b_n\left\|T_2y_n-T_1y_n\right\| \\ &\le b\|x_n-x_{n+1}\| + (1-a_n)\left\|T_2y_n-T_1y_n\right\| \\ &\le b\|x_n-x_{n+1}\| + (1-a)\left\|T_2y_n-T_1y_n\right\|. \end{aligned} \tag{3.24}$$

Hence, $\lim_{n\to\infty} a_n(1-a_n)\|x_{n+1}-T_2y_n\| = 0$. Since $\liminf_{n\to\infty} a_n(1-a_n) > 0$, we have

$$\lim_{n\to\infty}\left\|x_{n+1}-T_2y_n\right\| = 0. \tag{3.25}$$

Next, we prove that

$$\lim_{n\to\infty}\left\|y_n-T_1y_n\right\| = 0, \qquad \lim_{n\to\infty}\left\|y_n-T_2y_n\right\| = 0. \tag{3.26}$$

Since $S_{1,r}$ and $S_{2,\lambda}$ are firmly nonexpansive, it follows that

$$\left\|u_n-y^*\right\|^2 = \|S_{2,\lambda}x_n - S_{2,\lambda}x^*\|^2 \le \langle S_{2,\lambda}x_n - S_{2,\lambda}x^*, x_n-x^*\rangle = \langle u_n-y^*, x_n-x^*\rangle, \tag{3.27}$$

which gives that

$$\begin{aligned} \left\|(u_n-y^*)-(x_n-x^*)\right\|^2 &= \left\|u_n-y^*\right\|^2 - 2\langle u_n-y^*, x_n-x^*\rangle + \|x_n-x^*\|^2 \\ &\le \left\|u_n-y^*\right\|^2 - 2\left\|u_n-y^*\right\|^2 + \|x_n-x^*\|^2 \\ &= -\left\|u_n-y^*\right\|^2 + \|x_n-x^*\|^2. \end{aligned} \tag{3.28}$$

This implies that

$$\left\|u_n-y^*\right\|^2 \le \|x_n-x^*\|^2 - \left\|(u_n-y^*)-(x_n-x^*)\right\|^2. \tag{3.29}$$

By the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \|x_{n+1}-x^*\|^2 &= \left\|a_nx_n + (1-a_n)(b_nT_1y_n + (1-b_n)T_2y_n) - x^*\right\|^2 \\ &= \left\|a_n(x_n-x^*) + (1-a_n)(b_nT_1y_n + (1-b_n)T_2y_n - x^*)\right\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq a_n\|x_n - x^*\|^2 + (1-a_n)\left\|b_nT_1y_n + (1-b_n)T_2y_n - x^*\right\|^2 \\
&\leq a_n\|x_n - x^*\|^2 + (1-a_n)\left\|y_n - x^*\right\|^2 \\
&\leq a_n\|x_n - x^*\|^2 + (1-a_n)\left\|u_n - y^*\right\|^2 \\
&\leq a_n\|x_n - x^*\|^2 + (1-a_n)\left(\|x_n - x^*\|^2 - \left\|(u_n - y^*) - (x_n - x^*)\right\|^2\right) \\
&= \|x_n - x^*\|^2 - (1-a_n)\left\|(u_n - y^*) - (x_n - x^*)\right\|^2.
\end{aligned} \tag{3.30}$$

Thus,

$$(1-a_n)\left\|(u_n - y^*) - (x_n - x^*)\right\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.31}$$

Since $0 < a \leq a_n \leq b < 1$, we have

$$(1-b)\left\|(u_n - y^*) - (x_n - x^*)\right\|^2 \leq (1-a_n)\left\|(u_n - y^*) - (x_n - x^*)\right\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.32}$$

Since $\lim_{n\to\infty}\|x_n - x^*\|$ exists, we have

$$\lim_{n\to\infty}\left\|u_n - x_n + x^* - y^*\right\| = 0. \tag{3.33}$$

Similarly, we have

$$\left\|y_n - x^*\right\|^2 = \left\|S_{1,r_n}u_n - S_{1,r_n}y^*\right\|^2 \leq \langle S_{1,r_n}u_n - S_{1,r_n}y^*, u_n - y^*\rangle = \langle y_n - x^*, u_n - x^*\rangle, \tag{3.34}$$

which gives that

$$\begin{aligned}
\left\|(y_n - x^*) - (u_n - y^*)\right\|^2 &= \left\|y_n - x^*\right\|^2 - 2\langle y_n - x^*, u_n - x^*\rangle + \left\|u_n - y^*\right\|^2 \\
&\leq \left\|y_n - x^*\right\|^2 - 2\left\|y_n - x^*\right\|^2 + \left\|u_n - y^*\right\|^2 \\
&= -\left\|y_n - x^*\right\|^2 + \left\|u_n - y^*\right\|^2.
\end{aligned} \tag{3.35}$$

This implies that

$$\left\|y_n - x^*\right\|^2 \leq \left\|u_n - y^*\right\|^2 - \left\|y_n - u_n - x^* + y^*\right\|^2. \tag{3.36}$$

By the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|a_nx_n + (1-a_n)(b_nT_1y_n + (1-b_n)T_2y_n) - x^*\right\|^2 \\
&\leq a_n\|x_n - x^*\|^2 + (1-a_n)\left\|y_n - x^*\right\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq a_n\|x_n - x^*\|^2 + (1-a_n)\left(\|u_n - y^*\|^2 - \|y_n - u_n - x^* + y^*\|^2\right)\\
&\leq a_n\|x_n - x^*\|^2 + (1-a_n)\|x_n - x^*\|^2 - (1-a_n)\|y_n - u_n - x^* + y^*\|^2\\
&= \|x_n - x^*\|^2 - (1-a_n)\|(y_n - x^*) - (u_n - y^*)\|^2.
\end{aligned} \tag{3.37}$$

Thus,

$$(1-a_n)\|y_n - u_n - x^* + y^*\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.38}$$

Since $0 < a \leq a_n \leq b < 1$, we have

$$(1-b)\|y_n - u_n - x^* + y^*\|^2 \leq (1-a_n)\|y_n - u_n - x^* + y^*\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.39}$$

Since $\lim_{n\to\infty}\|x_n - x^*\|$ exists, we have

$$\lim_{n\to\infty}\|y_n - u_n - x^* + y^*\| = 0. \tag{3.40}$$

Hence,

$$\begin{aligned}
&\|(b_nT_1y_n + (1-b_n)T_2y_n) - y_n\|\\
&\quad = \|(b_nT_1y_n + (1-b_n)T_2y_n) - x_n + x_n - u_n + u_n - y^* + y^* - x^* + x^* - y_n\|\\
&\quad \leq \|(b_nT_1y_n + (1-b_n)T_2y_n) - x_n\| + \|x_n - u_n - x^* + y^*\| + \|u_n - y_n + x^* - y^*\|.
\end{aligned} \tag{3.41}$$

It follow from (3.10), (3.33), and (3.40) that

$$\lim_{n\to\infty}\|(b_nT_1y_n + (1-b_n)T_2y_n) - y_n\| = 0, \tag{3.42}$$

from which it follows that

$$\begin{aligned}
\|y_n - x_n\| &= \|y_n - (b_nT_1y_n + (1-b_n)T_2y_n) + (b_nT_1y_n + (1-b_n)T_2y_n) - x_n\|\\
&\leq \|y_n - (b_nT_1y_n + (1-b_n)T_2y_n)\|\\
&\quad + \|(b_nT_1y_n + (1-b_n)T_2y_n) - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{3.43}$$

that is,

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.44}$$

Thus,

$$\|y_n - T_1y_n\| \leq \|y_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - T_1y_n\| \longrightarrow 0. \tag{3.45}$$

Similarly, we have $\|y_n - T_2 y_n\| \to 0$. Since $\{y_n\}$ is bounded sequence, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup \overline{x}$ as $i \to \infty$. Since T_1 and T_2 are demiclosed at 0, we conclude that $\overline{x} \in F(T_1) \cap F(T_2)$. Let G be a mapping which is defined as in Lemma 3.1. Thus, we have

$$\|y_n - G(y_n)\| = \|S_{1,r} S_{2,\lambda} x_n - G(y_n)\| = \|G(x_n) - G(y_n)\| \le \|x_n - y_n\|, \tag{3.46}$$

and hence,

$$\begin{aligned} \|x_n - G(x_n)\| &= \|x_n - y_n + y_n - G(y_n) + G(y_n) - G(x_n)\| \\ &\le \|x_n - y_n\| + \|y_n - G(y_n)\| + \|G(y_n) - G(x_n)\| \\ &\le \|x_n - y_n\| + \|x_n - y_n\| + \|y_n - x_n\| \\ &= 3\|x_n - y_n\| \longrightarrow 0. \end{aligned} \tag{3.47}$$

This together with $x_{n_i} \rightharpoonup \overline{x}$ implies that $\overline{x} \in F(G) := \Omega$, if $\{y_{n_j}\}$ is another subsequence of $\{y_n\}$ such that $y_{n_j} \rightharpoonup \widehat{x}$ as $i \to \infty$. Since T_1 and T_1 are demiclosed at 0, we conclude that $\widehat{x} \in F(T_1) \cap F(T_2) \cap \Omega$. From $x_{n_i} \rightharpoonup \overline{x}$ and $x_{n_j} \rightharpoonup \widehat{x}$, we will show that $\overline{x} = \widehat{x}$. Assume that $\overline{x} \neq \widehat{x}$. Since $\lim_{n\to\infty}\|x_n - x^*\|$ exists for all $x^* \in F(T_1) \cap F(T_2) \cap \Omega$, by Opial's Theorem 2.2, we have

$$\begin{aligned} \lim_{n\to\infty}\|x_n - \overline{x}\| &= \liminf_{i\to\infty}\|x_{n_i} - \overline{x}\| \\ &< \liminf_{i\to\infty}\|x_{n_i} - \widehat{x}\| \\ &= \lim_{n\to\infty}\|x_n - \widehat{x}\| \\ &= \liminf_{j\to\infty}\left\|x_{n_j} - \widehat{x}\right\| \\ &< \liminf_{j\to\infty}\left\|x_{n_j} - \overline{x}\right\| \\ &= \lim_{n\to\infty}\|x_n - \overline{x}\|. \end{aligned} \tag{3.48}$$

This is a contradiction. Thus, we have $\overline{x} = \widehat{x}$. This implies that $y_n \rightharpoonup \overline{x} \in F(T_1) \cap F(T_2) \cap \Omega$. Since $\|x_n - y_n\| \to 0$, we have $x_n \rightharpoonup \overline{x}$. Put $z_n = P_{F(T_1)\cap F(T_2)\cap\Omega} x_n$. Finally, we show that $\overline{x} = \lim_{n\to\infty} z_n$. Now from (2.4) and $\overline{x} \in F(T_1) \cap F(T_2) \cap \Omega$, we have

$$\langle \overline{x} - z_n, z_n - x_n \rangle \ge 0. \tag{3.49}$$

Since $\{\|x_n - x^*\|\}$ is nonnegative and nonincreasing for all $x^* \in F(T_1) \cap F(T_2) \cap \Omega$, it follows by Lemma 2.1 that $\{z_n\}$ converges strongly to some $\widehat{x} \in F(T_1) \cap F(T_2) \cap \Omega$. By (3.49), we have

$$\langle \overline{x} - \widehat{x}, \widehat{x} - \overline{x} \rangle \ge 0. \tag{3.50}$$

Therefore, $\overline{x} = \widehat{x}$. □

Setting $T := T_1 = T_2$ in Theorem 3.2, we have the following result.

Corollary 3.3 (see [6]). *Let C be a closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4). Let $\lambda, \mu > 0$, and let $S_{1,\lambda}$ and $S_{2,\mu}$ be defined as in Lemma 2.5 associated to F_1 and F_2, respectively. Let $T : C \to C$ be a quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero and $F(T) \cap \Omega \neq \emptyset$. Suppose that $x_0 = x \in C$ and $\{x_n\}, \{y_n\}$, and $\{z_n\}$ are given by*

$$\begin{gathered} z_n \in C, \quad F_2(z_n, z) + \varphi(z) - \varphi(z_n) + \frac{1}{\mu}\langle z - z_n, z_n - x_n\rangle \geq 0, \quad \forall z \in C, \\ y_n \in C, \quad F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{\lambda}\langle z - y_n, y_n - z_n\rangle \geq 0, \quad \forall z \in C, \\ x_{n+1} = a_n x_n + (1 - a_n) T y_n, \end{gathered} \tag{3.51}$$

for all $n \in \mathbb{N}$, where $\{a_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, and satisfy $\liminf_{n\to\infty} a_n(1 - a_n) > 0$, then $\{x_n\}$ converges weakly to $\overline{x} = \lim_{n\to\infty} P_{F(T)\cap\Omega} x_n$ and $(\overline{x}, \overline{y})$ is a solution of problem (1.5)*, where $\overline{y} = S_{2,\mu}\overline{x}$.*

Setting $F_1 = F_2 \equiv 0$, $\varphi \equiv 0$ in Theorem 3.2, we have the following result.

Corollary 3.4 (see [9]). *Let H be a Hilbert space, let C be a nonempty, closed, and convex subset of H, and let T_1, T_2 be two quasi-nonexpansive mappings of C into itself such that $I - T_1$, $I - T_2$ are demiclosed at zero with $F(T_1) \cap F(T_2) \neq \emptyset$. For any x_1 in C, let $\{x_n\}$ be defined by*

$$x_{n+1} = (1 - a_n)x_n + a_n(b_n T_1 x_n + (1 - b_n)T_2 x_n), \tag{3.52}$$

where $\{a_n\}$ and $\{b_n\}$ are chosen so that

$$\liminf_{n\to\infty} a_n(1 - a_n) > 0 \qquad \liminf_{n\to\infty} b_n(1 - b_n) > 0, \tag{3.53}$$

then $x_n \rightharpoonup p \in F(T_1) \cap F(T_2)$.

4. Applications

In this section, we apply our results to approximate a common element of the set of common fixed points of an asymptotic nonspreading mapping and an asymptotic TJ mapping and the solution set of SMEP in a real Hilbert space. We recall the following definitions. A mapping $T : C \to C$ is called nonspreading [13] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \quad \forall x, y \in C. \tag{4.1}$$

Furthermore, Takahashi and Yao [14] also introduced two nonlinear mappings in Hilbert spaces. A mapping $T : C \to C$ is called a $TJ - 1$ mapping [14] if

$$2\|Tx - Ty\|^2 = \|x - y\|^2 + \|Tx - y\|^2, \tag{4.2}$$

for all $x, y \in C$. A mapping $T : C \to C$ is called a $TJ - 2$ [14] mapping if

$$3\|Tx - Ty\|^2 = 2\|Tx - y\|^2 + \|Ty - x\|^2, \tag{4.3}$$

for all $x, y \in C$. For these two nonlinear mappings, $TJ - 1$ and $TJ - 2$ mappings, Takahashi and Yao [14] studied the existence results of fixed points in Hilbert spaces. Very recently, Lin et al. [15] introduced the following definitions of new mappings.

Definition 4.1. Let C be a nonempty closed convex subset of a Hilbert space H. We say that $T : C \to C$ is an asymptotic nonspreading mapping if there exist two functions $\alpha : C \to [0, 2)$ and $\beta : C \to [0, k]$, $k < 2$, such that

(A1) $2\|Tx - Ty\|^2 \le \alpha(x)\|Tx - y\|^2 + \beta(x)\|Ty - x\|^2$ for all $x, y \in C$,

(A2) $0 < \alpha(x) + \beta(x) \le 2$ for all $x \in C$.

Remark 4.2. The class of asymptotic nonspreading mappings contains the class of nonspreading mappings and the class of $TJ - 2$ mappings in a Hilbert space. Indeed, in Definition 4.1, we know that

(i) if $\alpha(x) = \beta(x) = 1$ for all $x \in C$, then T is a nonspreading mapping,

(ii) if $\alpha(x) = 4/3$ and $\beta(x) = 2/3$ for all $x \in C$, then T is a $TJ - 2$ mapping.

Definition 4.3. Let C be a nonempty closed convex subset of a Hilbert space H. We say $T : C \to C$ is an asymptotic TJ mapping if there exists two functions $\alpha : C \to [0, 2]$ and $\beta : C \to [0, k]$, $k < 2$, such that

(B1) $2\|Tx - Ty\|^2 \le \alpha(x)\|x - y\|^2 + \beta(x)\|Tx - y\|^2$ for all $x, y \in C$,

(B2) $\alpha(x) + \beta(x) \le 2$ for all $x \in C$.

Remark 4.4. The class of asymptotic TJ mappings contains the class of $TJ - 1$ mappings and the class of nonexpansive mappings in a Hilbert space. Indeed, in Definition 4.3, we know that

(i) if $\alpha(x) = 2$ and $\beta(x) = 0$ for each $x \in C$, then T is a nonexpansive mapping,

(ii) if $\alpha(x) = \beta(x) = 1$ for each $x \in C$, then T is a $TJ - 1$ mapping.

It is well known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is a closed and convex set [1]. Hence, if $T : C \to C$ is an asymptotic nonspreading mapping (resp., asymptotic TJ mapping) with $F(T) \ne \emptyset$, then T is a quasi-nonexpansive mapping, and this implies that $F(T)$ is a nonempty closed convex subset of C.

Theorem 4.5 (see [15]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an asymptotic nonspreading mapping, then $I - T$ is demiclosed at 0.*

Theorem 4.6 (see [15]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an asymptotic TJ mapping, then $I - T$ is demiclosed at 0.*

Applying the above results, we have the following theorem.

Theorem 4.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4). Let $r, \lambda > 0$ and $S_{1,r}$ and $S_{2,\lambda}$ be defined as in Lemma 2.5 associated to F_1 and F_2, respectively. Let $T_i : C \to C$, $i = 1,2$, be any one of asymptotic nonspreading mapping and asymptotic TJ mapping such that $F(T_1) \cap F(T_2) \cap \Omega \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be given by*

$$\begin{gathered}x_1 \in C \text{ chosen arbitrary},\\ u_n \in C, \quad F_2(u_n, z) + \varphi(z) - \varphi(u_n) + \frac{1}{\lambda}\langle z - u_n, u_n - x_n\rangle \geq 0, \quad \forall z \in C,\\ y_n \in C, \quad 2F_1(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{r}\langle z - y_n, y_n - u_n\rangle \geq 0, \quad \forall z \in C,\\ x_{n+1} = a_n x_n + (1 - a_n)\big(b_n T_1 y_n + (1 - b_n) T_2 y_n\big), \quad \forall n \in \mathbb{N},\end{gathered} \tag{4.4}$$

where $\{a_n\}, \{b_n\} \subset [a, b]$ for some $a, b \in (0,1)$ and satisfying

$$\liminf_{n\to\infty} a_n(1 - a_n) > 0, \qquad \liminf_{n\to\infty} b_n(1 - b_n) > 0, \tag{4.5}$$

then $x_n \rightharpoonup \overline{x} := \lim_{n\to\infty} P_{F(T_1)\cap F(T_2)\cap\Omega} x_n$, and $(\overline{x}, \overline{y})$ is a solution of problem (1.5), where $\overline{y} = S_{2,\lambda_n}\overline{x}$.

Setting $F_1 = F_2 := F$ and $\varphi \equiv 0$ in the above theorem, we have the following result.

Corollary 4.8. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be the bifunctions from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4). Let $T_i : C \to C$, $i = 1,2$, be any one of asymptotic nonspreading mapping and asymptotic TJ mapping such that $\mathcal{F} := F(T_1) \cap F(T_2) \cap \mathrm{EP}(F) \neq \emptyset$. For given $u \in C$ and $r > 0$, let the sequences $\{x_n\}$ and $\{u_n\}$ be defined by*

$$\begin{gathered}x_1 \in C \text{ chosen arbitrary},\\ u_n \in C, \quad F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C,\\ x_{n+1} = a_n x_n + (1 - a_n)(b_n T_1 u_n + (1 - b_n) T_2 u_n) \quad \forall n \in \mathbb{N},\end{gathered} \tag{4.6}$$

where $\{a_n\}$, $\{b_n\}$ are two sequences in $(0,1)$ satisfying

$$\liminf_{n\to\infty} a_n(1 - a_n) > 0, \qquad \liminf_{n\to\infty} b_n(1 - b_n) > 0, \tag{4.7}$$

then $x_n \rightharpoonup w$ for some $w \in \mathcal{F}$.

Setting $F_1 = F_2 \equiv 0$ and $\varphi \equiv 0$ in Theorem 4.7, we have the following result.

Corollary 4.9 (see [15]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T_i : C \to C$, $i = 1, 2$, be any one of asymptotic nonspreading mapping and asymptotic TJ mapping. Let $\mathcal{F} := F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $(0, 1)$. Let $\{x_n\}$ be defined by*

$$\begin{aligned} &x_1 \in C \text{ chosen arbitrary}, \\ &x_{n+1} = a_n x_n + (1 - a_n)(b_n T_1 x_n + (1 - b_n) T_2 x_n). \end{aligned} \tag{4.8}$$

Assume that $\liminf_{n\to\infty} a_n(1 - a_n) > 0$ and $\liminf_{n\to\infty} b_n(1 - b_n) > 0$, then $x_n \rightharpoonup w$ for some $w \in \mathcal{F}$.

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