Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 280451, 10 pages
doi:10.1155/2012/280451

Research Article

Some Iterative Methods for Solving Nonconvex Bifunction Equilibrium Variational Inequalities

Muhammad Aslam Noor,[1] Khalida Inayat Noor,[1] and Saira Zainab[2]

[1] *Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan*
[2] *Mathematics Department, HITEC University, Texila Cantt, Islamabad, Pakistan*

Correspondence should be addressed to Muhammad Aslam Noor, noormaslam@hotmail.com

Received 29 March 2012; Accepted 21 April 2012

Academic Editor: Rudong Chen

We introduce and consider a new class of equilibrium problems and variational inequalities involving bifunction, which is called the nonconvex bifunction equilibrium variational inequality. We suggest and analyze some iterative methods for solving the nonconvex bifunction equilibrium variational inequalities using the auxiliary principle technique. We prove that the convergence of implicit method requires only monotonicity. Some special cases are also considered. Our proof of convergence is very simple. Results proved in this paper may stimulate further research in this dynamic field.

1. Introduction

Variational inequalities theory, which was introduced by Stampacchia [1], can be viewed as an important and significant extension of the variational principles. This theory combines the theory of extremal problems and monotone operators under a unified viewpoint. It is well known that the variational inequalities represent the optimality condition of the convex function. For the directional differentiable convex functions, we have another class of variational inequalities, which is known as the bifunction variational inequalities. Let K be a closed and convex set in the real Hilbert space H. For a given bifunction $B(\cdot,\cdot) : H\times H \rightarrow R$, we consider the problem of finding $u \in K$ such that

$$B(u, v-u) \geq 0, \quad \forall v \in K, \tag{1.1}$$

which is called the bifunction variational inequality. Crespi et al. [2–4], Fang and Hu [5], Lalitha and Mehta [6], and Noor [7] have studied various aspects of the bifunction variational inequalities. We would like to mention that that the bifunction variational inequality is quite different than the variational inequality.

For a given bifunction $F(\cdot,\cdot) : H\times H \rightarrow R$, Blum and Oettli [8] considered the problem of finding $u \in K$ such that

$$F(u,v) \geq 0, \quad \forall v \in K, \tag{1.2}$$

which is known as the equilibrium problem. It has been shown that the variational inequalities and fixed point problems are special cases of the equilibrium problems. We would like to emphasize that the bifunctions $B(\cdot,\cdot)$ and $F(\cdot,\cdot)$ are distinctly different from each other. Their properties are different from each other.

It is natural to consider the unification of these problems. This fact has motivated Noor et al. [9, 10] to consider a general and unified class, which is called the bifunction equilibrium variational inequality. They considered the problem of finding $u \in K$ such that

$$F(u,v) + B(u,v-u) \geq 0, \quad \forall v \in K. \tag{1.3}$$

Obviously, problem (1.3) includes the problems (1.1) and (1.2) as special cases. They have also discussed the numerical methods for solving such type of bifunction equilibrium variational inequalities using the auxiliary principle technique. For the applications and numerical methods for the bifunction equilibrium variational inequalities, see [2–30] and the references therein.

These problems have been studied in the convexity setting. This means that the underlying set is a convex set. Naturally a question arises as to whether or not these problems are well defined on the nonconvex sets. The answer to this question is positive. It is possible to consider these problems on the prox-regular sets. The prox-regular sets are nonconvex sets, see [11, 12, 24, 29]. Several authors have studied properties of these nonconvex sets related to a good behaviour of their boundary. See Sebbah and Thibault [30] and Noor [23] for the applications and projection characterization of the prox-regular sets. In recent years, Noor [7, 20–24] and Bounkhel et al. [11] have considered variational inequality in the context of uniformly prox-regular sets. In this paper, we introduce and consider the bifunction equilibrium variational inequalities on the prox-regular sets, which is called the nonconvex bifunction equilibrium variational inequality. This class is quite general and unifying one. One can easily show that the several classes of equilibrium problems and variational inequalities are special cases of this new class. There are a substantial number of numerical methods including projection technique and its variant forms, Wiener-Hopf equations, auxiliary principle and resolvent equations methods for solving variational inequalities. However, it is known that projection, Wiener-Hopf equations, and proximal and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving bifunction variational inequalities due to the nature of the problem. This fact has motivated the use of the auxiliary principle technique, which is mainly due to mainly due to Glowinski et al. [13]. This technique deals with finding the auxiliary problem and proving that the solution of the auxiliary problem is a solution of the original problem by using the fixed point problem. This technique is very useful and can be used to find the equivalent differentiable optimization problem. Glowinski et al. [13]

used this technique to study the existence of a solution of the mixed variational inequality. Noor [18, 19] has used this technique to develop some iterative schemes for solving various classes of variational inequalities. We point out that this technique does not involve the projection of the operator and is flexible. It is well known that a substantial number of numerical methods can be obtained as special cases from this technique. In this paper, we show that the auxiliary principle technique can be used to suggest and analyze a class of iterative methods for solving the nonconvex bifunction equilibrium variational inequalities. We also prove that the convergence of the implicit method requires only the monotonicity, which is a weaker condition than monotonicity. Since the nonconvex bifunction equilibrium variational inequalities included (nonconvex) bifunction variational inequalities and (nonconvex) equilibrium problems as special cases, results obtained in this paper continue to hold for these and related problems. Our method of proof is very simple as compared with other techniques.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty and convex set in H. We, first of all, recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis [12, 29]. Poliquin et al. [29] and Clarke et al. [12] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets.

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\}, \tag{2.1}$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}. \tag{2.2}$$

Here $d_K(\cdot)$ is the usual distance function to the subset K, that is,

$$d_K(u) = \inf_{v \in K} \|v - u\|. \tag{2.3}$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.2. *Let K be a nonempty, closed, and convex subset in H. Then $\zeta \in N_K^P(u)$, if and only if there exists a constant $\alpha > 0$ such that*

$$\langle \zeta, v - u \rangle \leq \alpha \|v - u\|^2, \quad \forall v \in K. \tag{2.4}$$

Definition 2.3. For a given $r \in (0,\infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal cone to K_r can be realized by an r-ball, that is, for all $u \in K_r$ and $0 \neq \xi \in N^P_{K_r}(u)$, one has

$$\left\langle \frac{\xi}{\|\xi\|}, v-u \right\rangle \leq \left(\frac{1}{2}r\right)\|v-u\|^2, \quad \forall v \in K_r. \tag{2.5}$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets; see [12, 29]. It is well known [11, 12, 29] that the union of two disjoint intervals $[a,b]$ and $[c,d]$ is a prox-regular set with $r = (c-b)/2$. For other examples of prox-regular sets, see M. A. Noor and K. I. Noor [24]. Obviously, for $r = \infty$, the uniformly prox-regularity of K_r is equivalent to the convexity of K. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

For the sake of simplicity, we take $\gamma = 1/r$. Then it is clear that, for $r = \infty$, we have $\gamma = 0$.

For given bifunctions $F(\cdot,\cdot), B(\cdot,\cdot) : H \times H \Rightarrow R$, we consider the problem of finding $u \in K_r$ such that

$$F(u,v) + B(u,v-u) + \gamma\|v-u\|^2 \geq 0, \quad \forall v \in K_r, \tag{2.6}$$

which is called the *nonconvex bifunction equilibrium variational inequality.*

We note that, if $K_r \equiv K$, the convex set in H, then problem (2.6) is equivalent to finding $u \in K$ such that

$$F(u,v) + B(u,v-u) \geq 0, \quad \forall v \in K. \tag{2.7}$$

Inequality of type (2.6) is called the *bifunction equilibrium variational inequality*, considered and studied by Noor et al. [9].

If $B(u,v-u) = \langle Tu, v-y \rangle$, where T is a nonlinear operator, then problem (2.6) is equivalent to finding $u \in K_r$ such that

$$F(u,v) + \langle Tu, v-u \rangle + \gamma\|v-u\|^2 \geq 0, \quad \forall v \in K_r, \tag{2.8}$$

which is called the nonconvex equilibrium variational inequality and appears to be a new one.

For a suitable and appropriate choice of the bifunctions and the spaces, one can obtain several new classes of equilibrium and variational inequalities, see [1–30] and the references therein. This shows that the problem (2.6) is quite general and includes several new and known classes of variational inequalities and equilibrium problems as special cases.

3. Main Results

In this section, we use the auxiliary principle technique of Glowinski et al. [13] as developed by Noor et al. [10, 26, 27] to suggest and analyze some iterative methods for solving the nonconvex equilibrium bifunction variational inequality (2.6). We would like to mention that this technique does not involve the concept of the projection and the resolvent, which is the main advantage of this technique.

For a given $u \in K_r$ satisfying (2.6), consider the problem of finding $w \in K_r$ such that

$$\rho F(w, v) + \rho B(w, v - w) + \langle w - u - \alpha(u - u), v - w\rangle + \rho\gamma\|v - w\|^2 \geq 0, \quad \forall v \in K_r, \tag{3.1}$$

where $\rho > 0$, $\alpha > 0$, and $\gamma > 0$ are constants. Inequality of type (3.1) is called the auxiliary nonconvex bifunction variational inequality. Note that if $w = u$, then w is a solution of (2.6). This simple observation enables us to suggest the following iterative method for solving the nonconvex bifunction variational inequalities (2.6).

Algorithm 3.1. For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned} &\rho F(u_{n+1}, v) + \rho B(u_{n+1}, v - u_{n+1}) + \langle u_{n+1} - u_n - \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \\ &\quad + \rho\gamma\|v - u_{n+1}\|^2 \geq 0, \quad \forall v \in K_r. \end{aligned} \tag{3.2}$$

Algorithm 3.1 is called the inertial proximal point method for solving the nonconvex bifunction equilibrium variatioanal inequalities (2.6).

If $\gamma = 0$, then the uniformly prox-regular set K_r reduces to the convex set K. Consequently, Algorithm 3.1 collapses to the following.

Algorithm 3.2. For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, v) + \rho B(u_{n+1}, v - u_{n+1}) + \langle u_{n+1} - u_n - \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in K_r. \tag{3.3}$$

Algorithm 3.2 is called the inertial proximal point method for solving the equilibrium bifunction variational inequalities 92.2) and appears to be a new one.

We note that, if $\alpha = 0$, then Algorithm 3.1 reduces to the following.

Algorithm 3.3. For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, v) + \rho B(u_{n+1}, v - u_{n+1}) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \rho\gamma\|v - u_{n+1}\|^2 \geq 0, \quad \forall v \in K_r. \tag{3.4}$$

Algorithm 3.3 is called the proximal point algorithm for solving nonconvex bifunction equilibrium variational inequality (2.6). In particular, if $\gamma = 0$, then the uniformly prox-regular set K_r becomes the convex set K and consequently Algorithm 3.3 reduces to the following algorithm.

Algorithm 3.4. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, v) + \rho B(u_{n+1}, v - u_{n+1}) + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K, \tag{3.5}$$

which is known as the proximal point algorithm for solving bifunction equilibrium variational inequalities (2.7) and has been studied extensively, see [10, 26, 27].

For suitable rearrangement and appropriate choice of the operators and spaces, one can obtain a numer of proximal point algorithms for solving various classes of bifunction variational inequalities, equilibrium problems, and optimization problems. This shows that Algorithm 3.1 is quite general and unifying one.

For the convergence analysis of Algorithm 3.3, we recall the following concepts and results.

Definition 3.5. A bifunction $B(\cdot,\cdot) : H \times H \to H$ is said to be *monotone*, if and only if

$$B(u, v - u) + B(v, u - v) \leq 0, \quad \forall u, v \in H. \tag{3.6}$$

Definition 3.6. A bifunction $F(\cdot,\cdot) : H \times H \to H$ is said to be *monotone*, if and only if

$$F(u, v) + F(v, u) \leq 0, \quad \forall u, v \in H. \tag{3.7}$$

Remark 3.7. We would like to point out that the bifunctions $F(\cdot,\cdot)$ and $B(\cdot,\cdot-\cdot)$ are different, that is $F(\cdot,\cdot) \neq F(\cdot,\cdot-\cdot)$. Due to this reason, one cannot define $G := F + B$. This is the reason that problem (2.6) is not equal to nonconvex bifunction equilibrium variational inequality problem.

We now consider the convergence criteria of Algorithm 3.3. The analysis is in the spirit of Noor [9, 18, 19]. In a similar way, one can consider the convergence analysis of other algorithms.

Theorem 3.8. *Let the bifunction $F(\cdot,\cdot), B(\cdot,\cdot) : K_r \times K_r \to H$ be monotone. If u_{n+1} is the approximate solution obtained from Algorithm 3.3 and $u \in K_r$ is a solution of* (2.6)*, then*

$$(1 - 4\gamma\rho)\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2. \tag{3.8}$$

Proof. Let $u \in K_r$ be a solution of (2.6). Then

$$-F(v, u) + -B(v, u - v) + \gamma\|v - u\|^2 \geq 0, \quad \forall v \in K_r, \tag{3.9}$$

since $B(\cdot,\cdot)$ and $F(\cdot,\cdot)$ are monotone operators.

Taking $v = u_{n+1}$ in (3.9), we have

$$-F(u_{n+1}, u) + -B(u_{n+1}, u - u_{n+1}) + \gamma\|u - u_{n+1}\|^2 \geq 0. \tag{3.10}$$

Setting $v = u$ in (3.4), and using (3.10), we have

$$\begin{aligned}\langle u_{n+1} - u_n, u - u_{n+1}\rangle &\geq -\rho F(u_{n+1}, u) - \rho B(u_{n+1}, u_{n+1} - u) \\ &\quad - \rho\gamma\|u_{n+1} - u\|^2 \geq 0.\end{aligned} \tag{3.11}$$

From this, one can easily obtain

$$(1 - 4\rho\gamma)\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2, \tag{3.12}$$

the required result (3.8). □

Theorem 3.9. *Let H be a finite dimension subspace, and let u_{n+1} be the approximate solution obtained from Algorithm 3.3. If $u \in K_r$ is a solution of* (2.6) *and $\rho < 1/4\gamma$, then $\lim_{n\to\infty} u_n = u$.*

Proof. Let $u \in K_r$ be a solution of (2.6). Then it follows from (3.5) that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty}\|u_n - u_{n+1}\|^2 \leq \|u_0 - u\|^2, \tag{3.13}$$

which implies that

$$\lim_{n\to\infty}\|u_n - u_{n+1}\| = 0. \tag{3.14}$$

Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$, and let the subsequence $\{u_j\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in K_r$. Replacing u_n by u_{n_j} in (3.4) and taking the limit $n_j \to \infty$ and using (3.14), we have

$$B(\hat{u}, v - \hat{u}) + \gamma\|v - \hat{u}\|^2 \geq 0, \quad \forall v \in K_r, \tag{3.15}$$

which implies that $\hat{u}$ solves the nonconvex bifunction equilibrium variational inequality (2.6) and

$$\|u_n - u_{n+1}\|^2 \leq \|\hat{u} - u_n\|^2. \tag{3.16}$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\to\infty} u_n = \hat{u}$, the required result. □

We note that, for $r = \infty$, the r-prox-regular set K becomes a convex set and the nonconvex bifunction equilibrium variational inequality (2.6) collapses to the bifunction equilibrium variational inequality (2.7). Thus our results include the previous known results as special cases.

It is well known that, to implement the proximal point methods, one has to calculate the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we suggest another iterative method, the convergence of which requires only partially relaxed strongly monotonicity, which is a weaker condition that of cocoercivity.

For a given $u \in K_r$ satisfying (2.6), consider the problem of finding $w \in K_r$ such that

$$\rho F(u,v) + \rho B(u,v-w) + \langle w-u, v-w\rangle + \gamma\|v-w\|^2 \geq 0, \quad \forall v \in K_r, \tag{3.17}$$

which is also called the auxiliary nonconvex bifunction equilibrium variational inequality. Note that problems (3.1) and (3.17) are quite different. If $w = u$, then clearly w is a solution of the nonconvex bifunction equilibrium variational inequality (2.6). This fact enables us to suggest and analyze the following iterative method for solving the nonconvex bifunction equilibrium variational inequality (2.6).

Algorithm 3.10. For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_n,v) + \rho B(u_n,v-u_{n+1}) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle + \gamma\|v-u_{n+1}\|^2 \geq 0, \quad \forall v \in K_r. \tag{3.18}$$

Note that, for $r = \infty$, the uniformly prox-regular set K_r becomes a convex set K and Algorithm 3.3 reduces to the following.

Algorithm 3.11. For a given $u_0 \in K$, calculate the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_n,v) + \rho B(u_n,v-u_{n+1}) + \langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq 0, \quad \forall v \in K, \tag{3.19}$$

which is known as the projection iterative method for solving bifunction equilibrium variational inequalities (2.7).

4. Conclusion

For appropriate and suitable choice of the operators and the spaces, one can suggest and analyze several iterative methods for solving the nonconvex bifunction equilibrium variational inequalities. This shows that the algorithms suggested in this paper are more general and unifying ones. Using essentially the technique of Theorems 3.8 and 3.9, one can study the convergence analysis of Algorithm 3.10. It is an interesting problem to compare these iterative methods with other numerical methods for solving the nonconvex bifunction equilibrium variational inequalities. The ideas and technique of this paper may stimulate further research in these interesting fields.

Acknowledgment

The authors would like to express their gratitude to Dr. S. M. Junaid Zaidi, Rector, CIIT, for providing excellent research facilities.

References

[1] G. Stampacchia, "Formes bilinéaires coercitives sur les ensembles convexes," *Comptes-rendus de l'Académie des Sciences de Paris*, vol. 258, pp. 4413–4416, 1964.
[2] G. P. Crespi, I. Ginchev, and M. Rocca, "Minty variational inequalities, increase-along-rays property and optimization," *Journal of Optimization Theory and Applications*, vol. 123, no. 3, pp. 479–496, 2004.
[3] G. P. Crespi, I. Ginchev, and M. Rocca, "Existence of solutions and star-shapedness in Minty variational inequalities," *Journal of Global Optimization*, vol. 32, no. 4, pp. 485–494, 2005.
[4] G. P. Crespi, I. Ginchev, and M. Rocca, "Some remarks on the Minty vector variational principle," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 165–175, 2008.
[5] Y.-P. Fang and R. Hu, "Parametric well-posedness for variational inequalities defined by bifunctions," *Computers & Mathematics with Applications*, vol. 53, no. 8, pp. 1306–1316, 2007.
[6] C. S. Lalitha and M. Mehta, "Vector variational inequalities with cone-pseudomonotone bifunctions," *Optimization*, vol. 54, no. 3, pp. 327–338, 2005.
[7] M. A. Noor, "Some new classes of nonconvex functions," *Nonlinear Functional Analysis and Applications*, vol. 11, no. 1, pp. 165–171, 2006.
[8] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.
[9] M. A. Noor, K. I. Noor, and E. Al-Said, "On nonconvex bifunction variational inequalities," *Optimization Letters*. In press.
[10] M. A. Noor, K. I. Noor, and E. Al-Said, "Iterative methods for solving nonconvex equilibrium variational inequalities," *Applied Mathematics and Information Science* , vol. 6, no. 1, pp. 65–69, 2012.
[11] M. Bounkhel, L. Tadj, and A. Hamdi, "Iterative schemes to solve nonconvex variational problems," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, pp. 1–14, 2003.
[12] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, vol. 178 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1998.
[13] R. Glowinski, J.-L. Lions, and R. Trémolières, *Numerical Analysis of Variational Inequalities*, vol. 8 of *Studies in Mathematics and Its Applications*, North-Holland, Amsterdam, The Netherlands, 1981.
[14] F. Giannessi, A. Maugeri, and P. M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*, vol. 58, Kluwer Academics Publishers, Dordrecht, The Netherlands, 2001.
[15] R. P. Gilbert, P. D. Panagiotopoulos, and P. M. Pardalos, *From Convexity to Nonconvexity*, vol. 55 of *Nonconvex Optimization and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.
[16] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, vol. 31 of *Classics in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, USA, 2000.
[17] M. A. Noor, "General variational inequalities," *Applied Mathematics Letters*, vol. 1, no. 2, pp. 119–122, 1988.
[18] M. A. Noor, "Some developments in general variational inequalities," *Applied Mathematics and Computation*, vol. 152, no. 1, pp. 199–277, 2004.
[19] M. A. Noor, "Iterative schemes for nonconvex variational inequalities," *Journal of Optimization Theory and Applications*, vol. 121, no. 2, pp. 385–395, 2004.
[20] M. A. Noor, "Extended general variational inequalities," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 182–186, 2009.
[21] M. A. Noor, "Implicit iterative methods for nonconvex variational inequalities," *Journal of Optimization Theory and Applications*, vol. 143, no. 3, pp. 619–624, 2009.
[22] M. A. Noor, "On an implicit method for nonconvex variational inequalities," *Journal of Optimization Theory and Applications*, vol. 147, no. 2, pp. 411–417, 2010.
[23] M. A. Noor, "Some iterative methods for general nonconvex variational inequalities," *Mathematical and Computer Modelling*, vol. 54, no. 11-12, pp. 2955–2961, 2011.
[24] M. A. Noor and K. I. Noor, "Iterative schemes for trifunction hemivariational inequalities," *Optimization Letters*, vol. 5, no. 2, pp. 273–282, 2011.

[25] M. A. Noor, K. I. Noor, and E. Al-Said, "Auxiliary principle technique for solving bifunction variational inequalities," *Journal of Optimization Theory and Applications*, vol. 149, no. 2, pp. 441–445, 2011.

[26] M. A. Noor, K. I. Noor, S. Zainab, and E. Al-Said, "Proximal algorithms for solving mixed bifunction variational inequalities," *International Journal of Physical Sciences*, vol. 6, no. 17, pp. 4203–4212, 2011.

[27] M. A. Noor, K. I. Noor, and Z. Huang, "Bifunction hemivariational inequalities," *Journal of Applied Mathematics and Computing*, vol. 35, no. 1-2, pp. 595–605, 2011.

[28] M. A. Noor, K. I. Noor, and T. M. Rassias, "Some aspects of variational inequalities," *Journal of Computational and Applied Mathematics*, vol. 47, no. 3, pp. 285–312, 1993.

[29] R. A. Poliquin, R. T. Rockafellar, and L. Thibault, "Local differentiability of distance functions," *Transactions of the American Mathematical Society*, vol. 352, no. 11, pp. 5231–5249, 2000.

[30] M. Sebbah and L. Thibault, "Metric projection and compatibly parameterized families of prox-regular sets in Hilbert space," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 75, no. 3, pp. 1547–1562, 2012.