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Research Article

Exact Asymptotic Expansion of Singular Solutions for the $(2+1)$-D Protter Problem

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We study three-dimensional boundary value problems for the nonhomogeneous wave equation, which are analogues of the Darboux problems in $\mathbb{R}^2$. In contrast to the planar Darboux problem the three-dimensional version is not well posed, since its homogeneous adjoint problem has an infinite number of classical solutions. On the other hand, it is known that for smooth right-hand side functions there is a uniquely determined generalized solution that may have a strong power-type singularity at one boundary point. This singularity is isolated at the vertex of the characteristic light cone and does not propagate along the cone. The present paper describes asymptotic expansion of the generalized solutions in negative powers of the distance to this singular point. We derive necessary and sufficient conditions for existence of solutions with a fixed order of singularity and give a priori estimates for the singular solutions.

1. Introduction

In the present paper some boundary value problems (BVPs) formulated by M. H. Protter for the wave equation with two space and one time variables are studied as a multidimensional analogue of the classical Darboux problem in the plane. While the Darboux BVP in $\mathbb{R}^2$ is well posed the Protter problem is not and its cokernel is infinite dimensional. Therefore the problem is not Fredholm and the orthogonality of the right-hand side function f to the cokernel is one necessary condition for existence of classical solution. Alternatively, to avoid infinite number of conditions the notion of generalized solution is introduced that allows the solution to have singularity on a characteristic part of the boundary. It is known that for smooth right-hand side functions there is unique generalized solution and it may have a strong power-type singularity that is isolated at one boundary point. In the present paper we prove asymptotic expansion formula for the generalized solutions in negative powers of the distance to the singular point in the case when f is trigonometric polynomial. We leave for the next section the precise formulation of the paper's main results and the comparisons with recent publications concerning Protter problems, including a semi-Fredholm solvability result in the general case of smooth f but for somewhat easier (3 + 1)-D wave equation problem. First we give here a short historical survey.

Protter arrived at the multidimensional problems for hyperbolic equations while examining BVPs for mixed type equations, starting with planar problems with strong connection to transonic flow phenomena. In the plane, the problems of Tricomi, Frankl, and Guderley-Morawetz are the classical boundary-value problems that appear in hodograph plane for 2D transonic potential flows (see, e.g., the survey of Morawetz [1]). The first two of these problems are relevant to flows in nozzles and jets, and the third problem occurs as an approximation to a respective "exact" boundary-value problem in the study of flows around airfoils. For the Gellerstedt equation of mixed type, Protter [2] proposes a 3D analogue to the two-dimensional Guderley-Morawetz problem. At the same time, he formulates boundary value problems in the hyperbolic part of the domain, which is bounded by two characteristics and one noncharacteristic surfaces of the equation. The planar Guderley-Morawetz mixed-type problem is well studied. Existence of weak solutions and uniqueness of strong solutions in weighted Sobolev spaces were first established by Morawetz by reducing the problem to a first order system which then gives rise to solutions to the scalar equation in the presence of sufficient regularity. The availability of such sufficient regularity follows from the work of Lax and Phillips [3] who also established that the weak solutions of Morawetz are strong. On the other hand, for the 3D Protter mixed-type problems a general understanding of the situation is not at hand—even the question of well posedness is surprisingly subtle and not completely resolved. One has uniqueness results for quasiregular solutions, a class of solutions introduced by Protter, but there are real obstructions to existence in this class. To investigate the situation, we study a simpler problem—the Protter problems in the hyperbolic part Ω of the domain for the mixed-type problem. For the wave equation

$$\Box u \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} = f(x,t), \tag{1.1}$$

this is the set

$$\Omega := \left\{ (x_1, x_2, t) : \ 0 < t < \frac{1}{2}, \ t < \sqrt{x_1^2 + x_2^2} < 1 - t \right\}. \tag{1.2}$$

It is bounded, see Figure 1, by two characteristic cones of (1.1)

$$\begin{gathered} S_1 = \left\{ (x_1, x_2, t) : \ 0 < t < \frac{1}{2}, \ \sqrt{x_1^2 + x_2^2} = 1 - t \right\}, \\ S_2 = \left\{ (x_1, x_2, t) : \ 0 < t < \frac{1}{2}, \ \sqrt{x_1^2 + x_2^2} = t \right\}, \end{gathered} \tag{1.3}$$

and the disk $S_0 = \{(x_1, x_2, t) : \ t = 0, x_1^2 + x_2^2 < 1\}$, centered at the origin $O(0,0,0)$.

One could think of the Protter problems in Ω as three-dimensional variant of the planar Darboux problem. The classic Darboux problem involves a hyperbolic equation in a characteristic triangle bounded by two characteristic and one noncharacteristic segments. The data are prescribed on the noncharacteristic part of the boundary and one of

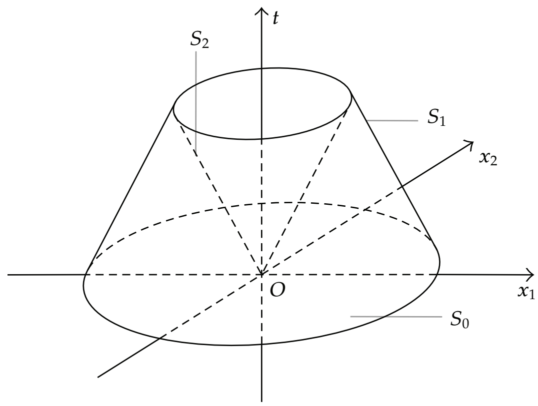

Figure 1: The domain Ω.

the characteristics. Actually, the set Ω could be produced via rotation around the t-axis in $\mathbb{R}^3$ of the flat triangle $\Omega_2 := \{(x_1, t) : 0 < t < 1/2;\ \ t < x_1 < 1 - t\} \subset \mathbb{R}^2$—a characteristic triangle for the corresponding string equation

$$u_{x_1x_1} - u_{tt} = g(x_1, t). \tag{1.4}$$

As mentioned before, the classical Darboux problem for (1.4) is to find solution in Ω_2 with data prescribed on $\{t = 0\}$ and $\{t = 1 - x_1\}$, for example. In conformity with this planar BVP, Protter [2, 4] formulated and studied the following problems.

Problems ($P1$) *and* ($P2$)

Find a solution of the wave equation (1.1) in which Ω satisfies one of the following boundary conditions:

$$u|_{S_0} = 0, \qquad u|_{S_1} = 0, \tag{P1}$$

or

$$u_t|_{S_0} = 0, \qquad u|_{S_1} = 0. \tag{P2}$$

Nowadays, it is known that the Protter Problems (P1) and (P2) are not well posed, in contrast to the planar Darboux problem. In fact, in 1957 Tong [5] proved the existence of infinite number nontrivial classical solutions to the corresponding homogeneous adjoint problem (P1*). The adjoint BVPs to Problems (P1) and (P2) were also introduced by Protter.

Problems ($P1^*$) *and* ($P2^*$)

Find a solution of the wave equation (1.1) in Ω which satisfies the boundary conditions:

$$u|_{S_0} = 0, \qquad u|_{S_2} = 0 \text{ (adjoint to Problem (P1))}, \tag{P1*}$$

or

$$u_t|_{S_0} = 0, \qquad u|_{S_2} = 0 \text{ (adjoint to Problem (P2))}. \tag{P2*}$$

Since [5], for each of the homogeneous Problems (P1*) and (P2*) (i.e., $f \equiv 0$ in (1.1)), an infinite number of classical solutions has been found (see Popivanov, Schneider [6], Khe [7]). According to this fact, a necessary condition for classical solvability of Problem (P1) or (P2) is the orthogonality in $L_2(\Omega)$ of the right-hand side function $f(x,t)$ to all the solutions of the corresponding homogenous adjoint problem (P1*) or (P2*). Although Garabedian proved [8] the uniqueness of a classical solution of Problem (P1) (for its analogue in $\mathbb{R}^4$), generally, Problems (P1) and (P2) are not classically solvable. Instead, Popivanov and Schneider [6] introduced the notion of generalized solution. It allows the solution to have singularity on the inner cone S_2 and by this the authors avoid the infinite number of necessary conditions in the frame of the classical solvability. In [6] some existence and uniqueness results for the generalized solutions are proved and some singular solutions of Protter Problems (P1) and (P2) are constructed.

In the present paper we study the properties of the generalized solution for Protter Problem (P2) in $\mathbb{R}^3$. From the results in [6] it follows that for $n \in \mathbb{N}$ there exists a smooth right-hand side function $f \in C^n(\overline{\Omega})$, such that the corresponding unique generalized solution of Problem (P2) has a strong power-type singularity at the origin O and behaves like $r^{-n}(P,O)$ there. This feature deviates from the conventional belief that such BVPs are classically solvable for very smooth right-hand side functions f. Another interesting aspect is that the singularity is isolated only at a single point the vertex O of the characteristic light cone, and does not propagate along the bicharacteristics which makes this case different from the traditional case of propagation of singularity (see, e.g., Hörmander [9], Chapter 24.5).

The Protter problems have been studied by different authors using various types of techniques like Wiener-Hopf method, special Legendre functions, a priori estimates, nonlocal regularization, and others. For recent known results concerning Protter's problems see the paper [6] and references therein. For further publications in this area see [7, 10–16]. On the other hand, Bazarbekov gives in Ω another analogue of the classical Darboux problem (see [17]) and analogously in $\mathbb{R}^4$ (see [18]) in the corresponding four-dimensional domain Ω. Some different statements of Darboux type problems in $\mathbb{R}^3$ or connected with them Protter problems for mixed type equations (also studied in [2]) can be found in [19–25]. Some results concerning the nonexistence principle for nontrivial solution of semilinear mixed type equations in multidimensional case, can be found in [26]. For recent existence results concerning closed boundary-value problems for mixed type equations see for example [27], and also [28] that studies an elliptic-hyperbolic equation which arises in models of electromagnetic wave propagation through zero-temperature plasma. The existence of bounded or unbounded solutions for the wave equation in $\mathbb{R}^3$ and $\mathbb{R}^4$, as well as for the Euler-Poisson-Darboux equation has been studied in [7, 13–16, 29].

Further, we aim to find some exact a priori estimates for the singular solutions of Problem (P2) and to outline the exact structure and order of singularity. For some other Protter problems necessary and sufficient conditions for existence of solutions with fixed order of singularity were found (see [15] in $\mathbb{R}^3$ and [16] in $\mathbb{R}^4$) and an asymptotic formula for the solution of Problem (P1) in $\mathbb{R}^4$ was obtained in [30].

Considering Protter Problems, Popivanov and Schneider [6] proved the existence of singular solutions for both wave and degenerate hyperbolic equation. First a priori estimates for singular solutions of Protter Problems, involving the wave equation in $\mathbb{R}^3$, were obtained in [6]. In [10] Aldashev mentioned the results of [6] and, for the case of the wave equation in $\mathbb{R}^{m+1}$, he notes the existence of solutions in the domain Ω_ε ($\Omega_\varepsilon \to \Omega$ and $S_{2,\varepsilon}$ approximates S_2 if $\varepsilon \to 0$), which blows up on the cone $S_{2,\varepsilon}$ like $\varepsilon^{-(n+m-2)}$, when $\varepsilon \to 0$. It is obvious that for $m = 2$ this results can be compared to the estimates in Corollary 2.4 here. Finally, we point out that in the case of an equation, which involves the wave operator and nonzero lower terms, Karatopraklיev [24] obtained a priori estimates, but only for the sufficiently smooth solutions of Protter Problem.

Regarding the ill-posedness of the Protter Problems, there have appeared some possible regularization methods in the case of the wave equation, involving either lower order terms ([11, 31]), or some other type perturbations, like integrodifferential term, or nonlocal one ([12]).

In Section 2 the result of the existence of infinite number of classical solutions to the homogeneous Problem (P2*) (Lemma 2.1) and the definition of *generalized solution* of Problem (P2) are given. The main results of the paper, concerning the asymptotic expansion of the unique *generalized solution* $u(x,t)$ of Problem (P2) (Theorem 2.3) are formulated and discussed. The expansion of $u(P)$ is given in negative powers of the distance $r(P,O)$ to the point O of singularity. An estimate for the remainder term and the exact behavior of the singularity under the orthogonality conditions imposed on the right-hand side function of the wave equation is found. Necessary and sufficient conditions for the existence of only bounded solutions are given in Corollary 2.4. In Section 3, the auxiliary 2D boundary value Problems (P2.1) and (P2.2), which correspond to the (2 + 1)-D Problem (P2), are considered. Actually, these 2D problems are transferred to an integral Volterra equation, which is invertible. Using the special Legendre functions P_ν, some exact formulas for the solution of the Problem (P2.2) are derived in Lemma 3.4. Some figures showing the effects appearing near the singularity point are also presented. Section 4 contains the most technical part of the paper. In this section the results concerning the asymptotic expansions of the generalized solution of the 2D Problem (P2.1) are proved and the proof of the main Theorem 2.3 is given.

2. Main Results on $(2+1)$-D Protter's Problem $(\mathbf{P}2)$

Define the functions

$$E_k^n(x,t) = \sum_{i=0}^{k} B_i^k \frac{\left(x_1^2 + x_2^2 - t^2\right)^{n-1/2-k-i}}{\left(x_1^2 + x_2^2\right)^{n-i}}, \quad n,k \in \mathbb{N} \cup \{0\}, \tag{2.1}$$

where the coefficients are

$$B_i^k := (-1)^i \frac{(k-i+1)_i (n+1/2-k-i)_i}{i!(n-i)_i}, \quad B_0^k = 1, \tag{2.2}$$

with $(a)_i := a(a+1)\cdots(a+i-1)$, $(a)_0 := 1$. Then for the functions

$$\begin{aligned} W_{k,1}^n(x,t) &:= E_k^n(x,t) \operatorname{Re}\left\{(x_1 + i x_2)^n\right\}, \\ W_{k,2}^n(x,t) &:= E_k^n(x,t) \operatorname{Im}\left\{(x_1 + i x_2)^n\right\}, \end{aligned} \tag{2.3}$$

we have the following lemma.

Lemma 2.1 (see [29]). *Let $n \in \mathbb{N}$, $n \geq 4$. For $k = 0,\dots,[(n-3)/2]$ and $i = 1,2$ the functions $W^n_{k,i}(x,t)$ are classical $C^2(\overline{\Omega}) \cap C^\infty(\Omega)$ solutions to the homogeneous Problem ($P2^*$).*

A necessary condition for the existence of classical solution for Problem (P2) is the orthogonality of the right-hand side function f to all functions $W^n_{k,i}(x,t)$, which are solutions of the homogeneous adjoint Problem (P2*). To avoid these infinite number necessary conditions in the framework of classical solvability, one needs to introduce some generalized solutions of Problems (P2) with possible singularities on the characteristic cone S_2, or only at its vertex O. Popivanov and Schneider in [6] give the following definition.

Definition 2.2. A function $u = u(x_1, x_2, t)$ is called a generalized solution of the Problem (P2) in Ω if:

(1) $u \in C^1(\overline{\Omega} \setminus O)$, $u_t|_{S_0 \setminus O} = 0$, $u|_{S_1} = 0$,

(2) the identity

$$\int_\Omega (u_t w_t - u_{x_1} w_{x_1} - u_{x_2} w_{x_2} - f w)\, dx_1 dx_2 dt = 0 \tag{2.4}$$

holds for all $w \in C^1(\overline{\Omega})$, $w_t = 0$ on S_0, and $w = 0$ in a neighborhood of S_2.

The uniqueness of the generalized solution of Problem (P2) and existence results for $f \in C^1(\overline{\Omega})$ can be found in [6].

Further, we fix the right-hand side function f as a trigonometric polynomial of order l with respect to the polar angle:

$$f(x_1, x_2, t) = \operatorname{Re}\left\{ \sum_{n=2}^{l} f_n(|x|, t)(x_1 + i x_2)^n \right\}, \tag{2.5}$$

with some complex-valued function-coefficients $f_n(|x|, t)$. For $n = 0,\dots,l;\ k = 0,\dots,[n/2]$ and $i = 1,2$, denote by $\beta^n_{k,i}$ the constants

$$\beta^n_{k,i} := \int_\Omega W^n_{k,i}(x,t) f(x,t)\, dx dt. \tag{2.6}$$

Note that actually $\beta^0_{k,i} = 0$ and $\beta^1_{k,i} = 0$ in cases of $n = 0$ and $n = 1$, due to the special form of the functions $W^n_{k,i}$ and the fact that in the representation (2.5) of the function f the sum starts from $n = 2$.

The main result is as follows.

Theorem 2.3. *Suppose that the function* $f(x,t) \in C^1(\overline{\Omega})$ *is a trigonometric polynomial* (2.5)*. Then there exist functions* $F^n(x,t)$, $F^n_{k,i}(x,t)$, $F(x,t) \in C^2(\overline{\Omega} \setminus O)$ *with the following properties:*

(i) *the unique generalized solution* $u(x,t)$ *of Problem* ($P2$) *exists, belongs to* $C^2(\overline{\Omega} \setminus O)$ *and has the asymptotic expansion at the origin* O*:*

$$u(x,t) = \sum_{m=0}^{l} \left(|x|^2 + t^2\right)^{-m/2} F^m(x,t) + \left(|x|^2 + t^2\right)^{1/4} F(x,t) \ln\left(|x|^2 + t^2\right), \tag{2.7}$$

(ii) *for the coefficient functions* $F^m(x,t)$ *the representation*

$$F^m(x,t) = \sum_{k=0}^{[(l-m)/2]} \sum_{i=1}^{2} \beta^{m+2k}_{k,i} F^{m+2k}_{k,i}(x,t), \quad m = 0,\ldots,l, \tag{2.8}$$

holds, where the functions $F^n_{k,i}(x,t)$ *are bounded and independent of* f*,*

(iii) *if in the expression* (2.8) *for* $F^m(x,t)$ *at least one of the constants* $\beta^{m+2k}_{k,i}$ *is different from zero (i.e., the corresponding orthogonality condition is not fulfilled), then there exists a direction* $(\alpha_1,\alpha_2,1)$ *with* $(\alpha_1,\alpha_2,1)t \in S_2$ *for* $0 < t < 1/2$*, such that* $\lim_{t\to+0} F^m(\alpha_1 t, \alpha_2 t, t) = c_m = const \neq 0$*,*

(iv) *if in the expression* (2.8) *for* $F^0(x,t)$ *at least one of the constants* $\beta^{2k}_{k,i}$ *is different from zero (i.e., the corresponding orthogonality condition is not fulfilled), then the generalized solution is not continuous at* O*,*

(v) *for the function* $F(x,t)$ *the estimate*

$$|F(x,t)| \le C\left\{\max_{\overline{\Omega}}\left|f(x,t)\right| + \max_{\overline{\Omega}}\left|f_t(x,t)\right|\right\}, \quad (x,t) \in \Omega, \tag{2.9}$$

holds with a constant C *independent of* f*.*

As a consequence of Theorem 2.3 one gets the following results that highlight the two extreme cases of the assertion. The first part gives rough estimate of the expansion (2.7) and describes the "worst" possible singularity. The second part shows that one could control the solution by making some of the defined by (2.6) constants $\beta^n_{k,i}$ in (2.8) to be zero, that is, by taking f to be orthogonal in $L_2(\Omega)$ to the corresponding functions $W^n_{k,i}$ defined in (2.3).

Corollary 2.4. *Suppose that* $f \in C^1(\overline{\Omega})$ *has the form* (2.5)*.*

(i) *Without any orthogonality conditions imposed, the unique generalized solution* u *of Problem* ($P2$) *satisfies the a priori estimate*

$$|u(x,t)| \le C\left(|x|^2 + t^2\right)^{-l/2} \|f\|_{C^1(\overline{\Omega})}, \quad (x,t) \in \Omega. \tag{2.10}$$

(ii) *Let the orthogonality conditions,*

$$\beta^n_{k,i} \equiv \int_\Omega W^n_{k,i}(x,t) f(x,t) dx\, dt \;=\; 0, \tag{2.11}$$

be fulfilled for all $n = 2, \dots, l;\ k = 0, \dots, [(n-1)/2]$ *and* $i = 1, 2$. *Then the generalized solution* $u(x,t)$ *belongs to* $C^2(\overline{\Omega} \setminus O)$, *is bounded and the a priori estimate*

$$\sup_\Omega |u| \le C\left\{ \|f\|_{C(\overline{\Omega})} + \|f_t\|_{C(\overline{\Omega})} \right\} \tag{2.12}$$

holds.

(iii) *In addition to (ii), if the conditions* (2.11) *are fulfilled for* $k = [n/2]$ *also, then* $u \in C(\overline{\Omega})$ *is a classical solution and* $u(O) = 0$.

Let us point out that in the case (ii), the generalized solution u is bounded if and only if the conditions (2.11) are fulfilled for $k \le [(n-1)/2]$ due to Theorem 2.3(iii). In addition, if all the conditions (2.11) are fulfilled for $k \le [(n-1)/2]$, but for some $k = [n/2]$ the corresponding orthogonality condition is not satisfied, then u is not continuous at O, according to Theorem 2.3 (iv). Such a solution is illustrated in Figure 4.

Notice that some of the functions $W^n_{k,i}(x,t)$ involved in the orthogonality conditions in Corollary 2.4(ii) and (iii) are not classical solutions of the homogenous adjoint Problem (P2*) in view of Lemma 2.1, although they satisfy the homogenous wave equation in Ω. In fact, for some k, $W^n_{k,i}$ or their derivatives may be discontinuous at S_2. For example when n is an odd number and $k = (n-1)/2$, the functions $W^n_{k,i}$ are not continuous at the origin O. On the other hand, when n is even and $k = n/2$, $W^n_{n/2,i}$ are singular on the cone S_2 and do not satisfy the homogeneous adjoint boundary condition there. However, this singularity is integrable in the domain Ω.

To explain the results in Theorem 2.3 and Corollary 2.4 we construct Table 1. It illustrates the connection between the singularity of the generalized solution and the functions $W^n_{k,i}$.

Both functions $W^n_{k,i}$, $i = 1, 2$ are located in column number n and row number $(n-2k)$ in Table 1. Thus, $W^n_{0,i}$ form the rightmost diagonal, the next one is empty—we put in these cells "diamonds" $\diamond$, $W^n_{1,i}$ constitute the third one, and so on. The row number designates the order of singularity of the generalized solution.

Corollary 2.4 shows that the generalized solution $u(x,t)$ is bounded, when the right-hand side function f is orthogonal to the functions in Table 1, except the ones in row number 0. If f is orthogonal to all the functions in the Table 1 (including the row 0), then u is continuous in $\overline{\Omega}$. When the right-hand side f satisfies orthogonal conditions (2.11) for all the functions from the rows in Table 1 with row-number larger then m, $0 < m < l$, but there is a function $W^p_{q,i}$ with $p - 2q = m$ from m th row which is not orthogonal to f (i.e., $\beta^p_{q,i} \neq 0$), then the solution behaves like r^{-m} at the origin, according to the expansion (2.7). If there are no orthogonality conditions, then the worst case with singularity r^{-l} appears.

Figures 2–5 are created using MATLAB and represent some numerical computations for singular solutions of Problem (P2) (actually the behaviour in (r,t)-domain D_1, not including the terms $\sin n\varphi$ and $\cos n\varphi$). They illustrate different cases according to the main results for the existence of a singularity at the origin O depending on orthogonality

Table 1: The order of singularity of the solution and the functions $W^n_{k,i}$.

	l	$l-1$	$l-2$	$l-3$	$\cdots$	p		$\cdots$	4	3	2
0	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	$W^4_{2,i}$	$\diamond$	$W^2_{1,i}$
1	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	$\diamond$	$W^3_{1,i}$	$\diamond$
2	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	$W^4_{1,i}$	$\diamond$	$W^2_{0,i}$
3	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	$\diamond$	$W^3_{0,i}$	
4	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	$W^4_{0,i}$		
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$			
$p-2q$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$W^p_{q,i}$	$\cdots$				
						$\cdots$					
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$						
$l-3$	$\diamond$	$W^{l-1}_{1,i}$	$\diamond$	$W^{l-3}_{0,i}$							
$l-2$	$W^l_{1,i}$	$\diamond$	$W^{l-2}_{0,i}$								
$l-1$	$\diamond$	$W^{l-1}_{0,i}$									
l	$W^l_{0,i}$										

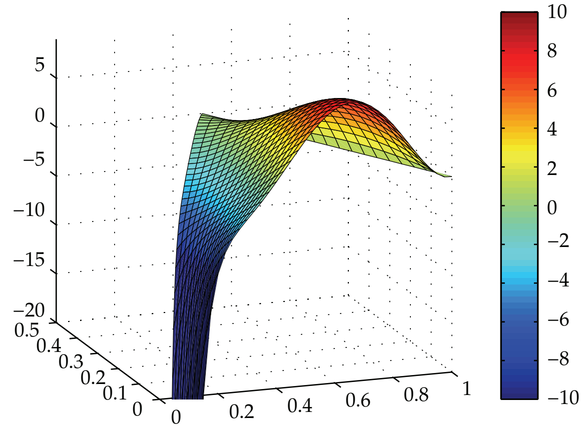

Figure 2: No orthogonality conditions.

conditions. Figure 2 is related to Corollary 2.4(i)—it gives the graph of the solution for the worst case without any orthogonality conditions fulfilled and the solution is going to $-\infty$ at the singular point O. In Figure 3, only one of orthogonality conditions (2.11) for $k \le [(n-1)/2]$ is not fulfilled and the solution tends to $\pm\infty$. Figures 4 and 5 are connected to Corollary 2.4(ii) and (iii): Figure 4 presents the case when all the orthogonality conditions (2.11) for $k \le [(n-1)/2]$ are satisfied and the solution is bounded but not continuous at $(0,0)$, while Figure 5 concerns the last part (iii) from Corollary 2.4, when conditions (2.11) are additionally fulfilled for $k = [n/2]$ and the solution is continuous.

Remark 2.5. We mention some differences between the results given here for the Problem (P2) and some other results in $\mathbb{R}^3$, but for the Problem (P1), like that from [15].

(i) In [15], assuming the right-hand side function f is smooth enough (i.e., $f \in C^l$) only the behavior of the singularities was studied using some weighted norms

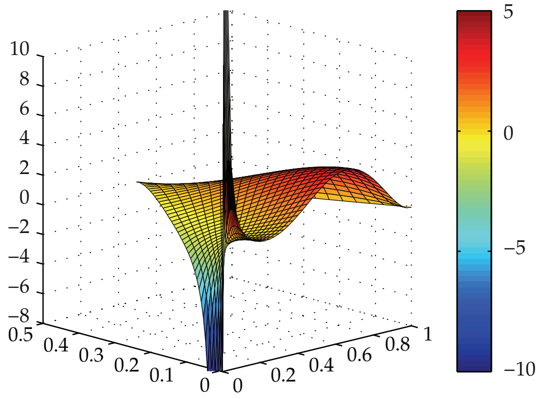

Figure 3: One orthogonality condition is not fulfilled.

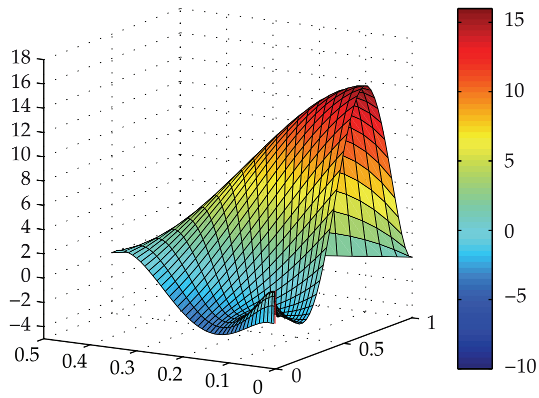

Figure 4: Orthogonality conditions fulfilled for $k \leq [(n-1)/2]$.

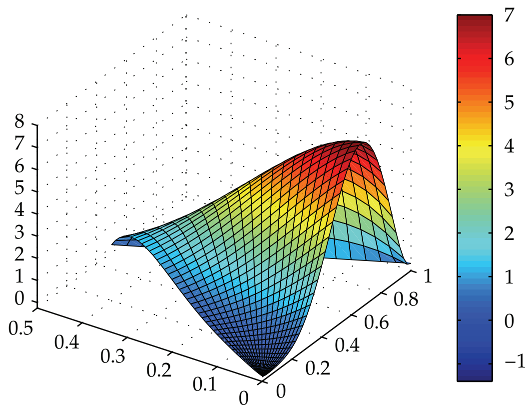

Figure 5: All orthogonality conditions fulfilled for $k \leq [n/2]$.

(analogous to the weighted Sobolev norms in corner domains). In the present paper we need only $f \in C^1$ and find in addition the explicit asymptotic expansion of the generalized solution. The bounded but not continuous at the origin solutions are also studied here.

(ii) Comparing the power of singularity of the *generalized solution* for Problem (P2) here and for Problem (P1) in [15] for the worst case without any orthogonality conditions one can see that the power in the estimate (2.10) from Corollary 2.4(i) is $(|x|^2 + t^2)^{-l/2}$, while in the analogous estimate in Conclusion 1 [15] it is $(|x|^2 + t^2)^{-(l-1)/2}$.

(iii) It is interesting to compare the results [14, 15], published in 2002. Going in a different way in both cases the authors asked for singular solutions of Problem (P1) in $\mathbb{R}^3$. However, in [14] there are absent any analogues to the orthogonality conditions presented in [15], and in contrast to [15] in [14] the dependence of the exact order of singularity on the data is not clarified.

Remark 2.6. Let us also compare the present expansion and the results in [30], where an asymptotic expansion of Problem (P1) is found for somewhat easier four-dimensional case.

(i) Both for Problem (P2) in $\mathbb{R}^3$ here and Problem (P1) in $\mathbb{R}^4$ as in [30], the study is based on the properties of the special Legendre functions. Instead of Legendre functions P_ν with non-integer indices $\nu = n - 1/2$ here, in the four-dimensional case one has to deal with integer indices $\nu = n$, that is, simply with the Legendre polynomials P_n. One can easily modify both these techniques to obtain similar results for the $(m + 1)$-dimensional problems: for even m (analogous to the present case $\mathbb{R}^3$) or for odd m (similarly to $\mathbb{R}^4$ case). Some different kind of results for the Protter problems in $\mathbb{R}^{m+1}$ are presented in [10, 11].

(ii) For the four-dimensional Problem (P1) in [30], the Corollary 3.3 gives only that the solution is bounded, it could be discontinuous at the origin. On the other hand, here Theorem 2.3 gives us also the control over the bounded but not continuous parts of the generalized solution (through the coefficient $F^0(x,t)$ for $m = 0$ in the expansion formula (2.7)). As a sequence, Corollary 2.4(iii) guarantees that the solution is continuous.

(iii) Based on the formulae and the computations from [30], the general case in $\mathbb{R}^4$ is also treated, when the right-hand side f is smooth enough, but not a finite harmonic polynomial analogous to (2.5). The results are announced and published in [32, 33]. For right-hand side functions $f \in C^{10}(\overline{\Omega})$ in [33] the necessary and sufficient conditions for the existence of bounded solution are found. They involve infinite number of orthogonality conditions for f that comes from the fact that this is not a Fredholm problem. On the other hand, the results from [33] show that the linear operator mapping the generalized solution u into f is a semi-Fredholm operator in $C^{10}(\overline{\Omega})$. Let us recall that a semi-Fredholm operator is a bounded operator that has a finite dimensional kernel or cokernel and closed range. Additionally, in [32] a right-hand side function is constructed such that the unique generalized solution of Protter Problem (P1) in $\mathbb{R}^4$ has exponential type singularity. One expects that similar results could also be obtained for the Problem (P2) in $\mathbb{R}^3$ studied here. These questions correspond to the Open Problem (1) below.

Remark 2.7. Let us mention one obvious consequence of Theorem 2.3 and all the arguments above, concerning construction of functions orthogonal to the solutions $W^n_{k,i}$ of the homogeneous adjoint Problem (P2*). Take an arbitrary $C^2(\overline{\Omega})$ function $U(x,t)$ satisfying the boundary conditions (P2). Then the function $F := \Box U$ with the wave operator $\Box$, is orthogonal to all the functions $W^n_{k,i}$, $n = 1, 2, \ldots$.

Finally, we formulate some still open questions, that naturally arise from the previous works on the Protter problem and the discussions above.

Open Problems

(1) To study the more general case when the right-hand side function $f \in C^k(\overline{\Omega})$, for an appropriate k. The smooth function f could be represented as a Fourier series rather than, the finite trigonometric polynomial (2.5) in the discussions here.

(i) Find some appropriate conditions for the function f under which there exists a generalized solution of the Protter problem (P2).

(ii) What kind of singularity can the generalized solution have? The a priori estimates, obtained in [6, 31], which indicate that the *generalized solutions* of Problem (P2) (including the singular ones), can have at most an exponential growth as $\rho \to 0$. The natural question is as follows: is there a singular solution of these problems with exponential growth as $\rho \to 0$ or do all such solutions have only polynomial growth?

(iii) Is it possible to prove some a priori estimates for *generalized solutions* of the Problem (P2) with smooth function f which is not a harmonic polynomial?

(iv) Find some appropriate conditions for the function f under which the Problem (P2) has only regular, bounded solutions, or even classical solutions.

(2) To study the Protter problems for degenerate hyperbolic equations. Up to now it is only known that some singular solutions exist.

(i) We do not know what is the exact behavior of the singular solution even when the right-hand side function f is a finite sum like (2.5). Can we prove some a priori estimates for *generalized solutions*?

(ii) Is it possible to find some orthogonality conditions for the function f, as here, under which only bounded solutions exist?

(3) Why does there appear a singularity for such smooth right-hand side even for the wave equation? Can we numerically model this phenomenon?

(4) What happens with the ill-posedness of the Protter problems in a more general domain (as in [2, 4]) when the maximal symmetry is lost if the cone S_2 is replaced by another light characteristic one with the vertex away from the origin.

3. Preliminaries

We have a relation between the functions $W^n_{k,i}$ and the Legendre functions P_ν. For $\nu > -1/2$, the functions P_ν could be defined by the equality (Section 3.7, formula (6), from Erdélyi et al. [34]),

$$P_\nu(z) = \frac{1}{\pi}\int_0^\pi \left(z + \sqrt{z^2-1}\cos t\right)^\nu dt, \quad z \geq 0, \tag{3.1}$$

where for $z < 1$ in this formula $\sqrt{z^2-1} := i\sqrt{1-z^2}$.

Let (ρ, φ, t) be the cylindrical coordinates in $\mathbb{R}^3$, that is, $x_1 = \rho\cos\varphi$, $x_2 = \rho\sin\varphi$. For simplicity, define the function $E^n_k(\rho,t) := E^n_k(x,t)|x|^n$. The following result is in connection with Lemma 5.1 from [15]. Actually, to prove this result one could formally follows the arguments of Lemma 2.3 from [16], where the four-dimensional analogue of Problem (P2) is treated.

Lemma 3.1. *For $n \in \mathbb{N}$ and $\nu = n - 1/2$ define the functions*

$$h^\nu_k(\xi,\eta) := \int_\eta^\xi s^k P_\nu\left(\frac{\xi\eta + s^2}{s(\xi+\eta)}\right)ds, \tag{3.2}$$

for $0 < \eta < \xi$. Then in $\{\rho > t\}$ the equality

$$\rho^{-1/2}\frac{\partial}{\partial t}h^\nu_{\nu-2k}\left(\frac{\rho+t}{2}, \frac{\rho-t}{2}\right) = a^n_k E^n_k(\rho,t) \tag{3.3}$$

holds for $k = 0, 1, \ldots, [n/2]$ with some constants $a^n_k \neq 0$.

Proof. Lemma 5.1 from [15] for $k \geq 0$ gives

$$\rho^{-1/2}h^\nu_{\nu-2k-2}\left(\frac{\rho+t}{2}, \frac{\rho-t}{2}\right) = C^n_k H^n_k(\rho,t), \tag{3.4}$$

where $C^n_k = \text{const} \neq 0$ and according to Lemma 2.2 from [29]

$$\frac{\partial}{\partial t}H^n_k(\rho,t) = 2(n-k-1)E^n_{k+1}(\rho,t). \tag{3.5}$$

Therefore the equality (3.3) holds for $k \geq 1$. We have to prove it for $k = 0$. In the proof of Lemma 5.1 from [15] the integrals h^ν_k were calculated using the Mellin transform. In order to compute $h^\nu_k(\xi,\eta)$ in the same way let us first introduce the variables x and z:

$$\xi = \frac{\rho+t}{2}; \qquad \eta = \frac{\rho-t}{2}; \qquad x = \frac{\rho^2}{\rho^2-t^2}; \qquad z = \frac{\left(\rho^2-t^2\right)^{1/2}}{2}. \tag{3.6}$$

As a consequence after some calculations, formulas (2.2.(4)), (1.10), and (1.4) from Samko et al. [35], show that

$$z^{-\nu-1}h_\nu^\nu(\xi,\eta) = C_\nu x^{(\nu+1)/2} I_{0+}^1\left(x^{-\nu-3/2}(x-1)_+^{1/2}\right)(x), \tag{3.7}$$

where $I_{0+}^\alpha(u)(s)$ is the Riemann-Liouville fractional integral (for its properties see e.g., [34, 35]); in our case we have $I_0^1(u)(s) = \int_0^s u(\tau)d\tau$. As usual, we denote also $\lambda_+(s) := \lambda(s)$ for $s > 0, \lambda_+(s) := 0$ for $s \le 0$. The substitution of (3.6) in (3.7) shows that

$$\begin{aligned}
\frac{\partial}{\partial t}\left\{z^{\nu+1}x^{(\nu+1)/2}I_{0+}^1\left(x^{-\nu-3/2}(x-1)_+^{-1/2}\right)\right\} &= \frac{\partial}{\partial t}\left\{z^{\nu+1}x^{(\nu+1)/2}\int_0^x \tau^{-\nu-3/2}(\tau-1)_+^{-1/2}d\tau\right\} \\
&= \frac{\partial}{\partial t}\left\{\rho^{\nu+1}2^{-\nu-1}\int_1^x \tau^{-\nu-3/2}(\tau-1)^{-1/2}d\tau\right\} = 2^{-\nu-1}\rho^{\nu+1}x^{-\nu-3/2}(x-1)^{-1/2}\frac{\partial x}{\partial t} \\
&= 2^{-\nu-1}\rho^{\nu+1}\frac{\left(\rho^2-t^2\right)^{\nu+3/2}}{\rho^{2\nu+3}}\frac{\left(\rho^2-t^2\right)^{1/2}}{t}\frac{2t\rho^2}{\left(\rho^2-t^2\right)^2} = 2^{-\nu}\frac{\left(\rho^2-t^2\right)^\nu}{\rho^\nu}
\end{aligned} \tag{3.8}$$

and thus

$$\rho^{-1/2}\frac{\partial}{\partial t}h_\nu^\nu\left(\frac{\rho+t}{2},\frac{\rho-t}{2}\right) = a_0^n E_0^n(\rho,t). \tag{3.9}$$

□

The next result is crucial for construction of solutions of Problem (P2) in the discussions later.

Lemma 3.2. *Let $\nu \in R$, $\nu > 1/2$ and the functions $F(\xi) \in C^1(0,1/2]$ satisfy $F(1/2) = 0$. Then all solutions $\lambda \in C^1(0,1/2]$ of the Volterra integral equation of first kind*

$$\int_{1/2}^{\xi}\lambda'(\xi_1)P_\nu\left(\frac{\xi}{\xi_1}\right)d\xi_1 = F(\xi) \tag{3.10}$$

are

$$\lambda(\xi) = \lambda\left(\frac{1}{2}\right) + F(\xi) + \int_\xi^{1/2} P_\nu'\left(\frac{\xi_1}{\xi}\right)\frac{F(\xi_1)}{\xi_1}d\xi_1. \tag{3.11}$$

Proof. Formulas (35.17) and (35.28) from Samko et al. [35] state that the solution of the integral equation (3.10) is given by

$$\lambda'(\xi) = -\xi\frac{d^2}{d\xi^2}\left(\xi\int_{\xi}^{1/2} P_\nu\left(\frac{\xi_1}{\xi}\right)\frac{F(\xi_1)}{\xi_1^2}d\xi_1\right)$$
$$= -\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\int_{\xi}^{1/2} P_\nu\left(\frac{\xi_1}{\xi}\right)\frac{F(\xi_1)}{\xi_1^2}d\xi_1\right). \tag{3.12}$$

Then, using that $F(1/2) = 0$, an integration gives (3.11). □

One could use the Mellin transform to calculate the following integral.

Lemma 3.3 (see [16]). *Let $\nu \in R$, $\nu > -1/2$, then*

$$\int_{\xi}^{1/2} P_\nu\left(\frac{\xi_1}{\xi}\right)\frac{P_\nu(2\xi_1)}{\xi_1^2}d\xi_1 = \frac{1-2\xi}{\xi}. \tag{3.13}$$

According to the existence and uniqueness results in [6], it is sufficient to study Problem (P2) when the right-hand side f of the wave equation is simply

$$f(\rho,t,\varphi) = f_n^1(\rho,t)\cos n\varphi + f_n^2(\rho,t)\sin n\varphi, \quad n \in \mathbb{N}\cup\{0\}. \tag{3.14}$$

Then we seek solutions for the wave equation of the same form:

$$u(\rho,t,\varphi) = u_n^1(\rho,t)\cos n\varphi + u_n^2(\rho,t)\sin n\varphi. \tag{3.15}$$

Thus Problem (P2) reduces to the following one.

Problem (*P*2.1)

Solve the equation

$$(u_n)_{\rho\rho} + \frac{1}{\rho}(u_n)_\rho - (u_n)_{tt} - \frac{n^2}{\rho^2}u_n = f_n(\rho,t) \tag{3.16}$$

in $D_1 = \{0 < t < 1/2;\ t < \rho < 1-t\} \subset \mathbb{R}^2$ with the boundary conditions

$$(u_n)_t(\rho,0) = 0 \quad \text{for } 0 < \rho \le 1, \qquad u_n(\rho,1-\rho) = 0 \quad \text{for } \frac{1}{2} \le \rho \le 1. \tag{P2.1}$$

Let us now introduce new coordinates

$$\xi = \frac{\rho+t}{2}; \qquad \eta = \frac{\rho-t}{2}, \tag{3.17}$$

and set

$$v(\xi,\eta) = \rho^{1/2}u_n(\rho,t); \qquad g(\xi,\eta) = \rho^{1/2}f_n(\rho,t). \tag{3.18}$$

Denoting $\nu = n - 1/2$, one transforms Problem (P2.1) into the following.

Problem (*P*2.2)

Find a solution $v(\xi,\eta)$ of the equation

$$v_{\xi\eta} - \frac{\nu(\nu+1)}{(\xi+\eta)^2}v = g(\xi,\eta) \tag{3.19}$$

in the domain $D = \{0 < \xi < 1/2;\ 0 < \eta < \xi\}$ with the following boundary conditions:

$$(v_\xi - v_\eta)(\eta,\eta) = 0, \qquad v\left(\frac{1}{2},\eta\right) = 0 \quad \text{for } \eta \in \left(0,\frac{1}{2}\right). \tag{P2.2}$$

Problems (P2.1) and (P2.2) were introduced in [6], although the change of coordinates $\xi = 1-\rho-t$ and $\eta = 1-\rho+t$ was used there instead of (3.17). Of course, because the solution of Problem (P2) may be singular, the same is true for the solutions of (P2.1) and (P2.2). For that reason, Popivanov and Schneider [6] defined and proved the existence and uniqueness of generalized solutions of Problems (P2.1) and (P2.2), which correspond to the generalized solution of Problem (P2). Further, by "solution" of Problem (P2.1) or (P2.2) we mean exactly this unique generalized solution.

Lemma 3.4. *Let $\nu \in R, \nu > 1/2$ and $g \in C^1(\overline{D})$. Then the solution $v(\xi,\eta)$ of Problem* (*P*2.2) *is given by the following formula:*

$$\begin{aligned} v(\xi,\eta) &= \tau(\xi) + \int_\xi^{1/2} \tau(\xi_1)\frac{\partial}{\partial\xi_1}P_\nu\left(\frac{(\xi-\eta)\xi_1 + 2\xi\eta}{\xi_1(\xi+\eta)}\right)d\xi_1 \\ &\quad - \int_\xi^{1/2}\left(\int_0^\eta P_\nu\left(\frac{(\xi-\eta)(\xi_1-\eta_1)+2\xi_1\eta_1+2\xi\eta}{(\xi_1+\eta_1)(\xi+\eta)}\right)g(\xi_1,\eta_1)d\eta_1\right)d\xi_1, \end{aligned} \tag{3.20}$$

where

$$\tau(\xi) = \int_{1/2}^{\xi} P_\nu\left(\frac{\xi_1}{\xi}\right)G(\xi_1)d\xi_1, \tag{3.21}$$

$$\begin{aligned} G(\xi) &= \int_\xi^{1/2}\int_0^\xi P_\nu\left(\frac{\xi_1\eta_1+\xi^2}{\xi(\xi_1+\eta_1)}\right)\left(\frac{\partial}{\partial\xi_1} - \frac{\partial}{\partial\eta_1}\right)g(\xi_1,\eta_1)d\eta_1 d\xi_1 \\ &\quad - \int_0^\xi P_\nu\left(\frac{\eta_1+2\xi^2}{\xi(2\eta_1+1)}\right)g\left(\frac{1}{2},\eta_1\right)d\eta_1 - \int_\xi^{1/2} P_\nu\left(\frac{\xi}{\xi_1}\right)g(\xi_1,0)d\xi_1. \end{aligned} \tag{3.22}$$

Proof. Notice that the function

$$R(\xi_1,\eta_1;\xi,\eta) = P_\nu\left(\frac{(\xi-\eta)(\xi_1-\eta_1)+2\xi_1\eta_1+2\xi\eta}{(\xi_1+\eta_1)(\xi+\eta)}\right) \tag{3.23}$$

is a Riemann function for (3.19) (Copson [36]). Therefore, following Aldashev [10], we can construct the function $v(\xi,\eta)$ as a solution of a Goursat problem in D with boundary conditions $v(1/2,\eta) = 0$ and $v(\xi,0) = \tau(\xi)$ with some unknown function $\tau(\xi) \in C^2(0,1/2]$, which will be determined later:

$$\begin{aligned} v(\xi,\eta) = \tau(\xi) &+ \int_\xi^{1/2}\tau(\xi_1)\frac{\partial}{\partial\xi_1}R(\xi_1,0;\xi,\eta)d\xi_1 \\ &- \int_\xi^{1/2}\int_0^\eta R(\xi_1,\eta_1;\xi,\eta)g(\xi_1,\eta_1)d\eta_1 d\xi_1. \end{aligned} \tag{3.24}$$

Now, the boundary condition

$$\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)v\bigg|_{\eta=\xi} = 0. \tag{3.25}$$

gives an integral equation for $\tau(\xi)$. For that reason, let us define the function $G(\xi)$:

$$\begin{aligned} G(\xi) &:= \left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)\int_\xi^{1/2}\left(\int_0^\eta R(\xi_1,\eta_1;\xi,\eta)g(\xi_1,\eta_1)d\eta_1\right)d\xi_1\bigg|_{\eta=\xi} \\ &= \int_\xi^{1/2}\left(\int_0^\xi P'_\nu\left(\frac{\xi_1\eta_1+\xi^2}{\xi(\xi_1+\eta_1)}\right)\frac{\xi_1-\eta_1}{\xi(\xi_1+\eta_1)}g(\xi_1,\eta_1)d\eta_1\right)d\xi_1 \\ &\quad - \int_0^\xi g(\xi,\eta_1)d\eta_1 - \int_\xi^{1/2} g(\xi_1,\xi)d\xi_1 = -\int_0^\xi g(\xi,\eta_1)d\eta_1 - \int_\xi^{1/2} g(\xi_1,\xi)d\xi_1 \\ &\quad - \int_\xi^{1/2}\left(\int_0^\xi g(\xi_1,\eta_1)\left(\frac{\partial}{\partial\xi_1}-\frac{\partial}{\partial\eta_1}\right)P_\nu\left(\frac{\xi_1\eta_1+\xi^2}{\xi(\xi_1+\eta_1)}\right)d\eta_1\right)d\xi_1 \\ &= \int_\xi^{1/2}\left(\int_0^\xi P_\nu\left(\frac{\xi_1\eta_1+\xi^2}{\xi(\xi_1+\eta_1)}\right)\left(\frac{\partial}{\partial\xi_1}-\frac{\partial}{\partial\eta_1}\right)g(\xi_1,\eta_1)d\eta_1\right)d\xi_1 \\ &\quad - \int_0^\xi P_\nu\left(\frac{\eta_1+2\xi^2}{\xi(2\eta_1+1)}\right)g\left(\frac{1}{2},\eta_1\right)d\eta_1 - \int_\xi^{1/2}P_\nu\left(\frac{\xi}{\xi_1}\right)g(\xi_1,0)d\xi_1. \end{aligned} \tag{3.26}$$

Obviously, $G \in C^2[0,1/2]$. The condition (3.25) leads us to the following equation:

$$\tau'(\xi) - \frac{1}{\xi}\tau(\xi)P'_\nu(1) - \int_\xi^{1/2}\frac{\tau(\xi_1)}{\xi_1^2}P''_\nu\left(\frac{\xi}{\xi_1}\right)d\xi_1 = G(\xi). \tag{3.27}$$

Then, using $\tau(1/2) = v(1/2, 0) = 0$, we have

$$\int_{1/2}^{\xi} \frac{d}{d\xi_1}\left\{\xi_1^2 \tau'(\xi_1)\right\} P_\nu\left(\frac{\xi}{\xi_1}\right) d\xi_1 = \xi^2 G(\xi) - \frac{\tau'(1/2)}{4} P_\nu(2\xi). \tag{3.28}$$

A necessary solvability condition for the unknown function $\tau \in C^2(0, 1/2]$ is: $\tau'(1/2) = G(1/2)$. One could solve this Volterra integral equation of the first kind, using Lemma 3.2. The result is

$$\xi^2 \tau'(\xi) - \frac{1}{4}\tau'\left(\frac{1}{2}\right) = \xi^2 G(\xi) - \frac{1}{4}\tau'\left(\frac{1}{2}\right) P_\nu(2\xi) + \int_{\xi}^{1/2} P_\nu'\left(\frac{\xi_1}{\xi}\right) \frac{4\xi_1^2 G(\xi_1) - \tau'(1/2) P_\nu(2\xi_1)}{4\xi_1} d\xi_1. \tag{3.29}$$

Integrate, we find

$$\begin{aligned} \tau(\xi) = & \int_{1/2}^{\xi} \left(G(z) + \frac{1}{z^2} \int_{z}^{1/2} P_\nu'\left(\frac{\xi_1}{z}\right) \xi_1 G(\xi_1) d\xi_1 \right) dz \\ & + \frac{\tau'(1/2)}{4} \int_{1/2}^{\xi} \left(\frac{1}{z^2} - \frac{P_\nu(2z)}{z^2} - \frac{1}{z^2} \int_{1/2}^{z} P_\nu'\left(\frac{\xi_1}{z}\right) P_\nu(2\xi_1) \xi_1^{-1} d\xi_1 \right) dz. \end{aligned} \tag{3.30}$$

Now, using Lemma 3.3 and the equality

$$\int_{1/2}^{\xi} \frac{1}{z^2} \left(\int_{z}^{1/2} P_\nu'\left(\frac{\xi_1}{z}\right) F(\xi_1) d\xi_1 \right) dz = \int_{1/2}^{\xi} \left(P_\nu\left(\frac{\xi_1}{\xi}\right) - 1 \right) \frac{F(\xi_1)}{\xi_1} d\xi_1, \tag{3.31}$$

for $F(\xi) = P_\nu(2\xi)\xi^{-1}$ one finds that the coefficient of $\tau'(1/2)$ in (3.30) is zero. Using again (3.31) for $F(\xi) = \xi G(\xi)$, τ is simply

$$\tau(\xi) = \int_{1/2}^{\xi} P_\nu\left(\frac{\xi_1}{\xi}\right) G(\xi_1) d\xi_1. \tag{3.32}$$

Obviously, $\tau \in C^2(0, 1/2]$ and $\tau(1/2) = 0, \tau'(1/2) = G(1/2)$. Finally, the solution of Problem (P2.2) is given by the formulae (3.20), (3.21), and (3.22). □

4. Proofs of Main Results

In order to study the behavior of the generalized solution of Problem (P2), in view of relations (3.18) and Lemma 3.4, we will examine the function $v(\xi, \eta)$ defined by the formulae (3.20), (3.21), and (3.22). It is not hard to see that the part "responsible" for the singularity is the integral in (3.21) for the function $\tau(\xi)$. In fact, $\tau(\xi)$ blows up at $\xi = 0$, since the argument ξ_1/ξ and thus the values of the Legendre function P_ν in (3.21) go to infinity when $\xi \to 0$. Actually,

$P_\nu(z)$ grows like $|z|^\nu$ at infinity. In the next lemma we find the dependance of the exact order of singularity of $\tau(\xi)$ on the function $G(\xi)$. It is governed by the constants

$$\gamma_k := \int_0^{1/2} \xi^{\nu-2k} G(\xi)\, d\xi \quad \text{for } k = 0, \ldots, \left[\frac{\nu+1}{2}\right]. \tag{4.1}$$

Actually, these numbers are closely connected to the constants $\beta^n_{k,i}$ from Theorem 2.3. We will clarify this relation later in Lemma 4.1 and the proof of Theorem 2.3.

Lemma 4.1. *Let $\nu = n - 1/2$, where $n \in \mathbb{N}$, $n \geq 2$, and let the function $G(\xi) \in C^1[0,1/2]$. Then the function*

$$\tau(\xi) = \int_{1/2}^{\xi} P_\nu\left(\frac{\xi_1}{\xi}\right) G(\xi_1) d\xi_1 \tag{4.2}$$

belongs to $C^2(0,1/2]$ and satisfies the representation

$$\tau(\xi) = \sum_{k=0}^{[(\nu+1)/2]} C_k^\nu \gamma_k \xi^{-(\nu-2k)} + \psi(\xi), \qquad \xi \in \left(0, \frac{1}{2}\right), \tag{4.3}$$

where the function $\psi(\xi) \in C^2(0,1/2]$, $|\psi(\xi)| \leq C\xi \max\{|G(\xi)| : \xi \in 0,1/2]\}$ and the nonzero constants C_k^ν and C are independent of $G(\xi)$.

Proof. The argument of the Legendre function P_ν in (4.2) satisfies the inequality $\xi_1/\xi \geq 1$, which allows us to apply the representation (3.1):

$$\tau(\xi) = \frac{1}{\pi}\frac{1}{\xi^\nu} \int_{1/2}^{\xi} \int_0^{\pi} \left(\xi_1 + \sqrt{\xi_1^2 - \xi^2}\cos t\right)^\nu G(\xi_1) dt\, d\xi_1. \tag{4.4}$$

We will study the expansion at $\xi = 0$ of the function

$$F(\xi) := \int_{\xi}^{1/2} \int_0^{\pi} \left(\xi_1 + \sqrt{\xi_1^2 - \xi^2}\cos t\right)^\nu G(\xi_1) dt\, d\xi_1. \tag{4.5}$$

Let us define the functions

$$M_k^\nu(\xi_1, \xi) := (-1)^k \frac{(\nu - 2k + 1)_{2k}}{2^k (1/2)_k} \int_0^{\pi} \left(\xi_1 + \sqrt{\xi_1^2 - \xi^2}\cos t\right)^{\nu-2k} \sin^{2k} t\, dt, \tag{4.6}$$

for $\xi \leq \xi_1 \leq 1/2$. Then, obviously

$$F(\xi) = \int_{\xi}^{1/2} M_0^\nu(\xi_1, \xi) G(\xi_1) d\xi_1. \tag{4.7}$$

First, we will examine the properties of the functions $M_k^\nu(\xi_1,\xi)$ and their derivatives with respect to ξ. We start with the equality

$$M_k^\nu(\xi,\xi) = a_k^\nu \xi^{\nu-2k}, \qquad a_k^\nu \neq 0. \tag{4.8}$$

Further, the index k will be less than ν. Notice that for $k < \nu + 1/2$ the integrals $\int_0^\pi (1 \pm \cos t)^{\nu-2k} \sin^{2k} t\,dt$ are convergent. Then, for $\xi \leq \xi_1$ we have the equality

$$M_k^\nu(\xi_1,0) = b_k^\nu \xi_1^{\nu-2k}, \quad b_k^\nu \neq 0, \tag{4.9}$$

and the inequality

$$\left| M_k^\nu(\xi_1,\xi) \right| \leq c_k^\nu \xi_1^{\nu-2k}. \tag{4.10}$$

Differentiating with respect to ξ one finds

$$\begin{aligned}
&(-1)^{k+1} \frac{\partial}{\partial \xi} M_k^\nu(\xi_1,\xi) \\
&\quad = \frac{(\nu-2k+1)_{2k}(\nu-2k)}{2^k(1/2)_k} \int_0^\pi \left(\xi_1 + \sqrt{\xi_1^2-\xi^2}\cos t \right)^{\nu-2k-1} \frac{\xi}{\sqrt{\xi_1^2-\xi^2}} \sin^{2k} t \cos t\,dt \\
&\quad = \frac{(\nu-2k)_{2k+1}(\nu-2k-1)}{2^k(1/2)_k(2k+1)} \int_0^\pi \xi \left(\xi_1 + \sqrt{\xi_1^2-\xi^2}\cos t \right)^{\nu-2k-2} \sin^{2k+2} t\,dt \\
&\quad = (-1)^{k+1} \xi M_{k+1}^\nu(\xi_1,\xi).
\end{aligned} \tag{4.11}$$

Therefore, for the derivatives of M_0^ν we find by induction

$$\frac{\partial^k M_0^\nu}{\partial \xi^k}(\xi_1,\xi) = \sum_{i=0}^{[k/2]} C_i^k \xi^{k-2i} M_{k-i}^\nu(\xi_1,\xi), \tag{4.12}$$

where the coefficients C_i^k are positive constants. We want to evaluate these derivatives of M_0^ν at $\xi = 0$. Let us estimate the terms in the last sum for $k < \nu$:

(i) when i is such that $\nu - 2(k-i) < 0$ the inequality (4.10) gives the estimate

$$\left| \xi^{k-2i} M_{k-i}^\nu(\xi_1,\xi) \right| \leq \xi^{k-2i} c_k^\nu \xi_1^{\nu-2(k-i)} \leq c_k^\nu \xi^{\nu-k}; \tag{4.13}$$

(ii) when $\nu - 2(k-i) \geq 0$ and $k/2 > i$, we have

$$\left| \xi^{k-2i} M_{k-i}^\nu(\xi_1,\xi) \right| \leq c_k^\nu \xi^{k-2i}. \tag{4.14}$$

Hence $\xi^{k-2i} M^{\nu}_{k-i}(\xi_1,\xi)|_{\xi=0} = 0$ for $2i < k$. Therefore, at the point $\xi = 0$ the only one nonzero term in the sum (4.12) is for $2i = k$, that is,

$$\frac{\partial^k M_0^{\nu}}{\partial \xi^k}(\xi_1, 0) = \begin{cases} 0, & \text{if } k \text{ is odd} \\ C^k_{k/2} b^{\nu}_{k/2} \xi_1^{\nu-k}, & \text{if } k \text{ is even.} \end{cases} \tag{4.15}$$

The last observation is that (4.8) and (4.12) imply

$$\left.\frac{\partial^k M_0^{\nu}}{\partial \xi^k}\right|_{\xi_1=\xi} = d^{\nu}_k \xi^{\nu-k}, \tag{4.16}$$

where $d^{\nu}_k = \sum_{i=0}^{[k/2]} C^k_i a^{\nu}_{k-i}$ are constants.

Now, we go back to the function $F(\xi)$. We want to differentiate $[\nu]$ times and evaluate at $\xi = 0$. Differentiating (4.7) we find the following:

$$F'(\xi) = -a^{\nu}_0 G(\xi)\xi^{\nu} + \int_{\xi}^{1/2} \frac{\partial}{\partial \xi} M^{\nu}_0(\xi_1,\xi) G(\xi_1) d\xi_1. \tag{4.17}$$

Next, since the assertion for $G(\xi)$ is only $G(\xi) \in C^1[0,1/2]$, instead of $F'(\xi)$ we will differentiate the function

$$F_1(\xi) := F'(\xi) + a^{\nu}_0 G(\xi)\xi^{\nu} = \int_{\xi}^{1/2} \frac{\partial}{\partial \xi} M^{\nu}_0(\xi_1,\xi) G(\xi_1) d\xi_1. \tag{4.18}$$

Notice that it belongs to $C[0,1/2] \cap C^1(0,1/2]$ and the derivative is

$$F_1'(\xi) = -d^{\nu}_1 G(\xi)\xi^{\nu-1} + \int_{\xi}^{1/2} \frac{\partial^2}{\partial \xi^2} M^{\nu}_0(\xi_1,\xi) G(\xi_1) d\xi_1. \tag{4.19}$$

In the same way, after denoting $F_0(\xi) \equiv F(\xi)$, define for $j = 1, \ldots, [\nu]$ the functions

$$F_j(\xi) := F'_{j-1}(\xi) + d^{\nu}_{j-1} G(\xi)\xi^{\nu-j+1} \tag{4.20}$$

with the constants d^{ν}_j from (4.16). Then, using (4.16), it follows by induction that F_j is continuous in $[0,1/2]$ and

$$F_j(\xi) = \int_{\xi}^{1/2} \frac{\partial^j}{\partial \xi^j} M^{\nu}_0(\xi_1,\xi) G(\xi_1) d\xi_1, \qquad j = 0, \ldots, [\nu]. \tag{4.21}$$

On the other hand,

$$F_j(0) = \int_0^{1/2} \frac{\partial^j}{\partial \xi^j} M^{\nu}_0(\xi_1,0) G(\xi_1) d\xi_1, \quad j = 0, \ldots, [\nu] - 1. \tag{4.22}$$

Hence, according to (4.15), for $j \leq [\nu] - 1$,

$$F_j(0) = \begin{cases} 0, & \text{if } j \text{ is odd} \\ \gamma_i', & \text{if } j \text{ is even.} \end{cases} \tag{4.23}$$

The next step is to evaluate the integral $F_{[\nu]}$. Using (4.12), one could rewrite it in the form

$$F_{[\nu]}(\xi) = \int_{\xi}^{1/2} \frac{\partial^{[\nu]}}{\partial \xi^{[\nu]}} M_0^{\nu}(\xi_1, \xi) G(\xi_1) d\xi_1 = \sum_{i=0}^{[[\nu]/2]} C_i^{[\nu]} \int_{\xi}^{1/2} \xi^{[\nu]-2i} M_{[\nu]-i}^{\nu}(\xi_1, \xi) G(\xi_1) d\xi_1. \tag{4.24}$$

For all the terms in the last sum, except one, the estimate is straightforward.
(1) When i is such that $[\nu] - 2i \geq 2$, for the corresponding terms we have

$$\left| \xi^{[\nu]-2i} M_{[\nu]-i}^{\nu}(\xi_1, \xi) \right| \leq \xi^2 \xi_1^{[\nu]-2i-2} c_{[\nu]-i}^{\nu} \xi_1^{\nu-2([\nu]-i)} \leq c_{[\nu]-i}^{\nu} \xi^2 \xi_1^{-3/2}, \tag{4.25}$$

and, therefore,

$$\left| \int_{\xi}^{1/2} \xi^{[\nu]-2i} M_{[\nu]-i}^{\nu}(\xi_1, \xi) G(\xi_1) d\xi_1 \right| \leq c_{[\nu]-i}^{\nu} A \xi^2 \int_{\xi}^{1/2} \xi_1^{-3/2} d\xi_1 \leq C A \xi^{3/2}, \tag{4.26}$$

where $A := \max\{|G(\xi)| : \xi \in 0, 1/2]\}$.
(2) For the last term in (4.24) with $i = [[\nu]/2]$ there are two cases:

(2a) when $[\nu] = 2m$ is an even number this is the integral

$$\int_{\xi}^{1/2} M_m^{\nu}(\xi_1, \xi) G(\xi_1) d\xi_1; \tag{4.27}$$

(2b) when $[\nu] = 2m - 1$ is an odd number, the integral is

$$\int_{\xi}^{1/2} \xi M_m^{\nu}(\xi_1, \xi) G(\xi_1) d\xi_1. \tag{4.28}$$

For simplicity let us define some constants γ_i'

$$\gamma_i' := \int_{0}^{1/2} M_i^{\nu}(\xi_1, 0) G(\xi_1) d\xi_1, \tag{4.29}$$

related to the constants γ_i given by (4.1). Indeed, due to (4.9) the equality $\gamma_i' = b_i^\nu \gamma_i$ holds. Let us begin with the following case.

(2a): $[\nu] = 2m$, that is, $\nu = 2m + 1/2$. We will evaluate the difference:

$$\begin{aligned}\left|\int_\xi^{1/2} M_m^\nu(\xi_1,\xi)G(\xi_1)d\xi_1 - \gamma_m'\right| &= \left|\int_\xi^{1/2} M_m^\nu(\xi_1,\xi)G(\xi_1)d\xi_1 - \int_0^{1/2} M_m^\nu(\xi_1,0)G(\xi_1)d\xi_1\right| \\ &\le \left|\int_\xi^{1/2}\{M_m^\nu(\xi_1,\xi) - M_m^\nu(\xi_1,0)\}G(\xi_1)d\xi_1\right| \\ &\quad + \left|\int_0^\xi M_m^\nu(\xi_1,0)G(\xi_1)d\xi_1\right|.\end{aligned} \tag{4.30}$$

For the first integral, using the estimate (4.10), we calculate

$$\begin{aligned}|M_m^\nu(\xi_1,\xi) - M_m^\nu(\xi_1,0)| &= \left|\int_0^\xi \frac{\partial}{\partial\xi_2} M_m^\nu(\xi_1,\xi_2)d\xi_2\right| \\ &= \left|\int_0^\xi \xi_2 M_{m+1}^\nu(\xi_1,\xi_2)d\xi_2\right| \le c_{m+1}^\nu \xi^2\xi_1^{\nu-2m-2} = c_{m+1}^\nu\xi^2\xi_1^{-3/2},\end{aligned} \tag{4.31}$$

and, therefore,

$$\left|\int_\xi^{1/2}\{M_m^\nu(\xi_1,\xi) - M_m^\nu(\xi_1,0)\}G(\xi_1)d\xi_1\right| \le C_1 A\xi^{3/2}. \tag{4.32}$$

For the second integral

$$\left|\int_0^\xi M_m^\nu(\xi_1,0)G(\xi_1)d\xi_1\right| \le c_m^\nu A\int_0^\xi \xi_1^{\nu-2m}d\xi_1 = CA\int_0^\xi \xi_1^{1/2}d\xi_1 = C_2A\xi^{3/2}. \tag{4.33}$$

From the last two inequalities we get the estimate

$$\left|\int_\xi^{1/2} M_m^\nu(\xi_1,\xi)G(\xi_1)d\xi_1 - \gamma_m'\right| \le CA\xi^{3/2}. \tag{4.34}$$

Therefore, in the case $[\nu] = 2m$, $m \in \mathbb{N}$,

$$F_{[\nu]}(\xi) = \gamma_m' + \psi_{[\nu]}(\xi), \tag{4.35}$$

where $|\psi_{[\nu]}(\xi)| \le CA\xi^{3/2}$.

(2b) When $[\nu] = 2m-1$, that is, $\nu = 2m-1/2$, we have to study the integral (4.28). Obviously,

$$\left|\int_{\xi}^{1/2} \xi M_m^{\nu}(\xi_1,\xi)G(\xi_1)d\xi_1 - \gamma_m'\xi\right| \le \xi\left|\int_{\xi}^{1/2}\{M_m^{\nu}(\xi_1,\xi) - M_m^{\nu}(\xi_1,0)\}G(\xi_1)d\xi_1\right| + \xi\left|\int_0^{\xi} M_m^{\nu}(\xi_1,0)G(\xi_1)d\xi_1\right|. \tag{4.36}$$

For the last integral we have

$$\left|\int_0^{\xi} M_m^{\nu}(\xi_1,0)G(\xi_1)d\xi_1\right| \le c_m^{\nu}A\int_0^{\xi}\xi_1^{\nu-2m}d\xi_1 = 2CA\xi^{1/2}. \tag{4.37}$$

Now, to estimate the first term in the right-hand side of (4.36), there are two cases:

(i) when $m \ge 2$, we have $\nu > [\nu] = 2m-1 \ge m+1$ and similarly to the previous case (2a) we can apply inequality (4.31). Thus, we estimate the difference:

$$|M_m^{\nu}(\xi_1,\xi) - M_m^{\nu}(\xi_1,0)| = \left|\int_0^{\xi}\frac{\partial}{\partial\xi_2}M_m^{\nu}(\xi_1,\xi_2)d\xi_2\right| = \left|\int_0^{\xi}\xi_2 M_{m+1}^{\nu}(\xi_1,\xi_2)d\xi_2\right| \le \int_0^{\xi} c_{m+1}^{\nu}\xi_2\xi_1^{\nu-2m-2}d\xi_2 \le c_{m+1}^{\nu}\xi^2\xi_1^{-5/2}, \tag{4.38}$$

(ii) when $m = 1$, denote for simplicity $p := \sqrt{1-\xi^2/\xi_1^2}$, then $p \in [0,1]$ and directly from the definition (4.6) of the functions M_m^{ν}, we get

$$\begin{aligned}
&\left|M_1^{3/2}(\xi_1,\xi) - M_1^{3/2}(\xi_1,0)\right| \\
&\quad = C\xi_1^{-1/2}\left|\int_{-1}^{1}\left\{(1-pz)^{-1/2} - (1-z)^{-1/2}\right\}\sqrt{1-z^2}dz\right| \\
&\quad \le C\xi_1^{-1/2}\int_{-1}^{1}\frac{(1-p)|z|\sqrt{1+z}}{\sqrt{1-pz}\left(\sqrt{1-pz}+\sqrt{1-z}\right)}dz \\
&\quad \le C_1\xi_1^{-1/2}(1-p) + C_1\xi_1^{-1/2}(1-p)\int_0^1\frac{1}{\sqrt{(1-pz)(1-z)}}dz \\
&\quad \le C_1\xi_1^{-1/2}(1-p)\left[1+\int_0^1\frac{1}{\sqrt{(1-p+p\tau)\tau}}d\tau\right] \\
&\quad \le C_2\xi_1^{-1/2}\sqrt{1-p} \le C_2\xi_1^{-1/2}\sqrt{1-p^2} = C\xi\xi_1^{-3/2}.
\end{aligned} \tag{4.39}$$

Thus for $m \geq 1$, both cases lead to

$$\left| \int_{\xi}^{1/2} \{ M_m^{\nu}(\xi_1, \xi) - M_m^{\nu}(\xi_1, 0) \} G(\xi_1) d\xi_1 \right| \leq CA\xi^{1/2}. \tag{4.40}$$

Finally, substituting (4.37) and (4.40) in (4.36), we find the estimate

$$\left| \int_{\xi}^{1/2} M_m^{\nu}(\xi_1, \xi) G(\xi_1) d\xi_1 - \gamma_m' \xi \right| \leq CA\xi^{3/2}. \tag{4.41}$$

Summarizing, in the case (2b) of odd $[\nu] = 2m - 1$, we have

$$F_{[\nu]}(\xi) = \gamma_m' \xi + \psi_{[\nu]}(\xi), \tag{4.42}$$

where $|\psi_{\nu]}(\xi)| \leq CA\xi^{3/2}$.

Now, we are ready to go backwards from $F_{[\nu]}(\xi)$ to $F_0(\xi)$. Integrating (4.20) we find

$$F_j(\xi) = F_j(0) + \int_0^{\xi} F_{j+1}(\xi_1) d\xi_1 + d_j^{\nu} \int_0^{\xi} \xi_1^{\nu - j} G(\xi_1) d\xi_1, \tag{4.43}$$

for $j = 0, \ldots, [\nu] - 1$. Using (4.43) and (4.23), one can find the relation between the functions $F(\xi) \equiv F_0(\xi)$ and $F_{[\nu]}(\xi)$. Starting from (4.43) with $j = 0$ one expresses $F_1(\xi)$ by applying again (4.43) in the right-hand side but for $j = 1$:

$$\begin{aligned} F(\xi) \equiv F_0(\xi) &= F_0(0) + \int_0^{\xi} F_1(\xi_1) d\xi_1 + d_0^{\nu} \int_0^{\xi} \xi_1^{\nu} G(\xi_1) d\xi_1 = F_0(0) + F_1(0)\xi \\ &+ \int_0^{\xi} \left(\int_0^t F_2(\xi_1) d\xi_1 \right) dt + \int_0^{\xi} \left\{ d_1^{\nu} \int_0^t \xi_1^{\nu-1} G(\xi_1) d\xi_1 + d_0^{\nu} t^{\nu} G(t) \right\} dt. \end{aligned} \tag{4.44}$$

Similarly, (4.43) gives a representation of $F_2(\xi)$ through $F_3(\xi)$ and so on. Finally, we get the sum

$$\begin{aligned} F(\xi) &= F_0(0) + \frac{F_1(0)}{1!}\xi + \frac{F_2(0)}{2!}\xi^2 + \cdots + d_0^{\nu} \int_0^{\xi} G(\xi_1) \xi_1^{\nu} d\xi_1 \\ &+ d_1^{\nu} \int_0^{\xi} G(\xi_1) \xi_1^{\nu-1} (\xi - \xi_1) d\xi_1 + \frac{d_2^{\nu}}{2} \int_0^{\xi} G(\xi_1) \xi_1^{\nu-2} (\xi - \xi_1)^2 d\xi_1 + \cdots, \end{aligned} \tag{4.45}$$

by consequently substituting $F_{k+1}(\xi)$ in the resulting expression at each step by applying the same formula (4.43) for $j = k + 1$. Thus, for $F(\xi)$ we find inductively

$$F(\xi) = \sum_j \frac{F_j(0)}{j!} \xi^j + \cdots = \sum_k \frac{\gamma_k'}{(2k)!} \xi^{2k} + \cdots, \tag{4.46}$$

since $F_{2i+1}(0) = 0$ and $F_{2i}(0) = \gamma_i'$, in view of (4.23). The process ends when $F_{[\nu]}$, integrated $[\nu]$ times, appears and we can apply (4.35) or (4.42) instead of (4.43). Therefore, according to (4.35), when $[\nu] = 2m$ is an even number, the last γ_i' in this sum will be γ_m', and its coefficient will be $(1/(2m)!)\xi^{2m}$, while when $[\nu] = 2m-1$ is an odd number, formula (4.42) shows that the last term will be also $(\gamma_m'/(2m)!)\xi^{2m}$. In both cases $m = [(\nu+1)/2]$ and the constant coefficients are independent of $G(\xi)$. Then, for the function $F(\xi)$ we have the representation

$$F(\xi) = \sum_{k=0}^{[(\nu+1)/2]} \frac{\gamma_k'}{(2k)!}\xi^{2k} + \Psi(\xi), \tag{4.47}$$

where the function $\Psi(\xi)$ is defined by

$$\Psi(\xi) := \sum_{j=0}^{[\nu]-1} \frac{d_j^\nu}{j!}\int_0^\xi G(\xi_1)\xi_1^{\nu-j}(\xi-\xi_1)^j d\xi_1 + \int_0^\xi \varphi_{[\nu]}(\xi_1)\frac{(\xi-\xi_1)^{[\nu]-1}}{([\nu]-1)!}d\xi_1. \tag{4.48}$$

Therefore $|\Psi(\xi)| \le CA\xi^{\nu+1}$, because $|G(\xi)| \le A$ and $|\varphi_{[\nu]}(\xi_1)| \le CA\xi_1^{3/2}$.

Finally, recall that $\gamma_k' = b_k^\nu \gamma_k$, with coefficients $b_k^\nu \neq 0$ from (4.9) and $\tau(\xi) = -\pi^{-1}\xi^{-\nu}F(\xi)$ from (4.4), and, therefore,

$$\tau(\xi) = \sum_{k=0}^{[(\nu+1)/2]} C_k\gamma_k\xi^{-(\nu-2k)} + \psi(\xi), \tag{4.49}$$

where $C_k = -\pi^{-1}b_k^\nu/(2k)! \neq 0$ and $|\psi(\xi)| \le CA\xi$. □

This lemma, due to formula (3.20), helps us to examine the solution $v(\xi,\eta)$ of Problem (P2.2) and therefore due to (3.18), the solution $u_n(\rho,t) = \rho^{-1/2}v$ of Problem (P2.1). First, for $k = 0,\dots,[n/2]$, denote by α_k^n the parameters

$$\alpha_k^n := \int_0^{1/2}\left(\int_t^{1-t} E_k^n(\rho,t)f_n(\rho,t)\rho d\rho\right)dt. \tag{4.50}$$

Theorem 4.2. *Let $f_n \in C^1(\overline{D}_1)$. Then the generalized solution $u_n(\rho,t)$ of Problem (P2.1) belongs to $C^2(\overline{D}_1 \setminus (0,0))$ and has the following asymptotic expansion at the origin $(0,0)$:*

$$u_n(\rho,t) = \sum_{k=0}^{[n/2]} \rho^{-1/2}(\rho+t)^{-(n-2k-1/2)}\alpha_k^n F_k^n(\rho,t) + \rho^{1/2}(\ln\rho)F^n(\rho,t), \tag{4.51}$$

where $F_k^n(\rho,t), F^n(\rho,t) \in C^2(\overline{D}_1 \setminus (0,0))$,

$$\left|F_k^n(\rho,t)\right| \le C, \qquad \left|F^n(\rho,t)\right| \le C\left\{\max_{\overline{D}_1}\left|f_n(\rho,t)\right| + \max_{\overline{D}_1}\left|(f_n)_t(\rho,t)\right|\right\}, \tag{4.52}$$

with functions F_k^n and a constant C independent of f_n and $\lim_{t\to+0}F_k^n(t,t) = const \neq 0$.

First, let us shortly outline the proof. In order to find the behavior of u_n we apply the relation (3.18) and study the function v. Actually, Lemma 3.4 uses Lemmas 3.2 and 3.3 to describe v by formulas (3.20)–(3.22) and the analysis passes to τ, G, and the Legendre function P_ν. This way the base for the asymptotic expansion (4.51) is the expansion found in Lemma 4.1 for τ given by (3.21).

Proof of Theorem 4.2. Denote

$$A := \max_{\overline{D}_1}\left|f_n(\rho,t)\right| + \max_{\overline{D}_1}\left|(f_n)_t(\rho,t)\right| \tag{4.53}$$

and thus $|g(\xi,\eta)| \leq CA, |G(\xi)| \leq CA$ with the constant C independent of f_n. Our goal is to apply Lemma 4.1.

The key of this will be the equality

$$\begin{aligned}
&\int_0^{1/2} \xi^{\nu-2k} G(\xi)d\xi \\
&\quad= \int_0^{1/2}\int_\xi^{1/2}\left(\int_0^\xi \xi^{\nu-2k} P_\nu\left(\frac{\xi_1\eta_1+\xi^2}{\xi(\xi_1+\eta_1)}\right)\left(\frac{\partial}{\partial\xi_1}-\frac{\partial}{\partial\eta_1}\right)g(\xi_1,\eta_1)d\eta_1\right)d\xi_1 d\xi \\
&\qquad- \int_0^{1/2}\left(\int_0^\xi \xi^{\nu-2k} P_\nu\left(\frac{\eta_1+2\xi^2}{\xi(1+2\eta_1)}\right)g\left(\frac{1}{2},\eta_1\right)d\eta_1\right)d\xi \\
&\qquad- \int_0^{1/2}\left(\int_\xi^{1/2} \xi^{\nu-2k} P_\nu\left(\frac{\xi}{\xi_1}\right)g(\xi_1,0)d\xi_1\right)d\xi \\
&\quad= \int_0^{1/2}\int_0^{\xi_1}\left(\int_{\eta_1}^{\xi_1} \xi^{\nu-2k} P_\nu\left(\frac{\xi_1\eta_1+\xi^2}{\xi(\xi_1+\eta_1)}\right)d\xi\right)\left(\frac{\partial}{\partial\xi_1}-\frac{\partial}{\partial\eta_1}\right)g(\xi_1,\eta_1)d\eta_1 d\xi_1 \\
&\qquad- \int_0^{1/2}\left(\int_{\eta_1}^{1/2} \xi^{\nu-2k} P_\nu\left(\frac{\eta_1+2\xi^2}{\xi(1+2\eta_1)}\right)d\xi\right)g\left(\frac{1}{2},\eta_1\right)d\eta_1 \\
&\qquad- \int_0^{1/2}\left(\int_0^{\xi_1} \xi^{\nu-2k} P_\nu\left(\frac{\xi}{\xi_1}\right)d\xi\right)g(\xi_1,0)d\xi_1 \\
&\quad= -\int_0^{1/2}\int_0^{\xi_1}\left(\frac{\partial}{\partial\xi_1}-\frac{\partial}{\partial\eta_1}\right)\left(\int_{\eta_1}^{\xi_1} \xi^{\nu-2k} P_\nu\left(\frac{\xi_1\eta_1+\xi^2}{\xi(\xi_1+\eta_1)}\right)d\xi\right)g(\xi_1,\eta_1)d\eta_1 d\xi_1 \\
&\quad= -\int_0^{1/2}\int_0^{\xi_1}\left(\frac{\partial}{\partial\xi_1}-\frac{\partial}{\partial\eta_1}\right)\left\{h^\nu_{\nu-2k}(\xi_1,\eta_1)\right\}g(\xi_1,\eta_1)d\eta_1 d\xi_1, \qquad k = 0,\dots,\left[\frac{n}{2}\right],
\end{aligned} \tag{4.54}$$

according to Definition (3.2) of functions h^ν_k in Lemma 3.1.

On the other hand, Lemma 3.1 gives

$$\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)(\xi+\eta)^{-1/2}h^\nu_{\nu-2k}(\xi,\eta) = a^n_k E^n_k(\xi+\eta,\xi-\eta), \tag{4.55}$$

and by definition $f = (\xi+\eta)^{-1/2}g$. Therefore, we have the following relation between α_k^n defined in (4.50) and the constants γ_k from Lemma 4.1:

$$\begin{aligned}\alpha_k^n &= \int_0^{1/2}\left(\int_t^{1-t} E_k^n(\rho,t)f(\rho,t)\rho d\rho\right)dt\\ &= \frac{2}{a_k^n}\int_0^{1/2}\left(\int_0^{\xi_1}\left(\frac{\partial}{\partial\xi_1}-\frac{\partial}{\partial\eta_1}\right)\left\{(\xi_1+\eta_1)^{-1/2}h_{\nu-2k}^{\nu}(\xi_1,\eta_1)\right\}g(\xi_1,\eta_1)(\xi_1+\eta_1)^{1/2}d\eta_1\right)d\xi_1\\ &= -\frac{2}{a_k^n}\int_0^{1/2}\xi_1^{\nu-2k}G(\xi_1)d\xi_1 = -\frac{2}{a_k^n}\gamma_k,\end{aligned} \tag{4.56}$$

with coefficients $a_k^n \neq 0$ from (3.3). Then Lemma 4.1 gives

$$\tau(\xi) = \sum_{k=0}^{[(\nu+1)/2]} C_k^n\alpha_k^n\xi^{-(\nu-2k)} + \psi(\xi), \tag{4.57}$$

where the constants $C_k^n \neq 0$ are independent of f and $|\psi(\xi)| \le CA\xi$. Hence, for the solution $v(\xi,\eta)$ of Problem (P2.1) we have

$$\begin{aligned}v(\xi,\eta) &= \tau(\xi) + \int_\xi^{1/2}\tau(\xi_1)\frac{\partial}{\partial\xi_1}P_\nu\left(\frac{(\xi-\eta)\xi_1+2\xi\eta}{\xi_1(\xi+\eta)}\right)d\xi_1\\ &\quad -\int_\xi^{1/2}\left(\int_0^\eta P_\nu\left(\frac{(\xi-\eta)(\xi_1-\eta_1)+2\xi_1\eta_1+2\xi\eta}{(\xi_1+\eta_1)(\xi+\eta)}\right)g(\xi_1,\eta_1)d\eta_1\right)d\xi_1\\ &= \sum_{k=0}^{[(\nu+1)/2]} C_k^n\beta_k^n G_k^n(\xi,\eta)\xi^{-(\nu-2k)} + \Psi_1(\xi,\eta),\end{aligned} \tag{4.58}$$

where

$$G_k^n(\xi,\eta) = 1 + \xi^{\nu-2k}\int_\xi^{1/2}\xi_1^{2k-\nu}\frac{\partial}{\partial\xi_1}P_\nu\left(\frac{(\xi-\eta)\xi_1+2\xi\eta}{\xi_1(\xi+\eta)}\right)d\xi_1, \tag{4.59}$$

$$\begin{aligned}\Psi_1(\xi,\eta) &= \psi(\xi) + \int_\xi^{1/2}\psi(\xi_1)\frac{\partial}{\partial\xi_1}P_\nu\left(\frac{(\xi-\eta)\xi_1+2\xi\eta}{\xi_1(\xi+\eta)}\right)d\xi_1\\ &\quad -\int_\xi^{1/2}\int_0^\eta P_\nu\left(\frac{(\xi-\eta)(\xi_1-\eta_1)+2\xi_1\eta_1+2\xi\eta}{(\xi_1+\eta_1)(\xi+\eta)}\right)g(\xi_1,\eta_1)d\xi_1 d\eta_1.\end{aligned} \tag{4.60}$$

Notice that the arguments of the Legendre's functions P_ν in (4.58), (4.59) and (4.60) vary in the interval $[0,1]$. Thus,

$$\left|G_k^n(\xi,\eta)\right| \le 1 + C_1 \frac{\eta\xi^{\nu-2k+1}}{\xi+\eta}\int_\xi^{1/2} \xi_1^{2k-\nu-2} d\xi_1 \le 1 + \frac{C_1}{\nu+1-2k} = C. \tag{4.61}$$

Therefore, the functions

$$F_k^n(\rho,t) := 2^{\nu-2k} C_k^n G_k^n\left(\frac{\rho+t}{2}, \frac{\rho-t}{2}\right) \tag{4.62}$$

are also bounded. On the other hand, $v(\xi,0) = \tau(\xi)$ and therefore

$$F_k^n(t,t) = 2^{\nu-2k} C_k^n G_k^n(t,0) = 2^{\nu-2k} C_k^n \tag{4.63}$$

with coefficients $C_k^n \neq 0$ from (4.57).

Let us now evaluate the function Ψ_1 defined in (4.60). We have $|\varphi(\xi)| \le CA\xi$,

$$\begin{gathered}\left|\int_\xi^{1/2} \varphi(\xi_1)\frac{\partial}{\partial \xi_1} P_\nu\left(\frac{(\xi-\eta)\xi_1 + 2\xi\eta}{\xi_1(\xi+\eta)}\right) d\xi_1\right| \le CA\frac{\xi\eta}{\xi+\eta}\left|\int_\xi^{1/2} \xi_1^{-1} d\xi_1\right| \le CA\xi|\ln\xi| \\ \left|\int_\xi^{1/2}\int_0^\eta P_\nu\left(\frac{(\xi-\eta)(\xi_1-\eta_1) + 2\xi_1\eta_1 + 2\xi\eta}{(\xi_1+\eta_1)(\xi+\eta)}\right) g(\xi_1,\eta_1) d\xi_1 d\eta_1\right| \le CA\xi.\end{gathered} \tag{4.64}$$

Finally, let us return to (ρ,t) coordinates using (3.17) and (3.18). The representation (4.58) gives (4.51), where the function

$$F^n(\rho,t) := \rho^{-1/2}(\ln\rho)^{-1}\Psi_1\left(\frac{\rho+t}{2}, \frac{\rho-t}{2}\right) \tag{4.65}$$

is continuous in $\overline{D}_1$ and the estimate $|F^n(\rho,t)| \le C_1 A\rho^{1/2}$, holds with $C_1 = \text{const}$. □

Finally, we are ready to prove our main result.

Proof of Theorem 2.3. The uniqueness and the existence of the generalized solution when $f \in C^1(\overline{\Omega})$ is a trigonometric polynomial, follows from the results in [6]. Now the right-hand side function satisfies (2.5), and thus it can be written in the form

$$f(x_1,x_2,t) = \sum_{n=2}^{l}\left\{f_n^1(|x|,t)\cos n\varphi + f_n^2(|x|,t)\sin n\varphi\right\}. \tag{4.66}$$

According to [6] the unique generalized solution $u(x_1,x_2,t)$ also has the form

$$u(x_1,x_2,t) = \sum_{n=2}^{l}\left\{u_n^1(\rho,t)\cos n\varphi + u_n^2(\rho,t)\sin n\varphi\right\}, \tag{4.67}$$

where the functions $u_n^i(\rho,t)$ are solutions of Problem (P2.1) with right-hand side function $f_n^i \in C^1(\overline{G})$ and are described in Theorem 4.2. Then, for the constants α_k^n from Theorem 4.2 and $\beta_{k,i}^n$ from (2.6), we have the following relation:

$$\alpha_k^n = \int_0^{1/2}\left(\int_t^{1-t} E_k^n(\rho,t) f_n^i(\rho,t)\rho d\rho\right)dt = \pi^{-1}\int_\Omega W_{k,i}^n(x,t) f(x,t)dx\,dt = \pi^{-1}\beta_{k,i}^n. \tag{4.68}$$

Therefore, from Theorem 4.2 it follows that

$$u_n^i(\rho,t) = \pi^{-1}\sum_{k=0}^{[n/2]}\rho^{-1/2}(\rho+t)^{-(n-2k-1/2)}\beta_{k,i}^n F_k^{n,i}(\rho,t) + \rho^{1/2}(\ln\rho)F^{n,i}(\rho,t), \tag{4.69}$$

where the functions $F_k^{n,i}$ are independent of f, $|F_k^{n,i}(\rho,t)| \le C$ and

$$\left|F^{n,i}(\rho,t)\right| \le C\left(\max_{\overline{D}_1}\left|f_n^i\right| + \max_{\overline{D}_1}\left|\left(f_n^i\right)_t\right|\right) \le C_1\left(\max_{\overline{\Omega}}\left|f(x,t)\right| + \max_{\overline{\Omega}}\left|f_t(x,t)\right|\right). \tag{4.70}$$

Summing over n and i in (4.67) we get the expansion

$$\begin{aligned} u(x,t) = {} & \sum_{n=2}^{l}\sum_{i=1}^{2}\sum_{k=0}^{[n/2]}\left(|x|^2+t^2\right)^{k-n/2}\beta_{k,i}^n F_{k,i}^n(x,t) \\ & + \left(|x|^2+t^2\right)^{1/4}F(x,t)\ln\left(|x|^2+t^2\right), \end{aligned} \tag{4.71}$$

where

$$\begin{aligned} F_{k,1}^n(x,t) &= \pi^{-1}\rho^{-1/2}\left(\rho^2+t^2\right)^{k-n/2}(\rho+t)^{2k-n+1/2}F_k^{n,1}(\rho,t)\cos n\varphi, \\ F_{k,2}^n(x,t) &= \pi^{-1}\rho^{-1/2}\left(\rho^2+t^2\right)^{k-n/2}(\rho+t)^{2k-n+1/2}F_k^{n,2}(\rho,t)\sin n\varphi, \\ F(x,t) &= \frac{|x|^{1/2}\ln|x|}{\left(|x|^2+t^2\right)^{1/4}\ln\left(|x|^2+t^2\right)}\sum_{n=2}^{l}\left(F^{n,1}(\rho,t)\cos n\varphi + F^{n,2}(\rho,t)\sin n\varphi\right). \end{aligned} \tag{4.72}$$

Obviously the functions $F_{k,i}^n(x,t)$ are bounded and independent of f. Also, we have

$$|F(x,t)| \le C\left(\max_{\overline{\Omega}}\left|f(x,t)\right| + \max_{\overline{\Omega}}\left|f_t(x,t)\right|\right). \tag{4.73}$$

Notice that the singularity $(|x|^2+t^2)^{-m/2}$ for fixed m appears in the sum for the solution $u(x,t)$ when $n = m, m+2, m+4, \ldots$, and the corresponding coefficients are $\beta^m_{0,i}F^m_{0,i}$; $\beta^m_{1,i}F^{m+2}_{1,i}$; $\beta^m_{2,i}F^{m+4}_{2,i}$; an so on, until $n \le l$. Therefore, (4.71) is equivalent to

$$u(x,t) = \sum_{m=0}^{l}\left(|x|^2+t^2\right)^{-m/2}\sum_{k=0}^{[(l-m)/2]}\sum_{i=1}^{2}\beta^{m+2k}_{k,i}F^{m+2k}_{k,i}(x,t) + \left(|x|^2+t^2\right)^{1/4}F(x,t)\ln\left(|x|^2+t^2\right). \tag{4.74}$$

Thus, the properties (i), (ii), and (v) are proved.

Finally, let us prove the properties (iii) and (iv). For a fixed direction $(\alpha_1,\alpha_2,1)t = (\cos\gamma,\sin\gamma,1)t \in S_2$, $0<t<1/2$, $\gamma\in[0,2\pi)$ we have the expressions

$$\begin{aligned} F^n_{k,1}(\alpha_1 t,\alpha_2 t,t) &= \pi^{-1}2^{2k-n+1/2}F^{n,1}_k(t,t)\cos n\gamma,\\ F^n_{k,2}(\alpha_1 t,\alpha_2 t,t) &= \pi^{-1}2^{2k-n+1/2}F^{n,2}_k(t,t)\sin n\gamma, \end{aligned} \tag{4.75}$$

with the functions $F^{n,i}_k(\rho,t)$ from (4.69) and $F^n_{k,i}(x_1,x_2,t)$ from (2.8). Therefore, according to (2.8) and (4.74),

$$\lim_{t\to+0}F^m(\alpha_1 t,\alpha_2 t,t) = \sum_{k=0}^{[(l-m)/2]}\left\{C^m_{k,1}\beta^{m+2k}_{k,1}\cos(m+2k)\gamma + C^m_{k,2}\beta^{m+2k}_{k,2}\sin(m+2k)\gamma\right\} \tag{4.76}$$

with some constants $C^m_{k,i}\ne 0$. Thus, this expression is zero for all $\gamma\in[0,2\pi]$ if and only if all the constants $\beta^{m+2k}_{k,i}$ involved are zero, because the trigonometric functions are linearly independent in $[0,2\pi]$. Thus, if at least one $\beta^{m+2k}_{k,i}\ne 0$, one could choose γ, that is, a direction $(\alpha_1,\alpha_2,1)$, such that $\lim_{t\to+0}F^m(\alpha_1 t,\alpha_2 t,t) = C_m = \text{const}\ne 0$, which proves (iii).

For (iv) in the case $m=0$ we have

$$\lim_{t\to+0}F^0(\alpha_1 t,\alpha_2 t,t) = \sum_{k=1}^{[l/2]}\left\{C^0_{k,1}\beta^{2k}_{k,1}\cos 2k\gamma + C^0_{k,2}\beta^{2k}_{k,2}\sin 2k\gamma\right\}, \tag{4.77}$$

and the sum starts at $k=1$ since $\beta^0_{0,1}=\beta^0_{0,2}=0$ according to Definition (2.6) and the special form (2.5) of f. Now, F^0 is continuous at $(0,0,0)$ only when the expression in the right-hand side of (4.77) is a constant. However, the constant 1 and the trigonometric functions involved in (4.77) are linearly independent. Therefore, if at least one $\beta^{2k}_{k,i}$ is not zero, then F^0 is not continuous at the origin. □

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