Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 275748, 9 pages
doi:10.1155/2012/275748

Research Article

Existence and Nonexistence of Positive Solutions for Quasilinear Elliptic Problem

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Received 18 April 2012; Accepted 21 June 2012

Academic Editor: Sergey Piskarev

Using variational arguments we prove some existence and nonexistence results for positive solutions of a class of elliptic boundary-value problems involving the p-Laplacian.

1. Introduction

In a recent paper, Rădulescu and Repovš [1] studied the existence and nonexistence of positive solutions of the nonlinear elliptic problem

$$\begin{gathered} -\Delta u = \lambda k(x) u^q \pm h(x) u^p \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0 \text{ in } \Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $\lambda > 0$ is a parameter, $0 < q < 1 < p$, and h, k in $L^\infty(\Omega)$ such that

$$\operatorname{ess}\inf_{x\in\Omega} k(x) > 0, \qquad \operatorname{ess}\inf_{x\in\Omega} h(x) > 0. \tag{1.2}$$

They showed using sub-supersolutions arguments and monotonicity methods that the problem $(1.1)_+$ has a minimal solution, provided that $\lambda > 0$ is small enough. The next result is concerned with problem $(1.1)_-$ and asserts that there is some $\lambda^* > 0$ such that $(1.1)_-$ has a nontrivial solution if $\lambda > \lambda^*$ and no solution exists provided that $\lambda < \lambda^*$.

In the present paper we consider that the corresponding quasilinear problem

$$-\Delta_p u = \lambda k(x) u^q \pm h(x) u^r \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0, \quad u > 0 \text{ in } \Omega, \tag{1.3}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, denotes the p-Laplacian operator, $1 < p < \infty$, $\lambda > 0$, $0 \leq q < p-1 < r < p^* - 1$, with $p^* = Np/(N-p)$ if $p < N$, and $p^* = +\infty$ otherwise, and h, k in $L^\infty(\Omega)$ such that

$$\operatorname*{ess\,inf}_{x\in\Omega} k(x) > 0, \qquad \operatorname*{ess\,inf}_{x\in\Omega} h(x) > 0. \tag{1.4}$$

We are concerned with the existence of weak solutions of problems $(1.3)_+$ and $(1.3)_-$, that is, for functions $u \in W_0^{1,p}(\Omega)$ satisfying $\operatorname{ess\,inf}_K u > 0$ over every compact set $K \subset \Omega$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \mathrm{d}x = \lambda \int_\Omega k(x) u^q \phi \mathrm{d}x \pm \int_\Omega h(x) u^r \phi \mathrm{d}x \tag{1.5}$$

for all $\phi \in C_c^\infty(\Omega)$. As usual, $C_c^\infty(\Omega)$ denotes the space of all C^∞ functions $\phi : \Omega \to \mathbb{R}$ with compact support. Using variational methods, we will prove the following theorems.

Theorem 1.1. *Assume $0 \leq q < p-1 < r < p^* - 1$. Then there exists a positive number Λ such that the following properties hold:*

(1) *for all $\lambda \in (0, \Lambda)$ problem $(1.3)_+$ has a minimal solution u_λ;*

(2) *Problem $(1.3)_+$ has a solution if $\lambda = \Lambda$;*

(3) *Problem $(1.3)_+$ does not have any solution if $\lambda > \Lambda$.*

Theorem 1.2. *Assume $0 \leq q < p-1 < r < p^* - 1$. Then there exists a positive number Λ such that the following properties hold:*

(1) *If $\lambda > \Lambda$, then problem $(1.3)_-$ has at least one solution;*

(2) *If $\lambda < \Lambda$, then problem $(1.3)_-$ does not have any solution.*

2. Proof of Theorem 1.1

At first, we give the definition of weak supersolution and subsolution of $(1.3)_+$. By definition $u \in W_0^{1,p}(\Omega)$ is a weak subsolution to $(1.3)_+$ if $u > 0$ in Ω and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \mathrm{d}x \leq \lambda \int_\Omega k(x) u^q \phi \mathrm{d}x \pm \int_\Omega h(x) u^r \phi \mathrm{d}x \tag{2.1}$$

for all $\phi \in C_c^\infty(\Omega)$. Similarly $u \in W_0^{1,p}(\Omega)$ is a weak supersolution to $(1.3)_+$ if in the above the reverse inequalities hold.

Let us define

$$\Lambda \stackrel{\text{def}}{=} \sup\{\lambda > 0 : (1.3)_{+}\text{has a weak solution}\} \tag{2.2}$$

and the energy functional $E_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} E_\lambda(u) = {} & \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x \\ & - \frac{\lambda}{q+1}\int_\Omega k(x)u^{q+1}\mathrm{d}x - \frac{1}{r+1}\int_\Omega h(x)u^{q+1}\mathrm{d}x \end{aligned} \tag{2.3}$$

in the Sobolev space $W_0^{1,p}(\Omega)$.

The proof of the theorem is organized in several steps.

Step 1 (existence of minimal solution for $0 < \lambda < \Lambda$). To show the existence of a solution to $(1.3)_+$, we construct a subsolution $\underline{u}_\lambda$, and a supersolution $\overline{u}_\lambda$, such that $\underline{u}_\lambda \le \overline{u}_\lambda$.

We introduce the following Dirichlet problem:

$$\begin{aligned} -\Delta_p \widetilde{u} &= \lambda k(x)\widetilde{u}^q \quad \text{in } \Omega, \\ \widetilde{u}|_{\partial\Omega} &= 0, \quad \widetilde{u} > 0 \text{ in } \Omega. \end{aligned} \tag{2.4}$$

From [2] we know there exists a unique solution, say $\widetilde{u}$, satisfying the problem (2.4). Define $\underline{u}_\lambda = \epsilon\widetilde{u}$. Then $-\Delta_p(\underline{u}_\lambda) = \lambda k(x)\epsilon^{p-1}\widetilde{u}^q$ and $\underline{u}_\lambda$ is a subsolution of the problem $(1.3)_+$ if

$$\lambda k(x)\epsilon^{p-1}\widetilde{u}^q \le \lambda k(x)\epsilon^q\widetilde{u}^q + h(x)\epsilon^r\widetilde{u}^r. \tag{2.5}$$

Indeed, for ϵ small enough we get

$$\lambda k(x)\epsilon^{p-1}\widetilde{u}^q \le \lambda k(x)\epsilon^q\widetilde{u}^q \le \lambda k(x)\epsilon^q\widetilde{u}^q + h(x)\epsilon^r\widetilde{u}^r. \tag{2.6}$$

(Since $q < p - 1$ and for $\epsilon \in (0,1)$). Then $\epsilon\widetilde{u}$ is a subsolution of the problem $(1.3)_+$.

On the other hand, let v the solution to the following problem be:

$$\begin{aligned} -\Delta_p v &= \lambda + 1 \quad \text{in } \Omega, \\ v|_{\partial\Omega} &= 0, \quad v > 0 \text{ in } \Omega. \end{aligned} \tag{2.7}$$

Then $0 < v < K$ in Ω. By simplicity of writing we call

$$F(u) = \lambda k(x)u^q + h(x)u^r. \tag{2.8}$$

Define $\overline{u}_\lambda(x) = Tv(x)$ where T is a constant that will be chosen in such a way that

$$-\Delta_p\overline{u}_\lambda \ge F(TM) \ge F(\overline{u}_\lambda), \tag{2.9}$$

where $M = \max\{1, \|v\|_\infty\}$. Now $-\Delta_p \overline{u}_\lambda = T^{p-1}(\lambda+1)$ and

$$F(\overline{u}_\lambda) \equiv \lambda k(x) T^q v^q + T^r v^r \leq \lambda c_1 T^q M^q + c_2 T^r M^r, \tag{2.10}$$

where $c_1 = \|k\|_{L^\infty}$ et $c_2 = \|h\|_{L^\infty}$. Then, it is sufficient to find T such that

$$(\lambda+1) \geq \lambda c_1 T^{q+1-p} M^q + c_2 T^{r+1-p} M^r. \tag{2.11}$$

We call

$$\varphi(T) = \lambda A T^{q+1-p} + B T^{r+1-p}, \tag{2.12}$$

with $A = c_1 M^q, B = c_2 M^r$. Then

$$\lim_{T\to 0^+} \varphi(T) = \lim_{T\to\infty} \varphi(T) = \infty, \tag{2.13}$$

because $q+1-p < 0 < r+1-p$; then φ attains a minimum in $[0,\infty)$. Elementary computations shows that this function attains its minimum for $T_0 = C\lambda^{1/(r-q)}$ where $C = [AB^{-1}(r-p+1)(p-q-1)^{-1}]^{1/(r-q)}$. For the validity of (2.11) it suffices that

$$\varphi(T_0) \leq \lambda + 1, \tag{2.14}$$

that is,

$$D\lambda^{(r+1-p)/(r-q)} < \lambda + 1, \tag{2.15}$$

where D is a constant, depends on p, q, and M. Then there exists λ_0 such that for $0 < \lambda < \lambda_0$, $\overline{u}(x) = T_0 v$ is a supersolution of problem $(1.3)_+$. It remains to show that $\epsilon\tilde{u} \leq T_0 v$. In turn, fix the supersolution, that is, T, for ϵ small enough, we get

$$-\Delta_p \underline{u}_\lambda = \lambda k(x) \epsilon^{p-1} \tilde{u}^q \leq \lambda \epsilon^{p-1} \leq -\Delta_p(\overline{u}_\lambda). \tag{2.16}$$

Consequently, we may apply the weak comparison principle (see Proposition 2.3 in [3]) in order to conclude that $\underline{u}_\lambda \leq \overline{u}_\lambda$. Thus, By the classical iteration method $(1.3)_+$ has a solution between the subsolution and supersolution.

Let us now prove that u_λ is a minimal solution of $(1.3)_+$. We use here the weak comparison principle (see Proposition 2.3 in Cuesta and Takáč [3]) and the following monotone iterative scheme:

$$\begin{gathered} -\Delta_p u_n = \lambda k(x) u_{n-1}^q + h(x) u_{n-1}^r \quad \text{in } \Omega; \\ u_n|_{\partial\Omega} = 0, \end{gathered} \tag{2.17}$$

where $u_0 = \underline{u}_\lambda$, the unique solution to (2.4). Note that u_0 is a weak subsolution to $(1.3)_+$. and $u_0 \leq U$ where U is any weak solution to $(1.3)_+$. Then, from the weak comparison principle, we get easily that $u_0 \leq u_1$ and $\{u_n\}_{n=1}^{\infty}$ is a nondecreasing sequence. Furthermore, $u_n \leq U$ and $\{u_n\}_{n=1}^{\infty}$ is uniformly bounded in $W_0^{1,p}(\Omega)$. Hence, it is easy to prove that $\{u_n\}$ converges weakly in $W_0^{1,p}(\Omega)$ and pointwise to $\widehat{u}_\lambda$, a weak solution to $(1.3)_+$. Let us show that $\widehat{u}_\lambda$ is the minimal solution to $(1.3)_+$ for any $0 < \lambda < \Lambda$. Let v_λ a weak solution to $(1.3)_+$ for any $0 < \lambda < \Lambda$. Then, $u_0 = \underline{u}_\lambda \leq v_\lambda$. From the weak comparison principle, $u_n \leq v_\lambda$ for any $n \geq 0$. Letting $n \to \infty$, we get $\widehat{u}_\lambda \leq v_\lambda$. This completes the proof of the Step 1.

Step 2 (there exists $\Lambda > 0$ such that $(1.3)_+$ has no positive solution for $\lambda > \Lambda$). From the definition of Λ, problem $(1.3)_+$ does not have any solution if $\lambda > \Lambda$. In what follows we claim that $\Lambda < \infty$. We argue by contradiction: suppose there exists a sequence $\lambda_n \to \infty$ such that $(1.3)_+$ admits a solution u_n. Denote

$$m := \min\left\{\operatorname{ess}\inf_{x\in\Omega} k(x), \operatorname{ess}\inf_{x\in\Omega} h(x)\right\} > 0. \tag{2.18}$$

There exists $\lambda_* > 0$ such that

$$m(\lambda t^q + t^r) \geq (\lambda_1 + \epsilon)t^{p-1} \quad \forall t > 0,\ \epsilon \in (0,1),\ \lambda > \lambda_*, \tag{2.19}$$

where λ_1 is the first Dirichlet eigenvalue of $-\Delta_p$ is positive and is given by

$$\lambda_1 = \min_{u\neq 0} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \tag{2.20}$$

(see Lindqvist [4]). Choose $\lambda_n > \lambda_*$. Clearly u_n is a supersolution of the problem

$$\begin{gathered} -\Delta_p u = (\lambda_1 + \epsilon)u^{p-1} \quad \text{in } \Omega, \\ u > 0, \quad u|_{\partial\Omega} = 0 \end{gathered} \tag{2.21}$$

for all $\epsilon \in (0,1)$. We now use the result in [2] to choose $\mu < \lambda_1 + \epsilon$ small enough so that $\mu\phi_1(x) < u_n(x)$ and $\mu\phi_1$ is a subsolution to problem (2.8). By a monotone interation procedure we obtain a solution to (2.8) for any $\epsilon \in (0,1)$, contradicting the fact that λ_1 is an isolated point in the spectrum of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ (see Anane [5]). This proves the claim and completes the proof of the Step 2.

Step 3 (there exists at least one positive-weak solution for $\lambda = \Lambda$ to $(1.3)_+$). Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be such that $\lambda_k \uparrow \Lambda$ as $k \to \infty$. Then, from Step 1, there exists $u_k = u_{\lambda_k} \geq \underline{u}_{\lambda_k}$ to a weak positive solution to $(1.3)_+$ for $\lambda = \lambda_k$. Therefore, for any $\phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla\phi \,\mathrm{d}x = \lambda_k k(x)\int_\Omega (u_k)^q \phi \,\mathrm{d}x + h(x)\int_\Omega u_k^r \phi \,\mathrm{d}x. \tag{2.22}$$

Since $u_k \in W_0^{1,p}(\Omega)$ and $u_k \geq \underline{u}_{\lambda_k}$ it is easy to see that (2.22) holds also for $\phi \in W_0^{1,p}(\Omega)$. Moreover, from above

$$E_{\lambda_k}(u_k) \leq E_{\lambda_k}\left(\underline{u}_{\lambda_k}\right) < \frac{1}{p}\int_\Omega \left|\nabla \underline{u}_{\lambda_k}\right|^p \mathrm{d}x - \frac{\lambda_k k(x)}{q+1}\int_\Omega \underline{u}_{\lambda_k}{}^{q+1}\mathrm{d}x < 0, \tag{2.23}$$

it follows that

$$\sup_k \|u_k\|_p < \infty. \tag{2.24}$$

Hence, there exists $u_\Lambda \geq \underline{u}_{\lambda_k}$ such that $u_k \rightharpoonup u_\Lambda$ in $W_0^{1,p}(\Omega)$ as $k \to \infty$ and then by Sobolev imbedding and using the fact that $k,\ h \in L^\infty(\Omega)$:

$$u_k \rightharpoonup u \quad \text{in } L^q(\Omega) \text{ and point wise a.e. as } k \longrightarrow \infty. \tag{2.25}$$

From (2.22), (2.24), and (2.25), we get for any $\phi \in W_0^{1,p}(\Omega)$

$$\int_\Omega |\nabla u_\Lambda|^{p-2}\nabla u_\Lambda \nabla\phi \mathrm{d}x = \lambda\int_\Omega k(x)u_\Lambda^q\phi\mathrm{d}x + \int_\Omega h(x)u_\Lambda^r\phi\mathrm{d}x \tag{2.26}$$

which completes the proof of the Step 3 and gives the proof of Theorem 1.1.

3. Proof of Theorem 1.2

At first, we introduce some notation which will be used throughout the proof. The norm in $W_0^{1,p}(\Omega)$ will be denoted by

$$\|u\|_p \stackrel{\text{def}}{=} \left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)^{1/p}. \tag{3.1}$$

The norm in $L^{q+1}(\Omega)$ will be denoted by

$$\|u\|_{q+1} \stackrel{\text{def}}{=} \left(\int_\Omega |u|^{q+1} \mathrm{d}x\right)^{1/q+1}. \tag{3.2}$$

The norm in $L^{r+1}(\Omega)$ will be denoted by

$$\|u\|_{r+1} \stackrel{\text{def}}{=} \left(\int_\Omega |u|^{r+1} \mathrm{d}x\right)^{1/r+1}. \tag{3.3}$$

Let us define the energy functional $J_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x - \frac{\lambda}{q+1}\int_\Omega k(x)u^{q+1}\mathrm{d}x + \frac{1}{r+1}\int_\Omega h(x)u^{r+1}\mathrm{d}x \tag{3.4}$$

in the Sobolev space $W_0^{1,p}(\Omega)$.

The proof of the theorem is organized in several steps.

Step 1 (coercivity of J_λ:). For any $u \in W_0^{1,p}(\Omega)$ and all $\lambda > 0$

$$J_\lambda(u) \geq \frac{1}{p}\|u\|^p - C_1\|u\|_{q+1}^{q+1} + C_2\|u\|_{r+1}^{r+1}, \tag{3.5}$$

where $C_1 = \lambda\|k\|_{L^\infty}/(q+1)$ and $C_2 = (r+1)^{-1}\text{ess inf}_{x\in\Omega}h(x)$ are positive constants. We call

$$\phi(T) = AT^{q+1-p} - BT^{r+1-p}. \tag{3.6}$$

Then

$$\lim_{T\to 0^+}\phi(T) = \lim_{T\to\infty}\phi(T) = \infty, \tag{3.7}$$

because $q+1-p < 0 < r+1-p$; then φ attains a minimum $m < 0$ in $[0,\infty)$. By elementary computations shows that this function attains its minimum for $T = [A(q+1-p)/(Br+1-p)]^{1/(r-q)}$.

Returning to (3.5), we deduce that

$$J_\lambda(u) \geq \frac{1}{p}\|u\|^p + m. \tag{3.8}$$

Hence, from (3.8), we get that

$$J_\lambda(u) \longrightarrow +\infty \quad \text{as } \|u\| \longrightarrow \infty. \tag{3.9}$$

Let $n \mapsto u_n$ be a minimizing sequence of J_λ in $W_0^{1,p}(\Omega)$, which is bounded in $W_0^{1,p}(\Omega)$ by Step 1. Without loss of generality, we may assume that $(u_n)_n$ is nonnegative, converges weakly to some u in $W_0^{1,p}(\Omega)$, and converges also pointwise. Moreover, by the weak lower semicontinuity of the norm $\|\cdot\|$ and the boundedness of $(u_n)_n$ in $W_0^{1,p}(\Omega)$ we get

$$J_\lambda(u) \leq \liminf_{n\to\infty} J_\lambda(u_n). \tag{3.10}$$

Hence u is a global minimizer of J_λ in $W_0^{1,p}(\Omega)$, which completes the proof of the Step 1.

Step 2 (the weak limit u is a nonnegative weak solution of $(1.3)_-$ if $\lambda > 0$ is sufficiently large). Firstly, observe that $J_\lambda(0) = 0$. Thus, to prove that the nonnegative solution is nontrivial, it suffices to prove that there exists $\lambda^* > 0$ such that

$$\inf_{u\in W_0^{1,p}(\Omega)} J_\lambda(u) < 0 \quad \forall\lambda > \lambda^*. \tag{3.11}$$

For this, we consider the constrained minimization problem

$$\lambda^* \overset{\text{def}}{=} \inf\left\{ \frac{1}{p}\int_\Omega |\nabla w|^p \mathrm{d}x + \frac{1}{r+1}\int_\Omega h(x)|w|^{r+1}\mathrm{d}x : w \in W_0^{1,p}(\Omega) \text{ and } \frac{1}{q+1} \times \int_\Omega k(x)|w|^{q+1}\mathrm{d}x = 1 \right\}. \tag{3.12}$$

Let $n \mapsto v_n$ be a minimizing sequence of (3.12) in $W_0^{1,p}(\Omega)$, which is bounded in $W_0^{1,p}(\Omega)$, so that we can assume, without loss of generality, that it converges weakly to some $v \in W_0^{1,p}(\Omega)$, with

$$\frac{1}{q+1}\int_\Omega k(x)|v|^{q+1}\mathrm{d}x = 1, \qquad \lambda^* = \frac{1}{p}\int_\Omega |\nabla v|^p \mathrm{d}x + \frac{1}{r+1}\int_\Omega h(x)|v|^{r+1}\mathrm{d}x. \tag{3.13}$$

Thus, $J_\lambda(v) = \lambda^* - \lambda < 0$ for any $\lambda > \lambda^*$.

Now put

$$\Lambda \overset{\text{def}}{=} \inf\{\lambda > 0 : (1.3)_- \text{ admits a non trivial weak solution}\}. \tag{3.14}$$

From above $\lambda^* \geq \Lambda$ and that problem $(1.3)_-$ has a solution for all $\lambda > \lambda^*$. The proof of the Step 2 is now completed.

Step 3 (problem $(1.3)_-$ has a weak solution for any $\lambda > \Lambda$). By the definition of Λ, there exists $\mu \in (\Lambda, \lambda)$ such that J_μ has a nontrivial critical point $u_\mu \in W_0^{1,p}(\Omega)$. Since $\mu < \lambda, u_\mu$ is a subsolution of the problem $(1.3)_-$. In order to find a super-solution of the problem $(1.3)_-$ which dominates u_μ, we consider the constrained minimization problem

$$\inf\left\{ J_\lambda(w); w \in W_0^{1,p}(\Omega) \text{ and } w \geq u_\mu. \right\}. \tag{3.15}$$

Arguments similar to those used in Step 2 show that the above minimization problem has a solution $u_\lambda \geq u_\mu$ which is also a weak solution of problem $(1.3)_-$, provided $\lambda > \Lambda$.

Using similar arguments as in [6]. Thus, from Theorem 2.2 in Pucci and Servadei [7], based on the Moser iteration, it is clear that $u \in L^\infty_{\text{loc}}$. Next, again by bootstrap regularity [Corollary on p. 830] due to DiBenedetto, [8] shows that the weak solution $u \in C^{1,\alpha}(\Omega)$ where $\alpha \in (0,1)$. Finally, the nonnegative follows immediately by the strong maximum principle since u is a C^1 nonnegative weak solution of the differential inequality $\nabla(|\nabla u|^{p-2}\nabla u) - h(x)u^r \leq 0$ in Ω, with $p-1 < r$, see, for instance, Section 4.8 of Pucci and Serrin [9]. Thus, $u > 0$ in Ω. The proof of the Step 3 is now completed.

Step 4 (nonexistence for $\lambda > 0$ is small). The same monotonicity arguments as in Step 3 show that $(1.3)_-$ does not have any solution if $\lambda < \Lambda$, which completes the proof of the Theorem 1.2.

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