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Research Article

Improving Results on Convergence of AOR Method

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We present some sufficient conditions on convergence of AOR method for solving $Ax = b$ with A being a strictly doubly α diagonally dominant matrix. Moreover, we give two numerical examples to show the advantage of the new results.

1. Introduction

Let us denote all complex square matrices by $C^{n\times n}$ and all complex vectors by C^n.

For $A = (a_{ij}) \in C^{n\times n}$, we denote by $\rho(A)$ the spectral radius of matrix A.

Let us consider linear system $Ax = b$, where $b \in C^n$ is a given vector and $x \in C^n$ is an unknown vector. Let $A = D - T - S$ be given and D is the diagonal matrix, $-T$ and $-S$ are strictly lower and strictly upper triangular parts of A, respectively, and denote

$$L = D^{-1}T, \qquad U = D^{-1}S, \tag{1.1}$$

where $\det(D) \neq 0$.

Then the AOR method [1] can be written as

$$x^{k+1} = M_{\sigma,\omega}x^k + d, \quad k = 0, 1, \ldots, x^0 \in C^n, \tag{1.2}$$

where

$$\begin{aligned} M_{\sigma,\omega} &= (I - \sigma L)^{-1}[(1-\omega)I + (\omega - \sigma)L + \omega U], \\ d &= \omega(I - \sigma L)^{-1}b, \quad \omega, \sigma \in R. \end{aligned} \tag{1.3}$$

2. Preliminaries

We denote

$$R_i(A) = \sum_{j \neq i} |a_{ij}|, \quad S_i(A) = \sum_{j \neq i} |a_{ji}|, \quad P_{i,\alpha}(A) = \alpha R_i(A) + (1-\alpha) S_i(A), \quad \forall i \in N. \tag{2.1}$$

For any matrix $A = (a_{ij}) \in C^{n \times n}$, the comparison matrix $M(A) = (m_{ij}) \in R^{n \times n}$ is defined by

$$m_{ii} = |a_{ii}|, \quad m_{ij} = -|a_{ij}|, \quad i, j \in N, \ i \neq j, \ N = \{1, 2, \ldots, n\}. \tag{2.2}$$

Definition 2.1 (see [2]). A matrix $A \in C^{n \times n}$ is called a strictly diagonally dominant matrix (SD) if

$$|a_{ii}| > R_i(A), \quad \forall i \in N. \tag{2.3}$$

A matrix $A \in C^{n \times n}$ is called a strictly doubly diagonally dominant matrix (DD) if

$$|a_{ii}| |a_{jj}| > R_i(A) R_j(A), \quad \forall i, j \in N, \ i \neq j. \tag{2.4}$$

Definition 2.2 (see [3]). A matrix $A \in C^{n \times n}$ is called a strictly α diagonally dominant matrix ($D(\alpha)$) if there exits $\alpha \in [0, 1]$, such that

$$|a_{ii}| > \alpha R_i(A) + (1-\alpha) S_i(A), \quad \forall i \in N. \tag{2.5}$$

Definition 2.3 (see [4]). Let $A = (a_{ij}) \in C^{n \times n}$, if there exits $\alpha \in [0, 1]$ such that

$$|a_{ii}| |a_{jj}| > \alpha R_i(A) R_j(A) + (1-\alpha) S_i(A) S_j(A), \quad \forall i, j \in N, \ i \neq j, \tag{2.6}$$

then A is called a strictly doubly α diagonally dominant matrix ($DD(\alpha)$).

In [3, 5, 6], some people studied the convergence of AOR method for solving linear system $Ax = b$ and gave the areas of convergence. In [5], Cvetković and Herceg studied the convergence of AOR method for strictly diagonally dominant matrices. In [3], Huang and Wang studied the convergence of AOR method for strictly α diagonally dominant matrices. In [6], Gao and Huang studied the convergence of AOR method for strictly doubly diagonally dominant matrices.

Theorem 2.4 (see [3]). *Let $A \in D(\alpha)$, then AOR method converges for*

$$
\begin{aligned}
&\text{(I)}\quad 0 \le \sigma < \frac{2}{(1+\rho(M_{0,1}(M(A))))} = s,\\
&\qquad 0 < \omega < \max\left\{\frac{2}{(1+\max_i P_{i,\alpha}(L+U))} = t, \frac{2\sigma}{(1+\rho(M_{\sigma,\sigma}))}\right\}, \quad \textit{or}\\
&\text{(II)}\quad \max_i \frac{(-\omega(1-P_{i,\alpha}(L+U)) + 2\max(0,\omega-1))}{2P_{i,\alpha}(L)} < \sigma < 0, \quad 0<\omega<t, \quad \textit{or}\\
&\text{(III)}\quad t \le \sigma < \min_i \frac{(\omega(1+P_{i,\alpha}(L)-P_{i,\alpha}(U)) + 2\min(0,1-\omega))}{2P_{i,\alpha}(L)}, \quad 0<\omega<t.
\end{aligned}
\tag{2.7}
$$

Theorem 2.5 (see [6]). *Let $A \in DD$, then AOR method converges for*

$$
\begin{aligned}
&\text{(I)}\quad 0 \le \sigma < \frac{2}{(1+\rho(M_{0,1}(M(A))))} = s,\\
&\qquad 0 < \omega < \max\left\{\frac{2\sigma}{(1+\rho(M_{\sigma,\sigma}))}, \min_{\substack{i,j\\ i\neq j}} \frac{2}{1+\sqrt{R_i(L+U)R_j(L+U)}} = t\right\}, \quad \textit{or}\\
&\text{(II)}\quad \max_{\substack{i,j\\ i\neq j}} \frac{\omega P_1 - \sqrt{\omega^2 P_2^2 + P_3 \min\left(\omega^2, (\omega-2)^2\right)}}{P_3} < \sigma < 0, \quad 0<\omega<t, \quad \textit{or}\\
&\text{(III)}\quad t \le \sigma < \min_{\substack{i,j\\ i\neq j}} \frac{\omega P_4 + \sqrt{\omega^2 P_5^2 + P_3 \min\left(\omega^2, (\omega-2)^2\right)}}{P_3}, \quad 0<\omega<t,
\end{aligned}
\tag{2.8}
$$

where

$$
\begin{aligned}
P_1 &= R_i(L)R_j(L+U) + R_i(L+U)R_j(L),\\
P_2 &= R_i(L)R_j(L+U) - R_i(L+U)R_j(L),\\
P_3 &= 4R_i(L)R_j(L),\\
P_4 &= R_i(L)\left(R_j(L)-R_j(U)\right) + R_j(L)(R_i(L)-R_i(U)),\\
P_5 &= R_i(L)\left(R_j(L)-R_j(U)\right) - R_j(L)(R_i(L)-R_i(U)).
\end{aligned}
\tag{2.9}
$$

3. Upper Bound for Spectral Radius of $M_{\sigma,\omega}$

In the following, we present an upper bound for spectral radius of AOR iterative matrix $M_{\sigma,\omega}$ for strictly doubly α diagonally dominant coefficient matrix.

Lemma 3.1 (see [4]). *If $A \in DD(\alpha)$, then A is a nonsingular H-matrix.*

Theorem 3.2. *Let $A \in DD(\alpha)$, if $1-\sigma^2[\alpha R_i(L)R_j(L)+(1-\alpha)S_i(L)S_j(L)] > 0$, for all $j \in N$, $i \neq j$, then*

$$\rho(M_{\sigma,\omega}) \leq \max_{\substack{i,j \\ i\neq j}} \frac{A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}, \tag{3.1}$$

where

$$\begin{aligned} A_1 &= 1-\sigma^2\left[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)\right], \\ A_2 &= 2|1-\omega| + 2|\omega-\sigma||\sigma|\left[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)\right] \\ &\quad + |\omega||\sigma|\left[\alpha R_i(L)R_j(U) + (1-\alpha)S_i(U)S_j(L)\right] \\ &\quad + |\omega|\sigma|\left|\left[\alpha R_i(U)R_j(L) + (1-\alpha)S_i(L)S_j(U)\right]\right., \\ A_3 &= (1-\omega)^2 - \alpha\left[|\omega-\sigma|R_i(L) + |\omega|R_i(U)\right]\left[|\omega-\sigma|R_j(L) + |\omega|R_j(U)\right] \\ &\quad - (1-\alpha)\left[|\omega-\sigma|S_i(L) + |\omega|S_i(U)\right]\left[|\omega-\sigma|S_j(L) + |\omega|S_j(U)\right]. \end{aligned} \tag{3.2}$$

Proof. Let λ be an eigenvalue of $M_{\sigma,\omega}$ such that

$$\det(\lambda I - M_{\sigma,\omega}) = 0, \tag{3.3}$$

that is,

$$\det(\lambda(I-\sigma L) - ((1-\omega)I + (\omega-\sigma)L + \omega U)) = 0. \tag{3.4}$$

If $\lambda(I-\sigma L) - ((1-\omega)I + (\omega-\sigma)L + \omega U) \in DD(\alpha)$, then by Lemma 3.1, $\lambda(I-\sigma L) - ((1-\omega)I + (\omega-\sigma)L + \omega U)$ is nonsingular and λ is not an eigenvalue of iterative matrix $M_{\sigma,\omega}$, that is, if

$$\begin{aligned} |\lambda-(1-\omega)|^2 > \alpha &\left[\sum_{t\neq i}|\lambda\sigma l_{it} - (\omega-\sigma)l_{it} - \omega u_{it}|\right]\left[\sum_{l\neq j}\left|\lambda\sigma l_{jl} - (\omega-\sigma)l_{jl} - \omega u_{jl}\right|\right] \\ &+ (1-\alpha)\left[\sum_{k\neq i}|\lambda\sigma l_{ki} - (\omega-\sigma)|l_{ki} - \omega u_{ki}\right] \\ &\times \left[\sum_{s\neq j}\left|\lambda\sigma l_{sj} - (\omega-\sigma)\right|l_{sj} - \omega u_{sj}\right], \quad \forall i,j \in N,\ i\neq j, \end{aligned} \tag{3.5}$$

then λ is not an eigenvalue of $M_{\sigma,\omega}$. Especially, if

$$\begin{aligned}(|\lambda| - |1-\omega|)^2 > \alpha &\left[\sum_{t\neq i}(|\lambda||\sigma||l_{it}| + |\omega-\sigma||l_{it}| + |\omega||u_{it}|)\right] \\ &\times \left[\sum_{l\neq j}(|\lambda||\sigma|\left|l_{jl}\right| + |\omega-\sigma|\left|l_{jl}\right| + |\omega|\left|u_{jl}\right|)\right] \\ &+ (1-\alpha)\left[\sum_{k\neq i}(|\lambda||\sigma||l_{ki}| + |\omega-\sigma||l_{ki}| + |\omega|u_{ki})\right] \\ &\times \left[\sum_{s\neq j}(|\lambda||\sigma|\left|l_{sj}\right| + |\omega-\sigma|\left|l_{sj}\right| + |\omega|\left|u_{sj}\right|)\right],\end{aligned} \tag{3.6}$$

then λ is not an eigenvalue of $M_{\sigma,\omega}$.

If λ is an eigenvalue of $M_{\sigma,\omega}$, then there exits at least a couple of i, j $(i \neq j)$, such that

$$\begin{aligned}(|\lambda| - |1-\omega|)^2 \leq \alpha &\left[\sum_{t\neq i}(|\lambda||\sigma||l_{it}| + |\omega-\sigma||l_{it}| + |\omega||u_{it}|)\right] \\ &\times \left[\sum_{l\neq j}(|\lambda||\sigma|\left|l_{jl}\right| + |\omega-\sigma|\left|l_{jl}\right| + |\omega|\left|u_{jl}\right|)\right] \\ &+ (1-\alpha)\left[\sum_{k\neq i}(|\lambda||\sigma||l_{ki}| + |\omega-\sigma||l_{ki}| + |\omega|u_{ki})\right] \\ &\times \left[\sum_{s\neq j}(|\lambda||\sigma|\left|l_{sj}\right| + |\omega-\sigma|\left|l_{sj}\right| + |\omega|\left|u_{sj}\right|)\right],\end{aligned} \tag{3.7}$$

that is,

$$A_1|\lambda|^2 - A_2|\lambda| + A_3 \leq 0. \tag{3.8}$$

Since $A_1 = 1-\sigma^2[\alpha R_i(L)R_j(L)+(1-\alpha)S_i(L)S_j(L)] > 0$, and the discriminant Δ of the quadratic in $|\lambda|$ satisfies $\Delta \geq 0$, then the solution of (3.8) satisfies

$$\frac{A_2 - \sqrt{A_2^2 - 4A_1A_3}}{2A_1} \leq |\lambda| \leq \frac{A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}. \tag{3.9}$$

So

$$\rho(M_{\sigma,\omega}) \leq \max_{\substack{i,j \\ i\neq j}} \frac{A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}. \tag{3.10}$$

□

4. Improving Results on Convergence of AOR Method

In this section, we present new results on convergence of AOR method.

Theorem 4.1. *Let $A \in DD(\alpha)$, then AOR method converges if ω, σ satisfy either*

$$\begin{aligned}
&\text{(I)}\quad 0 \leq \sigma < \frac{2}{(1+\rho(M_{0,1}(M(A))))} = s,\\
&\qquad 0 < \omega < \max\Bigg\{\frac{2\sigma}{(1+\rho(M_{\sigma,\sigma}))},\\
&\qquad\qquad \min_{\substack{i,j \\ i\neq j}} \frac{2}{1+\alpha R_i(L+U)R_j(L+U)+(1-\alpha)S_i(L+U)S_j(L+U)} = t\Bigg\},\quad \textit{or}\\
&\text{(II)}\quad \max_{\substack{i,j \\ i\neq j}} \frac{\omega P_1' - \sqrt{\omega^2 P_2'^2 + P_3' \min\left(\omega^2, (\omega-2)^2\right)}}{P_3'} < \sigma < 0,\quad 0<\omega<t,\quad \textit{or}\\
&\text{(III)}\quad t \leq \sigma < \min_{\substack{i,j \\ i\neq j}} \frac{\omega P_4' + \sqrt{\omega^2 P_5'^2 + P_3' \min\left(\omega^2, (\omega-2)^2\right)}}{P_3'},\quad 0<\omega<t,
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
P_1' &= \alpha R_i(L)R_j(L+U) + (1-\alpha)S_i(L)S_j(L+U) + \alpha R_i(L+U)R_j(L)\\
&\quad + (1-\alpha)S_i(L+U)S_j(L),\\
P_2' &= \alpha R_i(L)R_j(L+U) + (1-\alpha)S_i(L)S_j(L+U) - \alpha R_i(L+U)R_j(L)\\
&\quad - (1-\alpha)S_i(L+U)S_j(L),\\
P_3' &= 4\left[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)\right],\\
P_4' &= \alpha R_i(L)\left[R_j(L) - R_j(U)\right] + (1-\alpha)S_i(L)\left[S_j(L) - S_j(U)\right]\\
&\quad + \alpha R_j(L)[R_i(L) - R_i(U)] + (1-\alpha)S_j(L)[S_i(L) - S_i(U)],\\
P_5' &= \alpha R_i(L)\left[R_j(L) - R_j(U)\right] + (1-\alpha)S_i(L)\left[S_j(L) - S_j(U)\right]\\
&\quad - \alpha R_j(L)[R_i(L) - R_i(U)] - (1-\alpha)S_j(L)[S_i(L) - S_i(U)].
\end{aligned} \tag{4.2}$$

Proof. It is easy to verify that for each σ, which satisfies one of the conditions (I)–(III), we have

$$1 - \sigma^2\left[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)\right] > 0, \quad \forall i \neq j;\ i, j \in N. \tag{4.3}$$

Firstly, we consider case (I). Since A be a α diagonally dominant matrix, then by Lemma 3.1, we know that A is a nonsingular H-matrix; therefore, $M(A)$ is a nonsingular M-matrix, and it follows that from paper [7], $\rho(M_{\sigma,\sigma}) < 1$ holds for $0 \leq \sigma < s$ and for $\sigma \neq 0$,

$$M_{\sigma,\omega} = \left(1 - \frac{\omega}{\sigma}\right)I + \left(\frac{\omega}{\sigma}\right)M_{\sigma,\sigma}. \tag{4.4}$$

If $0 < \omega < \max\{2\sigma/(1+\rho(M_{\sigma,\sigma})), t\} = 2\sigma/(1+\rho(M_{\sigma,\sigma}))$, then

$$0 < \frac{\omega}{\sigma} < \frac{2}{(1+\rho(M_{\sigma,\sigma}))}, \tag{4.5}$$

by extrapolation theorem [6], we have $\rho(M_{\sigma,\omega}) < 1$.

If $0 < \omega < \max\{2\sigma/(1+\rho(M_{\sigma,\sigma})), t\} = t$, then it remains to analyze the case

$$\frac{2\sigma}{(1+\rho(M_{\sigma,\sigma}))} \leq \omega < t, \quad 0 \leq \sigma < s. \tag{4.6}$$

Since when $\rho(M_{\sigma,\sigma}) < 1$,

$$\sigma < \frac{2\sigma}{(1+\rho(M_{\sigma,\sigma}))}, \tag{4.7}$$

then $0 \leq \sigma < \omega$. From

$$\left.\begin{array}{ll} A_2 - A_3 < A_1 & \text{(a)} \\ A_2 < 2A_1 \leq 2 & \text{(b)} \\ \sqrt{A_2^2 - 4A_1A_3} < 2A_1 \leq 2 & \text{(c)} \end{array}\right. \Longleftrightarrow \frac{A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1} < 1, \tag{4.8}$$

we have $\rho(M_{\sigma,\omega}) \leq \max_{\substack{i,j \\ i\neq j}}((A_2 + \sqrt{A_2^2 - 4A_1A_3})/2A_1) < 1$.

(1) When $\omega \leq 1$, it easy to verify that (4.8) holds.

(2) When $\omega > 1$, since

$$
\begin{aligned}
A_1 &= 1 - \sigma^2\left[\alpha R_i(L) R_j(L) + (1-\alpha) S_i(L) S_j(L)\right],\\
A_2 &= 2(\omega - \sigma)\left[\alpha R_i(L) R_j(L) + (1-\alpha) S_i(L) S_j(L)\right] + \omega\sigma\left[\alpha R_i(L) R_j(U) + (1-\alpha) S_i(U) S_j(L)\right]\\
&\quad + \omega\sigma\left[\alpha R_i(U) R_j(L) + (1-\alpha) S_i(L) S_j(U)\right] + 2|1-\omega|,\\
A_3 &= (1-\omega)^2 - \alpha\left[(\omega-\sigma) R_i(L) + \omega R_i(U)\right]\left[(\omega-\sigma) R_j(L) + \omega R_j(U)\right]\\
&\quad - (1-\alpha)\left[(\omega-\sigma) S_i(L) + \omega S_i(U)\right]\left[(\omega-\sigma) S_j(L) + \omega S_j(U)\right],
\end{aligned}
\tag{4.9}
$$

then by $A_2 - A_3 < A_1$ and $1 - \sigma^2[\alpha R_i(L) R_j(L) + (1-\alpha) S_i(L) S_j(L)] > 0$, for all $i, j \in N,\ i \neq j$, we have

$$
\omega^2\left[1 - \alpha R_i(L+U) R_j(L+U) - (1-\alpha) S_i(L+U) S_j(L+U)\right] - 4\omega + 4 > 0. \tag{4.10}
$$

It is easy to verify that the discriminant Δ of the quadratic in ω satisfies $\Delta > 0$, and so there holds

$$
\omega_1 > \frac{2}{1 - \alpha R_i(L+U) R_j(L+U) - (1-\alpha) S_i(L+U) S_j(L+U)}, \tag{4.11}
$$

or

$$
\omega_2 < \frac{2}{1 + \alpha R_i(L+U) R_j(L+U) + (1-\alpha) S_i(L+U) S_j(L+U)}, \quad \forall i \neq j. \tag{4.12}
$$

For ω_1, we have $A_2 > 2$, it is in contradiction with ((4.8)b). Therefore, ω_1 should be deleted.

Secondly, we prove (II).

(1) When $0 < \omega \leq 1$, $\sigma < 0$,

$$
\begin{aligned}
A_2 &= 2(1-\omega) - 2(\omega-\sigma)\sigma\left[\alpha R_i(L) R_j(L) + (1-\alpha) S_i(L) S_j(L)\right]\\
&\quad - \omega\sigma\left[\alpha R_i(L) R_j(U) + (1-\alpha) S_i(U) S_j(L)\right]\\
&\quad - \omega\sigma\left[\alpha R_i(U) R_j(L) + (1-\alpha) S_i(L) S_j(U)\right],\\
A_3 &= (1-\omega)^2 - \alpha\left[(\omega-\sigma) R_i(L) + \omega R_i(U)\right]\left[(\omega-\sigma) R_j(L) + \omega R_j(U)\right]\\
&\quad - (1-\alpha)\left[(\omega-\sigma) S_i(L) + \omega S_i(U)\right]\left[(\omega-\sigma) S_j(L) + \omega S_j(U)\right].
\end{aligned}
\tag{4.13}
$$

By $A_2 - A_3 < A_1$, we have

$$\begin{aligned}4\sigma^2 & \left[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)\right] \\ & - 2\omega\sigma\left[\alpha R_i(L)R_j(L+U) + (1-\alpha)S_i(L)S_j(L+U)\right. \\ & \qquad \left. + \alpha R_i(L+U)R_j(L) + (1-\alpha)S_i(L+U)S_j(L)\right] \\ & + \omega^2\left[\alpha R_i(L+U)R_j(L+U) + (1-\alpha)S_i(L+U)S_j(L+U) - 1\right] < 0.\end{aligned} \tag{4.14}$$

It is easy to verify that the discriminant Δ of the quadratic in σ satisfies $\Delta > 0$, and so there holds

$$\frac{\omega P_1' - \omega\sqrt{P_2'^2 + P_3'}}{P_3'} < \sigma < \frac{\omega P_1' + \omega\sqrt{P_2'^2 + P_3'}}{P_3'}. \tag{4.15}$$

By $\sigma < 0$ and $1 - \sigma^2[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)] > 0$, for all $i, j \in N, i \neq j$, we obtain

$$\max_{\substack{i,j \\ i\neq j}} \frac{\omega P_1' - \omega\sqrt{P_2'^2 + P_3'}}{P_3'} < \sigma < 0. \tag{4.16}$$

(2) When $1 < \omega < t$, $\sigma < 0$,

$$\begin{aligned}A_2 &= 2(\omega-1) - 2(\omega-\sigma)\sigma\left[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)\right] \\ &\quad - \omega\sigma\left[\alpha R_i(L)R_j(U) + (1-\alpha)S_i(U)S_j(L)\right] - \omega\sigma\left[\alpha R_i(U)R_j(L) + (1-\alpha)S_i(L)S_j(U)\right], \\ A_3 &= (1-\omega)^2 - \alpha\left[(\omega-\sigma)R_i(L) + \omega R_i(U)\right]\left[(\omega-\sigma)R_j(L) + \omega R_j(U)\right] \\ &\quad - (1-\alpha)\left[(\omega-\sigma)S_i(L) + \omega S_i(U)\right]\left[(\omega-\sigma)S_j(L) + \omega S_j(U)\right].\end{aligned} \tag{4.17}$$

By $A_2 - A_3 < A_1$, we have

$$\begin{aligned}4\sigma^2 & \left[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)\right] \\ & - 2\omega\sigma\left[\alpha R_i(L)R_j(L+U) + (1-\alpha)S_i(L)S_j(L+U)\right. \\ & \qquad \left. + \alpha R_i(L+U)R_j(L) + (1-\alpha)S_i(L+U)S_j(L)\right] \\ & + \omega^2\left[\alpha R_i(L+U)R_j(L+U) + (1-\alpha)S_i(L+U)S_j(L+U)\right] \\ & - \omega^2 + 4\omega - 4 < 0.\end{aligned} \tag{4.18}$$

It is easy to verify that the discriminant Δ of the quadratic in σ satisfies $\Delta > 0$, and so there holds

$$\frac{\omega P_1' - \sqrt{\omega^2 P_2'^2 + (\omega-2)^2 P_3'}}{P_3'} < \sigma < \frac{\omega P_1' + \sqrt{\omega^2 P_2'^2 + (\omega-2)^2 P_3'}}{P_3'}. \tag{4.19}$$

By $\sigma < 0$ and $1 - \sigma^2[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)] > 0$, for all $i, j \in N$, $i \neq j$, we obtain

$$\max_{\substack{i,j \\ i\neq j}} \frac{\omega P_1' - \sqrt{\omega^2 P_2'^2 + (\omega-2)^2 P_3'}}{P_3'} < \sigma < 0. \tag{4.20}$$

Therefore, by (4.16) and (4.20), we get

$$\max_{\substack{i,j \\ i\neq j}} \frac{\omega P_1' - \sqrt{\omega^2 P_2'^2 + P_3' \min\left(\omega^2, (\omega-2)^2\right)}}{P_3'} < \sigma < 0. \tag{4.21}$$

Finally, we prove (III).

(1) When $0 < \omega \le 1$, $\sigma \ge t$,

$$\begin{aligned} A_2 &= 2(1-\omega) + 2(\omega-\sigma)\sigma\left[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)\right] \\ &\quad + \omega\sigma\left[\alpha R_i(L)R_j(U) + (1-\alpha)S_i(U)S_j(L)\right] + \omega\sigma\left[\alpha R_i(U)R_j(L) + (1-\alpha)S_i(L)S_j(U)\right], \\ A_3 &= (1-\omega)^2 - \alpha\left[(\sigma-\omega)R_i(L) + \omega R_i(U)\right]\left[(\sigma-\omega)R_j(L) + \omega R_j(U)\right] \\ &\quad - (1-\alpha)\left[(\sigma-\omega)S_i(L) + \omega S_i(U)\right]\left[(\sigma-\omega)S_j(L) + \omega S_j(U)\right]. \end{aligned} \tag{4.22}$$

By $A_2 - A_3 < A_1$, we have

$$\begin{aligned} &4\sigma^2\left[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)\right] \\ &\qquad - 2\omega\sigma\left[\alpha R_i(L)\left(R_j(L) - R_j(U)\right) + (1-\alpha)S_i(L)\left(S_j(L) - S_j(U)\right)\right] \\ &\qquad - 2\omega\sigma\left[\alpha R_j(L)(R_i(L) - R_i(U)) + (1-\alpha)S_j(L)(S_i(L) - S_i(U))\right] \\ &\qquad + \omega^2\left[\alpha(R_i(L) - R_i(U))\left(R_j(L) - R_j(U)\right)\right. \\ &\qquad\qquad \left. +(1-\alpha)(S_i(L) - S_i(U))\left(S_j(L) - S_j(U)\right)\right] - \omega^2 < 0. \end{aligned} \tag{4.23}$$

It is easy to verify that the discriminant Δ of the quadratic in σ satisfies $\Delta > 0$, and so there holds

$$\frac{\omega P_4' - \omega\sqrt{P_5'^2 + P_3'}}{P_3'} < \sigma < \frac{\omega P_4' + \omega\sqrt{P_5'^2 + P_3'}}{P_3'}. \tag{4.24}$$

By $\sigma \geq t$, we obtain

$$t \leq \sigma < \min_{\substack{i,j \\ i\neq j}} \frac{\omega P_4' + \omega\sqrt{P_5'^2 + P_3'}}{P_3'}. \tag{4.25}$$

(2) When $1 < \omega < t$, $\sigma \geq t$,

$$\begin{aligned} A_2 &= 2(\omega-1) + 2(\sigma-\omega)\sigma\left[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)\right] \\ &\quad + \omega\sigma\left[\alpha R_i(L)R_j(U) + (1-\alpha)S_i(U)S_j(L)\right] \\ &\quad + \omega\sigma\left[\alpha R_i(U)R_j(L) + (1-\alpha)S_i(L)S_j(U)\right], \\ A_3 &= (1-\omega)^2 - \alpha\left[(\sigma-\omega)R_i(L) + \omega R_i(U)\right]\left[(\sigma-\omega)R_j(L) + \omega R_j(U)\right] \\ &\quad - (1-\alpha)\left[(\sigma-\omega)S_i(L) + \omega S_i(U)\right]\left[(\sigma-\omega)S_j(L) + \omega S_j(U)\right]. \end{aligned} \tag{4.26}$$

By $A_2 - A_3 < A_1$, we have

$$\begin{aligned} &4\sigma^2\left[\alpha R_i(L)R_j(L) + (1-\alpha)S_i(L)S_j(L)\right] \\ &\quad - 2\omega\sigma\left[\alpha R_i(L)\left(R_j(L) - R_j(U)\right) + (1-\alpha)S_i(L)\left(S_j(L) - S_j(U)\right)\right] \\ &\quad - 2\omega\sigma\left[\alpha R_j(L)(R_i(L) - R_i(U)) + (1-\alpha)S_j(L)(S_i(L) - S_i(U))\right] \\ &\quad + \omega^2\left[\alpha(R_i(L) - R_i(U))\left(R_j(L) - R_j(U)\right)\right. \\ &\qquad \left. +(1-\alpha)(S_i(L) - S_i(U))\left(S_j(L) - S_j(U)\right)\right] - \omega^2 + 4\omega - 4 < 0. \end{aligned} \tag{4.27}$$

It is easy to verify that the discriminant Δ of the quadratic in σ satisfies $\Delta > 0$, and so there holds

$$\frac{\omega P_4 - \sqrt{\omega^2 P_5'^2 + (\omega-2)^2 P_3'}}{P_3'} < \sigma < \frac{\omega P_4' + \sqrt{\omega^2 P_5'^2 + (\omega-2)^2 P_3'}}{P_3'}. \tag{4.28}$$

By $\sigma \geq t$, we obtain

$$t \leq \sigma < \min_{\substack{i,j \\ i\neq j}} \frac{\omega P_4' + \sqrt{\omega^2 P_5'^2 + (\omega-2)^2 P_3'}}{P_3'}. \tag{4.29}$$

Therefore, by (4.25) and (4.29), we obtain

$$t \le \sigma < \min_{\substack{i,j \\ i \ne j}} \frac{\omega P_4' + \sqrt{\omega^2 P_5'^2 + P_3' \min\left(\omega^2, (\omega - 2)^2\right)}}{P_3'}. \tag{4.30}$$

□

We can obtain the following results easily.

Theorem 4.2. *Let* $A \in DD(\alpha)$*. If* $R_i(L+U)R_j(L+U) - S_i(L+U)S_j(L+U) > 0$*, when* $0 < \omega \le 1$*, the following conditions hold:*

$$\begin{aligned} &\text{(I)} \quad 0 \le \alpha < \frac{\sqrt{R_i(L+U)R_j(L+U)} - S_i(L+U)S_j(L+U)}{R_i(L+U)R_j(L+U) - S_i(L+U)S_j(L+U)}, \\ &\text{(II)} \quad \frac{\omega P_1' - \sqrt{\omega^2 P_2'^2 + \omega^2 P_3'}}{P_3'} < \frac{\omega P_1 - \sqrt{\omega^2 P_2^2 + \omega^2 P_3}}{P_3}, \\ &\text{(III)} \quad \frac{\omega P_4' + \sqrt{\omega^2 P_5'^2 + \omega^2 P_3'}}{P_3'} > \frac{\omega P_4 + \sqrt{\omega^2 P_5^2 + \omega^2 P_3}}{P_3}, \end{aligned} \tag{4.31}$$

or when $1 < \omega < t$*, the following conditions hold:*

$$\begin{aligned} &\text{(I)} \quad 0 \le \alpha < \frac{\sqrt{R_i(L+U)R_j(L+U)} - S_i(L+U)S_j(L+U)}{R_i(L+U)R_j(L+U) - S_i(L+U)S_j(L+U)}, \\ &\text{(II)} \quad \frac{\omega P_1' - \sqrt{\omega^2 P_2'^2 + (\omega-2)^2 P_3'}}{P_3'} < \frac{\omega P_1 - \sqrt{\omega^2 P_2^2 + (\omega-2)^2 P_3}}{P_3}, \\ &\text{(III)} \quad \frac{\omega P_4' + \sqrt{\omega^2 P_5'^2 + (\omega-2)^2 P_3'}}{P_3'} > \frac{\omega P_4 + \sqrt{\omega^2 P_5^2 + (\omega-2)^2 P_3}}{P_3}, \end{aligned} \tag{4.32}$$

then the area of convergence of AOR method obtained by Theorem 4.1 is larger than that obtained by Theorem 2.5.

Theorem 4.3. *Let $A \in DD(\alpha)$. If $R_i(L+U)R_j(L+U) - S_i(L+U)S_j(L+U) < 0$, when $0 < \omega \le 1$, the following conditions hold:*

$$
\begin{aligned}
&\text{(I)} \quad \frac{\sqrt{R_i(L+U)R_j(L+U)} - S_i(L+U)S_j(L+U)}{R_i(L+U)R_j(L+U) - S_i(L+U)S_j(L+U)} < \alpha \le 1,\\
&\text{(II)} \quad \frac{\omega P_1' - \sqrt{\omega^2 P_2'^2 + \omega^2 P_3'}}{P_3'} < \frac{\omega P_1 - \sqrt{\omega^2 P_2^2 + \omega^2 P_3}}{P_3},\\
&\text{(III)} \quad \frac{\omega P_4' + \sqrt{\omega^2 P_5'^2 + \omega^2 P_3'}}{P_3'} > \frac{\omega P_4 + \sqrt{\omega^2 P_5^2 + \omega^2 P_3}}{P_3},
\end{aligned}
\tag{4.33}
$$

or when $1 < \omega < t$, the following conditions hold:

$$
\begin{aligned}
&\text{(I)} \quad \frac{\sqrt{R_i(L+U)R_j(L+U)} - S_i(L+U)S_j(L+U)}{R_i(L+U)R_j(L+U) - S_i(L+U)S_j(L+U)} < \alpha \le 1,\\
&\text{(II)} \quad \frac{\omega P_1' - \sqrt{\omega^2 P_2'^2 + (\omega-2)^2 P_3'}}{P_3'} < \frac{\omega P_1 - \sqrt{\omega^2 P_2^2 + (\omega-2)^2 P_3}}{P_3},\\
&\text{(III)} \quad \frac{\omega P_4' + \sqrt{\omega^2 P_5'^2 + (\omega-2)^2 P_3'}}{P_3'} > \frac{\omega P_4 + \sqrt{\omega^2 P_5^2 + (\omega-2)^2 P_3}}{P_3},
\end{aligned}
\tag{4.34}
$$

then the area of convergence of AOR method obtained by Theorem 4.1 is larger than that obtained by Theorem 2.5.

5. Examples

In the following examples, we give the areas of convergence of AOR method to show that our results are better than ones obtained by Theorems 2.4 and 2.5.

Example 5.1 (see [6]). Let

$$
A = \begin{pmatrix} 5 & 3 & 2 \\ 2 & 6 & 3 \\ 2 & 1 & 9 \end{pmatrix} = D - T - S,
\tag{5.1}
$$

where

$$D=\begin{pmatrix}5&0&0\\0&6&0\\0&0&9\end{pmatrix},\qquad T=\begin{pmatrix}0&0&0\\-2&0&0\\-2&-1&0\end{pmatrix},\qquad S=\begin{pmatrix}0&-3&-2\\0&0&-3\\0&0&0\end{pmatrix},$$

$$L=D^{-1}T=\begin{pmatrix}0&0&0\\-\frac{1}{3}&0&0\\-\frac{2}{9}&-\frac{1}{9}&0\end{pmatrix},\qquad U=D^{-1}S=\begin{pmatrix}0&-\frac{3}{5}&-\frac{2}{5}\\0&0&-\frac{1}{2}\\0&0&0\end{pmatrix},\qquad L+U=\begin{pmatrix}0&-\frac{3}{5}&-\frac{2}{5}\\-\frac{1}{3}&0&-\frac{1}{2}\\-\frac{2}{9}&-\frac{1}{9}&0\end{pmatrix}. \tag{5.2}$$

Obviously, $A\notin SD$, but $A\in DD(1/2)$.

By Theorem 4.1, we have the following area of convergence:

$$\begin{aligned}
&(1)\quad 0\le\sigma<1.1896,\quad 0<\omega<\max\left\{1.2390,\frac{2\sigma}{(1+\rho(M_{\sigma,\sigma}))}\right\},\quad\text{or}\\
&(2)\quad 0<\omega\le 1,\quad -1.0266\omega<\sigma<0,\quad\text{or}\\
&\qquad\qquad 1<\omega<1.2390,\quad \frac{\left(22\omega-3\sqrt{201\omega^2-800\omega+800}\right)}{20}<\sigma<0,\quad\text{or}\\
&(3)\quad 0<\omega\le 1,\quad 1.2390<\sigma<2.0266\omega,\quad\text{or}\\
&\qquad\qquad 1<\omega<1.2390,\quad 1.2390\le\sigma<\frac{\left(-2\omega+3\sqrt{201\omega^2-800\omega+800}\right)}{20}.
\end{aligned} \tag{5.3}$$

Obviously, $A\in DD$.

By Theorem 2.5, we have the following area of convergence:

$$\begin{aligned}
&(1)\quad 0\le\sigma<1.1896,\quad 0<\omega<\max\left\{1.0455,\frac{2\sigma}{(1+\rho(M_{\sigma,\sigma}))}\right\},\quad\text{or}\\
&(2)\quad 0<\omega\le 1,\quad -0.6712\omega<\sigma<0,\quad\text{or}\\
&\qquad\qquad 1<\omega<1.0455,\ \frac{\left(7\omega-3\sqrt{17\omega^2-64\omega+64}\right)}{8}<\sigma<0,\quad\text{or}\\
&(3)\quad 0<\omega\le 1,\quad 1.0455\le\sigma<1.6712\omega,\quad\text{or}\\
&\qquad\qquad 1<\omega<1.0455,\quad 1.0455\le\sigma<\frac{\left(\omega+3\sqrt{17\omega^2-64\omega+64}\right)}{8}.
\end{aligned} \tag{5.4}$$

In addition, $A\in D(1/2)$.

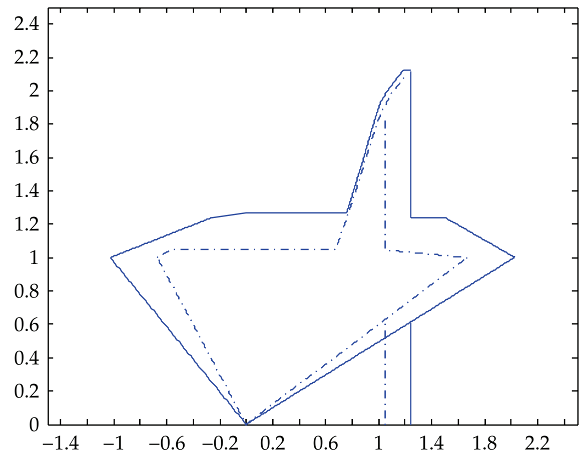

Figure 1

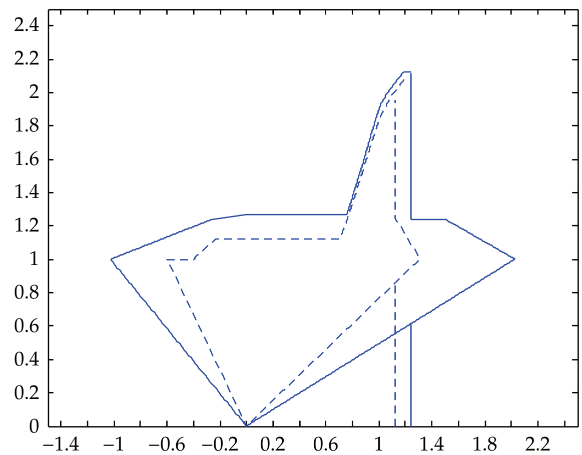

Figure 2

By Theorem 2.4, we have the following area of convergence:

$$(1)\quad 0 \le \sigma < 1.1896, \quad 0 < \omega < \max\left\{1.1250, \frac{2\sigma}{(1+\rho(M_{\sigma,\sigma}))}\right\}, \quad \text{or}$$

$$(2)\quad 0 < \omega \le 1, \quad -0.6\omega < \sigma < 0, \quad \text{or} \quad 1 < \omega < 1.1250, \quad \frac{7}{5}\omega - \frac{9}{5} < \sigma < 0, \quad \text{or}$$

$$(3)\quad 0 < \omega \le 1, \quad 1.1250 < \sigma < 1.31\omega, \quad \text{or} \quad 1 < \omega < 1.1250, \quad 1.1250 \le \sigma < 1.8 - 0.49\omega. \tag{5.5}$$

Now we give two figures. In Figure 1, we can see that the area of convergence obtained by Theorem 4.1 (real line) is larger than that obtained by Theorem 2.5 (virtual line). In Figure 2, we can see that the area of convergence obtained by Theorem 4.1 (real line) is larger than that obtained by Theorem 2.4 (virtual line). From above we know that the area of convergence obtained by Theorem 4.1 is larger than ones obtained by Theorems 2.5 and 2.4.

Example 5.2. Let

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}. \tag{5.6}$$

Obviously, $A \in DD(1/3)$, $A \notin D(\alpha)$, $A \notin DD$. So we cannot use Theorems 2.4 and 2.5. By Theorem 4.1, we have the following area of convergence:

$$\begin{aligned}
&(1) \quad 0 \le \sigma < 1.0724, \quad 0 < \omega < \max\left\{1.0693, \frac{2\sigma}{(1+\rho(M_{\sigma,\sigma}))}\right\}, \quad \text{or} \\
&(2) \quad 0 < \omega \le 1, \quad -0.1973\omega < \sigma < 0, \quad \text{or} \\
&\qquad 1 < \omega < 1.0693, \quad \frac{\left(37\omega - \sqrt{1945\omega^2 - 7776\omega + 7776}\right)}{36} < \sigma < 0, \quad \text{or} \\
&(3) \quad 0 < \omega \le 1, \quad 1.0693 < \sigma < 1.1973\omega, \quad \text{or} \\
&\qquad 1 < \omega < 1.0693, \quad 1.0693 \le \sigma < \frac{\left(\sqrt{1945\omega^2 - 7776\omega + 7776} - \omega\right)}{36}.
\end{aligned} \tag{5.7}$$

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