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Research Article

Positive Solutions for Some Nonlinear Elliptic Systems in Exterior Domains of $\mathbb{R}^2$

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Using some potential theory tools and the Schauder fixed point theorem, we prove the existence of positive continuous solutions with a precise global behavior for the competitive semilinear elliptic system $\Delta u = p(x)u^{\alpha}v^{r}$, $\Delta v = q(x)u^{s}v^{\beta}$ in an exterior domain D of $\mathbb{R}^2$, subject to some Dirichlet conditions, where $\alpha \geq 1$, $\beta \geq 1$, $r \geq 0$, $s \geq 0$ and the potentials p, q are nonnegative and satisfy some hypotheses related to the Kato class $K(D)$.

1. Introduction

The study of nonlinear elliptic systems has a strong motivation, and important research efforts have been made recently for these systems aiming to apply the results of existence and asymptotic behavior of positive solutions in applied fields. Coupled nonlinear Shrödinger systems arise in the description of several physical phenomena such as the propagation of pulses in birefringent optical fibers and Kerr-like photorefractive media, see [1, 2]. Stationary elliptic systems arise also in other physical models like non-Newtonian fluids: pseudoplastic fluids and dilatant fluids [3, 4], non-Newtonian filtration [5], and the turbulent flow of a gas in porous medium [6, 7]. They also describe other various nonlinear phenomena such as chemical reactions, pattern formation, population evolution where, for example, u and v represent the concentrations of two species in the process. As a consequence, positive solutions of such are of interest.

For some recent results on the qualitative analysis and the applications of positive solutions of nonlinear elliptic systems in both bounded and unbounded domains we refer to [8–15] and the references therein.

In these works various existence results of positive bounded solutions or positive blowing-up ones (called also large solutions) have been established, and a precise global behavior is given. We note also that several methods have been used to treat these nonlinear systems such as sub- and super-solutions method, variational method, and topological methods.

In this paper, we consider an unbounded domain D in $\mathbb{R}^2$ with a nonempty compact boundary ∂D consisting of finitely many Jordan curves and noncontaining zero. We fix two nontrivial nonnegative continuous functions φ and ψ on ∂D and some nonnegative constants, a, b, c, d such that $a + c > 0$ and $b + d > 0$, and we will deal with the existence of a positive continuous solution (in the sense of distributions) to the system:

$$\begin{gathered}
\Delta u = p(x) u^{\alpha} v^{r}, \quad \text{in } D \\
\Delta v = q(x) u^{s} v^{\beta}, \quad \text{in } D \\
u_{/\partial D} = a\varphi, \qquad v_{/\partial D} = b\psi \\
\lim_{|x| \to \infty} \frac{u(x)}{\ln|x|} = c, \qquad \lim_{|x| \to \infty} \frac{v(x)}{\ln|x|} = d,
\end{gathered} \tag{1.1}$$

where $\alpha \geq 1$, $\beta \geq 1$, $r \geq 0$, $s \geq 0$ and p, q are two nonnegative functions satisfying some hypotheses related to the Kato class $K(D)$ defined and studied in [16, 17] by means of the Green function $G(x, y)$ of the Dirichlet Laplacian in D.

Our method is based on some potential theory tools which we apply to give an existence result for equations by an approximation argument, then we use the result for equations to prove, by means of the Schauder fixed point theorem, the existence result for the system (1.1).

As far as we know, there are no results that contain existence of positive solutions to the elliptic system (1.1) in the case where $\alpha > 0$ and $\beta > 0$ and the weights $p(x)$ and $q(x)$ are singular functions.

The study of (1.1) is motivated by the existence results obtained in [18] to the following system

$$\begin{gathered}
\Delta u = \lambda p(x) g(v), \quad \text{in } D \\
\Delta v = \mu q(x) f(u), \quad \text{in } D \\
u_{/\partial D} = a\varphi, \qquad v_{/\partial D} = b\psi, \\
\lim_{|x| \to \infty} \frac{u(x)}{\ln(|x|)} = c, \qquad \lim_{|x| \to \infty} \frac{v(x)}{\ln(|x|)} = d,
\end{gathered} \tag{1.2}$$

where λ, μ are nonnegative constants, the functions $f, g : [0, \infty) \to [0, \infty)$ are nondecreasing and continuous.

More precisely, it was shown in [18] that if the functions $\widetilde{p} := pf(\theta)$ and $\widetilde{q} := qg(\omega)$ belong to the Kato class $K(D)$, then there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that for each $\lambda \in [0, \ \lambda_0)$ and $\mu \in [0, \ \mu_0)$, the system (1.2) has a positive continuous solution (u, v) having the global asymptotic behavior of the unique solution of the associated homogeneous system, where we have denoted by

$$\omega := aH_D\varphi + ch, \qquad \theta := bH_D\varphi + dh, \tag{1.3}$$

and the functions $H_D\varphi$ and h are the harmonic functions defined, respectively, by (1.4) and (1.5) below.

The system of two equations in (1.1) has been treated in exterior domains of $\mathbb{R}^n$, $n \geq 3$ in [19], and existence of positive bounded continuous solutions is established. The main difficulty in the present work is the case of the domain $D \subset \mathbb{R}^2$. More precisely, the function h defined by (1.5) behaves as $\ln(|x|)$ at infinity for $n = 2$, unlike the case $n \geq 3$ where this function is bounded at infinity.

Throughout this paper, we denote by $H_D\varphi$ the unique bounded continuous solution of the Dirichlet problem

$$\begin{gathered} \Delta w = 0 \quad \text{in } D, \\ w_{/\partial D} = \varphi, \\ \lim_{|x|\to+\infty} \frac{w(x)}{h(x)} = 0, \end{gathered} \tag{1.4}$$

where φ is a nonnegative continuous function on ∂D.

The function h is defined on D by

$$h(x) = 2\pi \lim_{|y|\to+\infty} G_D(x,y). \tag{1.5}$$

First, we recall the following result about this function h.

Proposition 1.1 (see [16]). *The function h defined by* (1.5) *is harmonic and positive in D and satisfies*

$$\lim_{x\to z\in\partial D} h(x) = 0, \qquad \lim_{|x|\to+\infty} \frac{h(x)}{\ln|x|} = 1. \tag{1.6}$$

Taking into account these notations, we use some potential theory tools and an approximating sequence in order to prove the following first result concerning the existence of a unique positive continuous solution to the boundary value problem:

$$\begin{gathered} \Delta u = p(x)u^{\gamma}, \quad \text{in } D \\ u_{/\partial D} = a\varphi \\ \lim_{x\to\infty} \frac{u(x)}{\ln|x|} = c, \end{gathered} \tag{1.7}$$

where $\gamma \geq 1$, φ is a nontrivial nonnegative continuous function on ∂D and a, c are two nonnegative constants with $a + c > 0$. More precisely we establish the following.

Theorem 1.2. *Let p be a nonnegative function such that the function $\widetilde{p} = \gamma p \omega^{\gamma-1}$ belongs to the Kato class $K(D)$. Then problem* (1.7) *has a unique positive continuous solution satisfying for each $x \in D$*

$$c_0 \omega(x) \leq u(x) \leq \omega(x), \tag{1.8}$$

where ω is defined in (1.3) *and the constant $c_0 \in (0,1]$.*

Next we exploit this result to prove the existence of a positive continuous solution (u,v) to the system (1.1). For this aim we denote by

$$\omega_0 := a + ch, \tag{1.9}$$

$$\theta_0 := b + dh, \tag{1.10}$$

and we need to assume the following hypothesis on the functions p and q.

(**H**) p and q are nonnegative measurable functions in D such that

$$x \longrightarrow p(x)\theta_0^r(x)\omega_0^{\alpha-1}(x), \qquad x \longrightarrow q(x)\omega_0^s(x)\theta_0^{\beta-1}(x) \tag{1.11}$$

are in $K(D)$.

Using the Schauder fixed point, we prove the following main result.

Theorem 1.3. *Under the hypothesis* (**H**)*, the problem* (1.1) *has a positive continuous solution (u,v) satisfying for each x in D*

$$\begin{gathered} c_1 \omega(x) \leq u(x) \leq \omega(x), \\ c_2 \theta(x) \leq v(x) \leq \theta(x), \end{gathered} \tag{1.12}$$

where ω, θ are defined by (1.3) *and c_1, $c_2 \in (0,1]$.*

In order to state these results and for the sake of completeness, we give in the sequel some notations, and we recall some properties of the Kato class $K(D)$ studied in [16, 17].

Let us denote by $B(D)$ the set of Borel measurable functions in D and by $B^+(D)$ the set of nonnegative ones. We denote also by $C_0(D)$ the set of continuous functions in D having limit zero at ∂D, by $C(\overline{D} \cup \{\infty\}) = \{f \in C(\overline{D}) : \lim_{|x| \to \infty} f(x) \text{ exists}\}$ and by $C_0(\overline{D}) = \{f \in C(\overline{D}) : \lim_{|x| \to \infty} f(x) = 0\}$. We note that $C(\overline{D} \cup \{\infty\})$ is a Banach space endowed with the uniform norm $\|f\|_\infty = \sup_{x \in D} |f(x)|$.

First we recall that if φ is a nonnegative continuous function on ∂D, then from [20, page 427] the function $H_D \varphi \in C(\overline{D} \cup \{\infty\})$ and satisfies $\lim_{x \to \infty} H_D \varphi(x) = C > 0$.

For any f in $B^+(D)$, we denote by Vf the Green potential of f defined on D by

$$Vf(x) := \int_D G(x,y) f(y) dy, \tag{1.13}$$

and we recall that if $f \in L^1_{\mathrm{loc}}(D)$ and $Vf \in L^1_{\mathrm{loc}}(D)$, then we have in the distributional sense (see [21, page 52])

$$\Delta(Vf) = -f \quad \text{in } D. \tag{1.14}$$

Furthermore, we recall that for $f \in B^+(D)$, the potential Vf is lower semicontinuous in D and if $f = f_1 + f_2$ with $f_1, f_2 \in B^+(D)$ and $Vf \in C^+(D)$, then $Vf_i \in C^+(D)$ for $i \in \{1,2\}$.

Let $(X_t, t > 0)$ be the Brownian motion in $\mathbb{R}^2$ and P^x be a probability measure on the Brownian continuous paths starting at x. For any function $q \in B^+(D)$, we define the kernel V_q by

$$V_q f(x) = E^x\left(\int_0^{\tau_D} e^{-\int_0^t q(X_s)ds} f(X_t)dt\right), \tag{1.15}$$

where E^x is the expectation on P^x and $\tau_D = \inf\{t > 0 : X_t \notin D\}$.

If q is a nonnegative function in D such that $Vq < \infty$, the kernel V_q satisfies the following resolvent equation (see [21, 22])

$$V = V_q + V_q(qV) = V_q + V(qV_q). \tag{1.16}$$

So for each $u \in B(D)$ such that $V(q|u|) < \infty$, we have

$$(I - V_q(q.))(I + V(q.))u = (I + V(q.))(I - V_q(q.))u = u, \tag{1.17}$$

and for each $u \in B^+(D)$, we have

$$0 \le V_q(u) \le V(u). \tag{1.18}$$

Now we recall the definition of the Kato class which contains in particular a wider class of singular functions near the boundary of the domain D.

Definition 1.4 (see [16]). A Borel measurable function s in D belongs to the Kato class $K(D)$ if

$$\begin{gathered}
\lim_{\alpha \to 0} \sup_{x \in D} \int_{D \cap B(x,\alpha)} \frac{\rho(y)}{\rho(x)} G(x,y)|s(y)|dy = 0, \\
\lim_{M \to \infty} \sup_{x \in D} \int_{D \cap \{|y| \ge M\}} \frac{\rho(y)}{\rho(x)} G(x,y)|s(y)|dy = 0,
\end{gathered} \tag{1.19}$$

where $\rho(x) = \min(1, \delta(x))$ and $\delta(x)$ denotes the Euclidian distance from x to the boundary ∂D of D.

This Kato class is rich enough as it can be seen in the following example.

Example 1.5 (see [16]). Let $q(x) = 1/(1+|x|)^{\mu-\lambda}(\delta(x))^{\lambda}$ *for* $x \in D$. Then

$$q \in K(D) \quad \text{if and only if } \lambda < 2 < \mu. \tag{1.20}$$

Remark 1.6. Let $p > 1$ and $\lambda, \mu \in \mathbb{R}$ such that $\lambda < 2 - (2/p) < \mu$. Then using the Hölder inequality and the same arguments as in the proof of the precedent example it follows that for each $f \in L^p(D)$, the function defined in D by $f(x)/(1+|x|)^{\mu-\lambda}(\delta(x))^{\lambda}$ belongs to $K(D)$.

Next, we recall some properties of $K(D)$.

Proposition 1.7 (see [16, 17]). *Let q be a nonnegative function in $K(D)$. Then one has*

(i) $\alpha_q := \sup_{x,y\in D} \int_D (G(x,z)G(z,y)/G(x,y))q(z)dz < \infty$.

(ii) *The function* $x \longrightarrow (\delta(x)/(1+|x|))q(x)$ *is in* $L^1(D)$. *In particular* $q \in L^1_{loc}(D)$.

(iii) $Vq \in C_0(D)$.

(iv) *For any nonnegative superharmonic function v in D and all $x \in D$, one has*

$$\int_D G(x,y)v(y)q(y)dy \le \alpha_q v(x). \tag{1.21}$$

The following compactness results will be used and they are proved, respectively, in [17] and [16].

Proposition 1.8 (see [17, Lemma 3.1]). *Let h_0 be a positive harmonic function in D, which is continuous and bounded in $\overline{D}$ and let q be a nonnegative function belonging to $K(D)$. Then the family of functions:*

$$F_q = \left\{ \int_D G_D(\cdot,y)h_0(y)p(y)dy : |p| \le q \right\} \tag{1.22}$$

is uniformly bounded and equicontinuous on $\overline{D} \cup \{\infty\}$. Consequently, it is relatively compact in $C(\overline{D} \cup \{\infty\})$.

Proposition 1.9 (see [16, Lemma 4.3]). *Let h be the function defined by* (1.5) *and let q be a nonnegative function in $K(D)$. Then the family of functions:*

$$\mathfrak{F}_q = \left\{ \frac{1}{h(\cdot)} \int_D G(\cdot,y)h(y)p(y)dy ; |p| \le q \right\} \tag{1.23}$$

is uniformly bounded and equicontinuous on $\overline{D} \cup \{\infty\}$, and consequently it is relatively compact in $C_0(\overline{D})$.

As a consequence of these Propositions, we obtain the following.

Corollary 1.10. *Let q be a nonnegative function in $K(D)$. Then the family of functions:*

$$\left\{x \longrightarrow \frac{1}{\omega_0(x)} \int_D G(x,y)\omega_0(y)p(y)dy; |p| \leq q\right\} \tag{1.24}$$

is relatively compact in $C(\overline{D} \cup \{\infty\})$.

Proof. Since

$$\frac{\omega_0(y)}{\omega_0(x)} = \frac{a+ch(y)}{a+ch(x)} \leq \max\left(1, \frac{h(y)}{h(x)}\right) \leq 1 + \frac{h(y)}{h(x)}, \tag{1.25}$$

then the result follows from Propositions 1.8 and 1.9. □

The following result will play an important role in the proofs of Theorems 1.2 and 1.3.

Proposition 1.11 (see [17, Proposition 2.9]). *Let v be a nonnegative superharmonic function in D and q be a nonnegative function in $K(D)$. Then for each $x \in D$ such that $0 < v(x) < \infty$, one has*

$$\exp(-\alpha_q)v(x) \leq v(x) - V_q(qv)(x) \leq v(x). \tag{1.26}$$

2. Proof of Theorem 1.2

First we give two Lemmas that will be used for uniqueness.

Lemma 2.1 (see [23]). *Let ζ be a function in $B^+(D)$ and ϑ be a nonnegative superharmonic function in D. Then for all $z \in B(D)$ such that $V(\zeta|z|) < \infty$ and $z + V(\zeta z) = \vartheta$, one has $0 \leq z \leq \vartheta$.*

Lemma 2.2. *Let u be a nonnegative continuous function in $\overline{D} \cup \{\infty\}$. Then*

$$u \text{ is a solution of } (1.7) \text{ if and only if } u = \omega - V(pu^\gamma) \quad \text{in } D. \tag{2.1}$$

Proof. Let u be a nonnegative continuous solution of (1.7). First, we will prove that $u(x) \leq \omega(x)$ in $\overline{D}$. Since $H_D\varphi$ is bounded, then $\lim_{x\to\infty}((u(x) - aH_D\varphi(x))/h(x)) = \lim_{x\to\infty}((u(x) - aH_D\varphi(x))/\ln|x|) = c$. Consequently, for $\varepsilon > 0$ there exists $M > 0$ such that

$$u(x) - aH_D\varphi(x) \leq (c+\varepsilon)h(x) \quad \text{for } |x| \geq M. \tag{2.2}$$

This implies that the function $v_\varepsilon = \omega + \varepsilon h - u$ satisfies

$$\begin{gathered} \Delta v_\varepsilon = -p(x)\, u^\gamma \leq 0 \quad \text{in } D \\ v_\varepsilon = 0 \quad \text{on } \partial D \\ \liminf_{x\to\infty} v_\varepsilon(x) \geq 0. \end{gathered} \tag{2.3}$$

Hence by [20, page 465], we get $u(x) \leq \omega(x) + \varepsilon h(x)$ in $\overline{D}$. Since ε is arbitrary, this implies that $u(x) \leq \omega(x)$ for each $x \in D$. Now, since $\widetilde{p} = \gamma p \omega^{\gamma-1} \in K(D)$, then $pu^{\gamma-1} \in K(D)$. Hence it follows from Propositions 1.8 and 1.9 that $V(aH_D pu^{\gamma-1})$ and $V(chpu^{\gamma-1})$ belong to $C(\overline{D})$ with boundary value zero, which implies that $V(p\omega u^{\gamma-1})$ belongs to $C(\overline{D})$ with boundary value zero. So, $V(pu^{\gamma})$ belongs to $C(\overline{D})$ with boundary value zero. Consequently, using Corollary 7 page 294 in [20], we deduce that the function $u - \omega + V(pu^{\gamma})$ is a classical harmonic in D with boundary value zero and satisfying $\lim_{|x|\to\infty}((u(x) - \omega(x) + V(pu^{\gamma})(x))/\ln|x|) = 0$. Thus by [20, page 419], we have $u - \omega + V(pu^{\gamma}) = 0$ in D. So $u = \omega - V(pu^{\gamma})$ and this proves necessity.

Now, we prove sufficiency. Let u be a nonnegative continuous function in $\overline{D}$ satisfying the integral equation $u = \omega - V(\mathrm{p}u^{\gamma})$. Since p is nonnegative and $\widetilde{p} = \gamma p \omega^{\gamma-1} \in K(D)$, then $u \leq \omega$ and $pu^{\gamma-1} \in K(D)$. This implies, by using Propositions 1.8 and 1.9 that $V(aH_D\varphi pu^{\gamma-1})$ and $V(chpu^{\gamma-1})$ are in $C(\overline{D})$ with boundary value zero. Consequently, $V(pu^{\gamma})$ is in $C(\overline{D})$ with boundary value zero. Hence, $\Delta u = \Delta\omega - \Delta(V(pu^{\gamma})) = pu^{\gamma}$ (in the sense of distributions) and u is a solution of (1.7). □

Now we prove Theorem 1.2

Proof of Theorem 1.2. First we show that problem (1.7) has at most one continuous solution. Let u, v be two continuous solutions of (1.7). Then, by Lemma 2.2 we have $u = \omega - V(p\ u^{\gamma})$ and $v = \omega - V(pv^{\gamma})$ in D. Put $z = v - u$ and $\zeta(x) = (v^{\gamma}(x) - u^{\gamma}(x))/(v(x) - u(x))$ if $u(x) \neq v(x)$ and $\zeta(x) = 0$ whenever $u(x) = v(x)$. Then we have $\zeta \geq 0$ and $z + V(p\zeta z) = 0$ in D. Using Lemma 2.1, we deduce that $z = 0$ and so $u = v$.

Next, we prove the existence of a positive continuous solution to (1.7). We recall that $\omega = aH_D\varphi + ch$ and $\widetilde{p} = \gamma p \omega^{\gamma-1} \in K(D)$. Put $c_0 = e^{-\alpha_{\widetilde{p}}}$ where the constant $\alpha_{\widetilde{p}}$ is defined in Proposition 1.7. We define the nonempty closed bounded convex set Λ by

$$\Lambda = \{u \in B^{+}(D) : c_0\ \omega \leq u \leq \omega\}. \tag{2.4}$$

Let T be the operator defined on Λ by

$$Tu := \omega - V_{\widetilde{p}}(\widetilde{p}\omega) + V_{\widetilde{p}}(\widetilde{p}u - pu^{\gamma}). \tag{2.5}$$

We will prove that T maps Λ to itself. Indeed, for each $u \in \Lambda$, we have

$$\begin{aligned} Tu &= \omega - V_{\widetilde{p}}(\widetilde{p}\omega) + V_{\widetilde{p}}(\widetilde{p}u - pu^{\gamma}) \\ &\leq \omega - V_{\widetilde{p}}(pu^{\gamma}) \\ &\leq \omega. \end{aligned} \tag{2.6}$$

On the other hand, since the function $\widetilde{p}u - pu^{\gamma} \geq 0$, we deduce by Proposition 1.11 that $Tu \geq \omega - V_{\widetilde{p}}(\widetilde{p}\omega) \geq c_0\omega$. Hence, $T\Lambda \subset \Lambda$.

Next, we prove that T is nondecreasing on Λ. Let $u_1,\ u_2 \in \Lambda$ such that $u_1 \leq u_2$. Since for each $y \in D$, the function $t \rightarrow \widetilde{p}(y)\ t - p(y)t^{\gamma}$ is nondecreasing on $[0, \omega(y)]$ we deduce that

$$\begin{aligned} Tu_2 - Tu_1 &= V_{\widetilde{p}}\left(\widetilde{p}u_2 - pu_2^{\gamma}\right) - V_{\widetilde{p}}\left(\widetilde{p}u_1 - pu_1^{\gamma}\right) \\ &= V_{\widetilde{p}}\left[\left(\widetilde{p}u_2 - pu_2^{\gamma}\right) - \left(\widetilde{p}u_1 - pu_1^{\gamma}\right)\right] \\ &\geq 0. \end{aligned} \tag{2.7}$$

Now, we consider the sequence $(u_k)_k$ defined by $u_0 = \omega - V_{\widetilde{p}}(\widetilde{p}\omega)$ and $u_{k+1} = Tu_k$. Clearly $u_0 \in \Lambda$ and $u_1 = Tu_0 \geq u_0$. Thus, using the fact that Λ is invariant under T and the monotonicity of T, we deduce that

$$c_0\omega \leq u_0 \leq u_1 \leq \cdots \leq u_k \leq \omega. \tag{2.8}$$

Hence, the sequence $(u_k)_k$ converges to a measurable function $u \in \Lambda$. Therefore, by applying the monotone convergence theorem, we deduce that u satisfies the following equation:

$$u = \omega - V_{\widetilde{p}}(\widetilde{p}\omega) + V_{\widetilde{p}}(\widetilde{p}u - pu^{\gamma}) \tag{2.9}$$

or equivalently

$$u - V_{\widetilde{p}}(\widetilde{p}u) = \omega - V_{\widetilde{p}}(\widetilde{p}\omega) - V_{\widetilde{p}}(pu^{\gamma}). \tag{2.10}$$

Applying the operator $(I + V(\widetilde{p}.))$ on both sides of (2.10), we deduce by using (1.16) and (1.17) that

$$u = \omega - V(pu^{\gamma}). \tag{2.11}$$

Now, let us verify that u is a solution of the problem (1.7). Since $\widetilde{p} = \gamma p\omega^{\gamma-1} \in K(D)$, then by Proposition 1.7, we have $\widetilde{p} \in L^1_{\text{loc}}(D)$.

Now, using the following inequality:

$$pu^{\gamma} \leq p\omega^{\gamma} \leq \widetilde{p}\omega \tag{2.12}$$

and the continuity of ω in D, we obtain that $pu^{\gamma} \in L^1_{\text{loc}}(D)$. Using Proposition 1.7 and (2.12), we obtain for each $x \in D$

$$V(pu^{\gamma})(x) \leq \int_D G(x,y)\widetilde{p}(y)\omega(y)dy \leq \alpha_{\widetilde{p}}\omega(x), \tag{2.13}$$

which gives $V(pu^{\gamma}) \in L^1_{\text{loc}}(D)$. Thus, by applying Δ on both sides of (2.11), we deduce that u is a solution of

$$\Delta u = pu^{\gamma} \quad \text{(in the sense of distributions)}. \tag{2.14}$$

Using (2.12), we obtain that pu^γ and $\widetilde{p}\omega$ are in $B^+(D)$. So $V(pu^\gamma)$ and $V(\widetilde{p}\omega - pu^\gamma)$ are two lower semicontinuous functions. On the other hand, by Proposition 1.8 we have $V(\widetilde{p}aH_D\varphi) \in C(D)$ and by Proposition 1.9 the function $(1/h)V(\widetilde{p}h) \in C_0(\overline{D})$. So $V(\widetilde{p}\omega) \in C(D)$. Thus $V(\widetilde{p}\omega - pu^\gamma) = V(\widetilde{p}\omega) - V(pu^\gamma)$ is also an upper semicontinuous function. Consequently, $V(\widetilde{p}\omega - pu^\gamma) \in C(D)$. Thus $V(pu^\gamma) = V(\widetilde{p}\omega) - V(\widetilde{p}\omega - pu^\gamma) \in C(D)$. Therefore $u \in C(D)$. Now, using Propositions 1.1 and 1.9, we deduce that $\lim_{x\to\partial D} V(\widetilde{p}h)(x) = 0$. In addition, since $H_D\varphi$ is bounded in $\overline{D}$, we deduce from Proposition 1.7 $\lim_{x\to\partial D} V(\widetilde{p})(x) = 0$. So that $\lim_{x\to\partial D} V(\widetilde{p}\omega)(x) = 0$. This in turn implies that $\lim_{x\to\partial D} V(pu^\gamma) = 0$. Which together with (2.11) imply that $u_{/\partial D} = a\varphi$. On the other hand, we have

$$\frac{1}{h}V(pu^\gamma) \leq a\|\varphi\|_\infty \frac{1}{h}V(\widetilde{p}) + \frac{c}{h}V(\widetilde{p}h). \tag{2.15}$$

Using Propositions 1.9, 1.8, and 1.1, we deduce that $(1/\ln|x|)V(pu^\gamma)(x)$ tends to zero as $|x| \to \infty$ and consequently $\lim_{|x|\to\infty}(u(x)/\ln|x|) = c$. This implies that u is a positive continuous solution of (1.7). This completes the proof of Theorem 1.2. □

Remark 2.3. Let $\widetilde{p_0} = \gamma\max(1, \|\varphi\|_\infty) p\omega_0^{\gamma-1}$, where ω_0 is given by (1.9). Then we have $0 \leq \widetilde{p} \leq \widetilde{p_0}$. So if we assume that $\widetilde{p_0} \in K(D)$, then $\widetilde{p} \in K(D)$ and $\alpha_{\widetilde{p}} \leq \alpha_{\widetilde{p_0}}$. Moreover, the solution u of (1.7) satisfies also the inequality:

$$e^{-\alpha_{\widetilde{p_0}}}\,\omega \leq u \leq \omega \quad \text{in } D. \tag{2.16}$$

Next we give the proof of Theorem 1.3.

3. Proof of Theorem 1.3

We recall that $\omega_0 = a + ch$, $\theta_0 = b + dh\omega = aH_D\varphi + ch$ and $\theta = bH_D\psi + dh$. Define $m = \max(1, \|\varphi\|_\infty, \|\psi\|_\infty)$, $\widetilde{f} = \alpha m^{r+\alpha-1}\; p\theta_0^r\omega_0^{\alpha-1}$ and $\widetilde{g} = \beta m^{s+\beta-1}\; q\omega_0^s\theta_0^{\beta-1}$. Put $c_1 = e^{-\alpha_{\widetilde{f}}}$ and $c_2 = e^{-\alpha_{\widetilde{g}}}$, where the nonnegative constants $\alpha_{\widetilde{f}}$ and $\alpha_{\widetilde{g}}$ are defined in Proposition 1.7.

In order to use a fixed point theorem, we consider the nonempty closed convex set Γ defined by

$$\Gamma = \left\{(\lambda, \chi) \in \left(C\left(\overline{D} \cup \{\infty\}\right)\right)^2 : 0 \leq \lambda \leq (1-c_1)\frac{\omega}{\omega_0},\ 0 \leq \chi \leq (1-c_2)\frac{\theta}{\theta_0}\right\}. \tag{3.1}$$

For $(\lambda, \chi) \in \Gamma$, we consider the following system:

$$\begin{gathered}
\Delta y = p(\theta - \theta_0\chi)^r y^\alpha, \quad \text{in } D \\
\Delta z = q(x)(\omega - \omega_0\ \lambda)^s z^\beta, \quad \text{in } D \\
y_{/\partial D} = a\varphi, \qquad z_{/\partial D} = b\psi, \\
\lim_{x\to\infty}\frac{y(x)}{\ln|x|} = c, \qquad \lim_{x\to\infty}\frac{z(x)}{\ln|x|} = d.
\end{gathered} \tag{3.2}$$

Then by Theorem 1.2, the system 21 has a unique positive continuous solution (y,z) satisfying the integral equations

$$y(x) = \omega(x) - V\left(p(\theta-\theta_0\chi)^r y^\alpha\right)(x), \tag{3.3}$$

$$z(x) = \theta(x) - V\left(q(\omega-\omega_0\lambda)^s z^\beta\right)(x). \tag{3.4}$$

Moreover, we have the following global inequalities:

$$c_1\omega \le y \le \omega, \quad c_2\theta \le z \le \theta, \quad \text{in } D, \tag{3.5}$$

$$p(\theta-\theta_0\chi)^r y^\alpha \le p\omega^\alpha\theta^r \le \frac{m}{\alpha}\omega_0\widetilde{f}, \tag{3.6}$$

$$q(\omega-\omega_0\lambda)^s z^\beta \le q\theta^\beta\omega^s \le \frac{m}{\beta}\theta_0\widetilde{g}. \tag{3.7}$$

Let T be the operator defined on Γ by $T(\lambda,\chi) = ((\omega-y)/\omega_0, (\theta-z)/\theta_0)$.

Since y, z satisfy (3.3) and (3.4) we deduce from (3.5), (3.6), hypothesis (**H**) and Corollary 1.10 that the family of functions:

$$T\Gamma = \left\{\left(\frac{1}{\omega_0}V\left(py^\alpha(\theta-\theta_0\chi)^r\right), \frac{1}{\theta_0}V\left(qz^\beta(\omega-\omega_0\lambda)^s\right)\right);\ (\lambda,\chi)\in\Gamma\right\} \tag{3.8}$$

is relatively compact in $(C(\overline{D}\cup\{\infty\}))^2$. This together with (3.5) imply that $T\Gamma\subset\Gamma$.

Next, we will prove the continuity of T with respect to the norm $\|\cdot\|$ defined on Γ by $\|(\lambda,\chi)\| = \|\lambda\|_\infty + \|\chi\|_\infty$. Let $(\lambda_k,\ \chi_k)_k$ be a sequence in Γ that converges to $(\lambda,\ \chi)\in\Gamma$ with respect to $\|\cdot\|$. Put $(y_k,z_k) = T(\lambda_k,\chi_k)$ and $(y,z) = T(\lambda,\chi)$. Then we have

$$\left\|(y_k,z_k)-(y,z)\right\| = \left\|\frac{y_k-y}{\omega_0}\right\|_\infty + \left\|\frac{z_k-z}{\omega_0}\right\|_\infty. \tag{3.9}$$

Using (3.3), we obtain

$$\begin{aligned} y_k - y &= V\left(py^\alpha(\theta-\theta_0\chi)^r\right) - V\left(py_k^\alpha\ (\theta-\theta_0\chi_k)^r\right) \\ &= V\left(p\left[y^\alpha\left((\theta-\theta_0\chi)^r-(\theta-\theta_0\chi_k)^r\right)\right) + V\left(p\left(y^\alpha-y_k^\alpha\right)(\theta-\theta_0\chi_k)^r\right). \end{aligned} \tag{3.10}$$

Now using the fact that

$$t^\alpha - \eta^\alpha = (t-\eta)\left[\alpha\int_0^1\left(\eta+\xi(t-\eta)\right)^{\alpha-1}d\xi\right] \quad \text{for } t\ge 0,\ \eta\ge 0, \tag{3.11}$$

we deduce that

$$(y_k-y) + V\left(p_k(y_k-y)\right) = V\left(py^\alpha\left((\theta-\theta_0\chi)^r-(\theta-\theta_0\chi_k)^r\right)\right), \tag{3.12}$$

where $p_k(x) = \alpha p(x)(\theta(x) - \theta_0(x)\chi_k(x))^r \int_0^1 [\xi y_k(x) + (1-\xi)y(x)]^{\alpha-1} d\xi$.

Clearly, we have

$$\begin{aligned} p_k(x) &\le \alpha p(x)\left(\theta(x) - \theta_0(x)\chi_k(x)\right)^r \omega^{\alpha-1}(x) \\ &\le \alpha p(x)\theta^r(x)\omega^{\alpha-1}(x) \\ &\le m^{r+\alpha-1}\alpha p(x)\ \theta_0^r(x)\ \omega_0^{\alpha-1}(x) = m^{r+\alpha-1}\ \tilde{f}. \end{aligned} \tag{3.13}$$

Now, since $\tilde{f} \in K(D)$ then $p_k \in K(D)$ and $Vp_k < \infty$. Moreover, we obtain from (3.6) and Proposition 1.7(iv) that

$$\begin{aligned} V\left(p_k|y_k - y|\right) &\le V\left(p y_k^{\alpha}(\theta - \theta_0\chi_k)^r\right) + V\left(p y^{\alpha}(\theta - \theta_0\chi_k)^r\right) \\ &\le \frac{2m}{\alpha}\omega_0\alpha_{\tilde{f}}. \end{aligned} \tag{3.14}$$

So we can apply $(I - V_{p_k}(p_k\cdot))$ to (3.12) to obtain from (1.16) and (1.17) that

$$y_k - y = V_{p_k}\left(p y^{\alpha}\left((\theta - \theta_0\chi)^r - (\theta - \theta_0\chi_k)^r\right)\right). \tag{3.15}$$

On the other hand, we have

$$\begin{aligned} p y^{\alpha}\left|(\theta - \theta_0\chi)^r - (\theta - \theta_0\chi_k)^r\right| &\le p y^{\alpha}\left((\theta - \theta_0\chi)^r + (\theta - \theta_0\chi_k)^r\right) \\ &\le 2p y^{\alpha}\theta^r \\ &\le \frac{2m}{\alpha}\omega_0\tilde{f}. \end{aligned} \tag{3.16}$$

So from hypothesis (**H**), Proposition 1.7(iv) and the dominated convergence theorem, we deduce that for each $x \in D$, we have

$$\lim_{k\to\infty} V\left(p y^{\alpha}\left((\theta - \theta_0\chi)^r - (\theta - \theta_0\chi_k)^r\right)\right)(x) = 0. \tag{3.17}$$

This together with (1.18) and (3.15) implies that for each $x \in D$, $(y_k(x))_k$ converges to $y(x)$ as $k \to \infty$. Similarly we prove that for each $x \in D$, $(z_k(x))_k$ converges to $z(x)$ as $k \to \infty$. Consequently, as $T\Gamma$ is relatively compact in $C(\overline{D} \cup \{\infty\})$, we deduce that the pointwise convergence implies the uniform convergence. Namely, $\|(y_k - y)/\omega_0\|_\infty + \|(z_k - z)/\theta_0\|_\infty$ converges to 0 as $k \to \infty$.

From the Schauder fixed point theorem there exists $(\lambda, \chi) \in \Gamma$ such that $T(\lambda, \chi) = (\lambda, \chi)$ or equivalently $((\omega - y)/\omega_0, (\theta - z)/\theta_0) = (\lambda, \chi)$. Put $u = \omega - \omega_0\lambda$ and $v = \theta - \theta_0\chi$. Then (u, v) is a positive continuous solution of the system (1.1) in the sense of distributions satisfying for each $x \in D$

$$c_1\omega(x) \le u(x) \le \omega(x), \qquad c_2\theta(x) \le v(x) \le \theta(x). \tag{3.18}$$

This completes the proof.

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