Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 270823, 11 pages
doi:10.1155/2012/270823

Research Article

Peculiarities of Bounds on States through the Concept of Linear Superposition

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Received 2 May 2012; Accepted 11 June 2012

Academic Editor: Shan Zhao

We investigate the effect of superposition of states on local conversion of pure bipartite states under deterministic LOCC. We also investigate the entanglement behaviour of such classes of states, specifically their monotone nature. Finally we are able to construct some counterintuitive situations, on the bounds of different measures of entanglement, emphasis on the idea of comparability and incomparability under deterministic LOCC.

1. Introduction

Quantum entanglement is one of the most puzzling, useful yet experimentally verified feature of quantum states. Quantum entanglement is also useful for performing many informational and computational tasks like teleportation, dense coding, and so forth [1–3], which are otherwise impossible. Now, to understand behaviour of quantum entanglement better, we need to probe different aspects of entanglement properly [4, 5]. Physicists have tried to observe the underline physics of quantum entanglement [6, 7] and suggested many algorithms and concepts to prove some new results.

In the paper [8], Linden et al. have raised the following problem: *suppose a bipartite quantum state* $|\Gamma\rangle$ *and a certain decomposition of it as a superposition of two other states are given. In* $|\Gamma\rangle = \alpha|\psi\rangle + \beta|\phi\rangle$ *what is the relation between the entanglement of* $|\Gamma\rangle$ *and those of the two constituent states in the superposition?* They also considered the following two examples to illustrate the above problem. One is $|\gamma\rangle = (1/\sqrt{2})|00\rangle + (1/\sqrt{2})|11\rangle$ and the other is $|\gamma'\rangle = (1/\sqrt{2})|\phi^{+}\rangle + (1/\sqrt{2})|\phi^{-}\rangle$, where $|\phi^{\pm}\rangle = (1/\sqrt{2})|00\rangle \pm (1/\sqrt{2})|11\rangle$ are two common Bell states. The first one clearly explains that $|\gamma\rangle$ is a maximally entangled state but each constituent state is fully separable [9, 10]. That is, superposition of fully separable states may form a maximally entangled state and the second example shows exactly the opposite to that of the first; that is, $|\gamma'\rangle$ is separable but each constituent state is maximally entangled. Therefore, through superposition of states one can find new physical insights as well as differently correlated states. In [8], Linden et al. employed von-Neumann entropy of the reduced system as the entanglement measure (ER) and found some upper bound on ER of the superposed state in terms of those of the states being superposed. This problem has been actively and extensively studied. Yu et al. [11] have studied the concurrence of superposition and presented both upper bound and lower bound on the concurrence of superposition. Ou and Fan [12] given an upper bound on the negativity of superposition. Niset and Cerf [13] gave lower and upper bounds simpler form. Many people have considered the problem of two superposed coherent states. Cavalcanti et al. [14], Song et al. [15], and Yu et al. [16] have investigated the entanglement of superpositions for multipartite quantum states by employing different entanglement measures. Gour [17] reconsidered the question in [8] and presented tighter upper and lower bounds. Finally, we have observed some new bounds of some different entanglement measures like negativity (N), logarithmic negativity (LN), Reyni entropy (S_δ) and also the entanglement of the superposed states $|\Gamma\rangle_{AB}$, and $|\Gamma'\rangle_{AB}$ as a direct function of Schmidt's coefficients as well as functions of different measures of the component of states.

This paper is organized as follows: firstly, in Section 2, we will discuss some useful notions regarding superposition of states and entanglement. In Section 3, we will discuss the concept of incomparability. Sections 4 and 5 are devoted to discuss the main results and some nice illustrations on the bounds of different measures of entanglement. The paper is ended with a brief conclusion of our results.

2. Superposition of States and Different Entanglement Measures

Quantum mechanics is inherently a linear theory and superposition is deeply related to this linear structure of quantum systems. Entanglement is a manifestation of quantum superposition whenever one deals with composite systems. Superposition of two pure product state may could give rise to an entangled state and quite contrary to this fact one could find pure product states with the superposition of entangled states only. It is clear that if someone tries to explain superposition of states as a physical process, then it should not be local, as entanglement may be created or increased in this process.

Now to understand the structural complexity of the class of equally entangled states we try to search for a suitable measure that may discriminate all the states of one equi-entangled class. It leads us to the search for different measures that discriminates such states. Here we consider some of well-known correlation measures like concurrence, linear entropy, logarithmic negativity, and so forth. Any pure bipartite state $|\psi\rangle$ of the joint Hilbert space $H = H_A \otimes H_B$, has the Schmidt representation form:

$$|\psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i} |i_A\rangle \otimes |i_B\rangle, \tag{2.1}$$

where $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ are orthonormal bases of the local Hilbert spaces H_A and H_B, respectively. The set of real numbers $\{\sqrt{\lambda_i}\}$, for $i = 1, 2, \ldots, d$, known as Schmidt coefficients of the state, are just the square-root of eigenvalues of reduced density matrices of the state, satisfying $0 \le \lambda_i \le 1$ for all i and $\sum_{i=1}^{d} \lambda_i = 1$. The number of Schmidt coefficients

$d \le \min\{\dim(H_A), \dim(H_B)\}$ is known as the Schmidt rank of the pure bipartite state. The Schmidt coefficients remain invariant under any local unitary transformations $U_A \otimes U_B$ on the pure bipartite state. Thus they are expected to serve well as ingredients of any good measure of entanglement. Concurrence is an important measure to quantify entanglement, functionally related to entanglement of formation [18] in 2×2 systems. For any pure bipartite state ρ in the Hilbert space $H_A \otimes H_B$ of two subsystems A, B it is defined by $C(\rho) = \sqrt{2(1-{\rho_A}^2)}$, where ρ_A is the reduced density matrix of ρ, after tracing out the subsystem B. The entanglement of formation for a general state of two-qubit system can be expressed by [19]

$$E_F(\rho) = \zeta\left(\frac{1+\sqrt{1-C^2(\rho)}}{2}\right), \tag{2.2}$$

where the function ζ is defined as $\zeta(x) = -x\log_2 x - (1-x)\log_2(1-x)$. For higher dimensional pure bipartite state (say $m \times n$, for $m \ge n$), concurrence is given by [20]

$$C(|\psi\rangle) = \sqrt{4\sum_{i<j}\lambda_i\lambda_j} = \sqrt{2\left(1-\sum_{i=1}^{d}\lambda_i^2\right)} \tag{2.3}$$

which varies smoothly from 0 for product states to $\sqrt{2(d-1)/d}$ for maximally entangled pure states of Schmidt rank d. For mixed bipartite states it is defined by convex roof extension.

Logarithmic negativity is a computable measure of entanglement. It has functional relation with another important quantification scheme, known as negativity. Negativity is defined from the Peres-Horodecki criteria [21, 22] by

$$N(\rho) \equiv \frac{\|\rho^{T_A}\|_1 - 1}{2}, \tag{2.4}$$

where $\|\rho^{T_A}\|_1$ denote the trace norms of ρ^{T_A}, partial transpose of the bipartite mixed state ρ with respect to the subsystem A, which corresponds to the absolute value of the sum of the negative eigenvalues of ρ^{T_A} and will vanish for unentangled states. For a pure state $|\psi\rangle$ negativity is $N(|\psi\rangle) \equiv (1/2)((\sum_i \sqrt{\lambda_i})^2 - 1)$. Logarithmic negativity is defined by

$$\mathrm{LN}(\rho) \equiv \log_2\|\rho^{T_A}\|_1 = \log_2\{2N(\rho)+1\}. \tag{2.5}$$

It is an entanglement monotone [23], related to the PPT entanglement cost, $E_{\mathrm{PPT}}(\rho) = \log_2\{N(\rho)+1\}$ of the state ρ, defined as the cost of exact preparing under PPT preserving operations. For pure bipartite states this measure is calculated by

$$\mathrm{LN}(|\psi\rangle) = 2\log_2\left(\sum_{i=1}^{d}\sqrt{\lambda_i}\right). \tag{2.6}$$

It is found that negativity is a convex function [24] of the state, though logarithmic negativity is not.

A series of correlation measures known as Réyni entropy [25] or Alpha entropy (S_α) are proposed by generalizing the concept of von-Neumann entropy

$$S_\alpha = \frac{1}{1-\alpha} \ln\left[\sum_{i=1}^{d} \lambda_i^\alpha\right]. \tag{2.7}$$

All the alpha entropy measures (naturally excluding the von-Neumann entropy function itself) are suitable to discriminate between any class of incomparable states with same entanglement. Here we only consider the Linear entropy (S_2) and alpha entropy for $\alpha = 3 (S_3)$.

Linear entropy measure for the pure bipartite state in the form (2.1) is given by

$$S_2(|\psi\rangle) = -\log_2\left(\sum_i \lambda_i^2\right). \tag{2.8}$$

Giampaolo and Elluminati show that [26], for all nonmaximally entangled states of $3 \times d$ system, there exists a range of values of linear entropy with same entanglement.

Alpha entropy for $\alpha = 3$, that is, (S_3) of the state (2.1), is computed by the formula

$$S_3(|\psi\rangle) = -\log_2\left(\sum_i \lambda_i^3\right). \tag{2.9}$$

Concurrence hierarchy [27] is a series of correlation measures generalized from the concept of concurrence, in finite dimensional bipartite pure states. For a general bipartite pure state of rank d in the Schmidt form (2.1), the precise definition of the concurrence hierarchy is

$$C_k(|\psi\rangle) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le d} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, \quad k = 1, \ldots, d. \tag{2.10}$$

For 3×3 system, there is only one concurrence hierarchy for $k = 3$, that is, $C_3(|\psi\rangle) = \lambda_1\lambda_2\lambda_3$.

The maximum fidelity is a convex function of the generalized entropy, $F_{\max} = \exp\{H_{1/2}(\rho)\}/d$, where $H_{1/2}$ is the Rényi entropy for $\alpha = 1/2$. Maximum fidelity for a pure state of the form (2.1) is given by

$$F_{\max}(|\psi\rangle) = \frac{\exp\left(2 \ln \sum_{i=1}^{d} \sqrt{\lambda_i}\right)}{d}. \tag{2.11}$$

The correlation measure robustness of entanglement [28], denoted by $R(\rho)$, examines how much mixing can take place between an entangled state and any other state, so that the convex combination of these two states is separable. In the characterization of the state space in terms of entangled and separable states, we observe some interesting properties of this measure. Robustness $R(\rho)$ is convex function of ρ, that is, for any two states ρ_1 and ρ_2 we have the following inequality $R(t\rho_1 + (1-t)\rho_2) \leq tR(\rho_1) + (1-t)R(\rho_2)$. Robustness of entanglement remains unchanged under unitary transformation of state, that is, $R(\rho) = R(U_L \rho U_L)$, where U_L is a local unitary transformation of the form $U_L = U_1 \otimes U_2$. Now for the pure state (2.1), we can define robustness of entanglement as follows:

$$R(|\psi\rangle) = \exp\left(2\ln\sum_{i=1}^{d}\lambda_i^{1/2}\right) - 1. \tag{2.12}$$

Now, for pure bipartite states apart from the entropy of entanglement calculated by the von-Neumann entropy of the reduced density matrices, there is an useful measure of entanglement, called generalized concurrence (C). For a separable state it is zero. For a two-qubit state ρ_{AB} it is calculated by $C(\rho_{AB}) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$, where λ_i's, $i = 1, 2, 3, 4$ are the square root of the eigenvalues of $\rho\widetilde{\rho}$ in decreasing order where $\widetilde{\rho} = \sigma_y \otimes \sigma_y \rho_{AB}^* \sigma_y \otimes \sigma_y$ and $*$ denotes conjugate operation. For higher order systems generalized concurrence is defined by

$$C(\rho_{AB}) = \sqrt{2(1 - Tr\rho_A{}^2)}, \tag{2.13}$$

where ρ_A is the reduced density matrix, obtained by tracing out the subsystem B. For a pure bipartite state $|\xi\rangle_{AB}$ of $d_1 \times d_2$ system with Schmidt form $|\xi\rangle_{AB} = \sum_i^{\min\{d_1,d_2\}} \sqrt{\mu_i}|i\rangle_A|i\rangle_B$, where $\{\mu_i;\ i = 1, 2, \ldots\}$ are nonnegative Schmidt coefficients and $\{|i\rangle_A\}$, $\{|i\rangle_B\}$ the orthonormal bases for subsystems A and B, respectively, the generalized concurrence $C(|\xi\rangle_{AB})$ turns out to be

$$C^2(|\xi\rangle_{AB}) = 4\sum_{i<j}\mu_i\mu_j = 2\left(1 - \sum_{i=1}^{\min\{d_1,d_2\}}\mu_i^2\right) \tag{2.14}$$

varies smoothly from 0 for pure product states to $2((d-1)/d)$ for maximally entangled pure bipartite states of Schmidt rank d.

3. Notion of Incomparability

Now before going to present our results we first mention the condition for a pair of pure bipartite states to be incomparable with each other. The notion of incomparability of a pair of bipartite pure states is a consequence of Nielsen's [2, 29] famous majorization criterion. To illustrate it, we consider the deterministic local conversion of the pure bipartite state $|\chi\rangle$ to $|\eta\rangle$ shared between two parties, say, Alice and Bob. We write the pair $(|\chi\rangle, |\eta\rangle)$ in their Schmidt bases $\{|i_A\rangle, |i_B\rangle\}$ with decreasing order of Schmidt coefficients: $|\chi\rangle = \sum_{i=1}^{d}\sqrt{\gamma_i}|i_A i_B\rangle$, $|\eta\rangle = \sum_{i=1}^{d}\sqrt{\delta_i}|i_A i_B\rangle$. The Schmidt vectors corresponding to the states $|\chi\rangle$ and $|\eta\rangle$ are $\lambda_\chi \equiv (\gamma_1, \gamma_2, \ldots, \gamma_d)$ and $\lambda_\eta \equiv (\delta_1, \delta_2, \ldots, \delta_d)$. From Nielsen's criterion $|\chi\rangle \rightarrow |\eta\rangle$ is possible with certainty under LOCC if and only if λ_χ is majorized by λ_η, (denoted by $\lambda_\chi \prec \lambda_\eta$), that is,

$$\sum_{i=1}^{k}\gamma_i \leq \sum_{i=1}^{k}\delta_i \quad \forall k = 1, 2, \ldots, d. \tag{3.1}$$

The above result has a direct consequence in the entanglement behaviour of the states involved. If $|\chi\rangle \to |\eta\rangle$ is possible under deterministic LOCC, then $E(|\chi\rangle) \geq E(|\eta\rangle)$ (where $E(\cdot)$ is the entropy of entanglement). Now in case of failure of the above criterion, we denote it as $|\chi\rangle \nrightarrow |\eta\rangle$. But it may happen that $|\eta\rangle \to |\chi\rangle$ under deterministic LOCC. If both $|\chi\rangle \nrightarrow |\eta\rangle$ and $|\eta\rangle \nrightarrow |\chi\rangle$ hold, we denote it as $|\chi\rangle \nleftrightarrow |\eta\rangle$ and call $(|\chi\rangle, |\eta\rangle)$ as a pair of incomparable states. The existence of incomparable pair of states starts from 3×3 systems. For our purpose, we require explicitly the criterion of incomparability for a pair of pure bipartite states $|\chi\rangle, |\eta\rangle$ of 3×3 system. Suppose the Schmidt vectors corresponding to the two states are $(\gamma_1, \gamma_2, \gamma_3)$ and $(\delta_1, \delta_2, \delta_3)$, respectively, where $\gamma_1 > \gamma_2 > \gamma_3 > 0$, $\delta_1 > \delta_2 > \delta_3 > 0$, $\gamma_1 + \gamma_2 + \gamma_3 = 1 = \delta_1 + \delta_2 + \delta_3$. Then $|\chi\rangle, |\eta\rangle$ are incomparable whenever [30] either of the following relations hold:

$$\begin{aligned} &\text{(i) } \gamma_1 > \delta_1, \qquad \gamma_3 > \delta_3, \\ &\text{(ii) } \delta_1 > \gamma_1, \qquad \delta_3 > \gamma_3. \end{aligned} \tag{3.2}$$

4. Main Results: Observations on the Bounds of Superposed States

Consider the states shared between two parties, say, A and B,

$$|\Gamma\rangle_{AB} = \alpha|\psi\rangle_{AB} + \beta|\phi\rangle_{AB}, \tag{4.1}$$

where $\alpha^2 + \beta^2 = 1$ and α, β are nonnegative real number, and also consider the state

$$|\Gamma'\rangle_{AB} = \alpha'|\psi'\rangle_{AB} + \beta'|\phi'\rangle_{AB}, \tag{4.2}$$

where $\alpha'^2 + \beta'^2 = 1$ with nonnegative real α', β', and further assume that $\langle\psi \mid \phi\rangle_{AB} = 0$; $\langle\psi' \mid \phi'\rangle_{AB} = 0$. Explicitly, suppose $|\psi\rangle_{AB}$, $|\psi'\rangle_{AB}$, $|\phi\rangle_{AB}$ and $|\phi'\rangle_{AB}$ may be expressed as follows:

$$\begin{aligned} |\psi\rangle_{AB} &= \sum_{i=0}^{2}\sqrt{a_i}|ii\rangle_{AB}, \\ |\phi\rangle_{AB} &= \sum_{j=0}^{2}\sqrt{b_j}|jj\rangle_{AB}, \\ |\psi'\rangle_{AB} &= \sum_{i=0}^{2}\sqrt{\alpha_i}|ii\rangle_{AB}, \\ |\phi'\rangle_{AB} &= \sum_{j=0}^{2}\sqrt{\beta_j}|jj\rangle_{AB}. \end{aligned} \tag{4.3}$$

We will now discuss the entanglement behaviour of the superposed states imposing some restrictions on α, β, α', β', and also on a_i, b_i, α_i, β_i for all $i = 0, 1, 2$, case by case.

In this section, our purpose is to find some new bounds of some different entangle measures like negativity (N), logarithmic negativity (LN), Reyni entropy (S_δ) and also the entanglement of the superposed states $|\Gamma\rangle_{AB}$, and $|\Gamma'\rangle_{AB}$. Here we find some tight bounds and also observe the behaviour of the bounds of the corresponding measures in the notion of incomparability under deterministic LOCC.

In this work, according to the basic restrictions of the states $|\Gamma\rangle_{AB}$ and $|\Gamma'\rangle_{AB}$ is that their components states are orthogonal, that is, $\langle\psi \mid \phi\rangle_{AB} = 0$. For negativity ($N$) we found the following forms of upper and lower bounds of the superposed state $|\psi\rangle_{AB}$ in terms of those states (i.e., $|\psi\rangle_{AB}$ and $|\phi\rangle_{AB}$) being superposed and also in terms Schmidt coefficients of the states.

Theorem 4.1. $\alpha^2 N(|\psi\rangle_{AB}) + \beta^2 N(|\phi\rangle_{AB}) \leq N(|\Gamma\rangle_{AB}) \leq \alpha^2 N(|\psi\rangle_{AB}) + \beta^2 N(|\phi\rangle_{AB}) + \alpha\beta$.

Proof. From definition, we have

$$\begin{aligned} N(|\Gamma\rangle_{AB}) &= \frac{1}{2}\left[\alpha^2\left(\sum_{i=0}^{2}\sqrt{a_i}\right)^2 + \beta^2\left(\sum_{i=0}^{2}\sqrt{b_i}\right)^2 + 2\alpha\beta\left(\sum_{i=0}^{2}\sqrt{a_i}\right)\left(\sum_{i=0}^{2}\sqrt{b_i}\right) - 1\right] \\ &\geq \frac{1}{2}\left[\alpha^2\left(\sum_{i=0}^{2}\sqrt{a_i}\right)^2 + \beta^2\left(\sum_{i=0}^{2}\sqrt{b_i}\right)^2 - 1\right] \\ &= \frac{1}{2}\left[\alpha^2\left\{\left(\sum_{i=0}^{2}\sqrt{a_i}\right)^2 - 1\right\} + \beta^2\left\{\left(\sum_{i=0}^{2}\sqrt{b_i}\right)^2 - 1\right\}\right]. \end{aligned} \tag{4.4}$$

So,

$$N(|\Gamma\rangle_{AB}) \geq \alpha^2 N\left(|\psi\rangle_{AB}\right) + \beta^2 N\left(|\phi\rangle_{AB}\right). \tag{4.5}$$

Again, we have

$$\begin{aligned} N(|\Gamma\rangle_{AB}) &= \frac{1}{2}\left[\alpha^2\left(\sum_{i=0}^{2}\sqrt{a_i}\right)^2 + \beta^2\left(\sum_{i=0}^{2}\sqrt{b_i}\right)^2 + 2\alpha\beta\left(\sum_{i=0}^{2}\sqrt{a_i}\right)\left(\sum_{i=0}^{2}\sqrt{b_i}\right) - 1\right] \\ &\leq \frac{1}{2}\left[\alpha^2\left(\sum_{i=0}^{2}\sqrt{a_i}\right)^2 + \beta^2\left(\sum_{i=0}^{2}\sqrt{b_i}\right)^2 - \left(\alpha^2 + \beta^2\right) + 2\alpha\beta\left(\sum_{i=0}^{2}a_i\right)\left(\sum_{i=0}^{2}b_i\right)\right], \\ N(|\Gamma\rangle_{AB}) &\leq \alpha^2 N\left(|\psi\rangle_{AB}\right) + \beta^2 N\left(|\phi\rangle_{AB}\right) + \alpha\beta. \end{aligned} \tag{4.6}$$

These equations directly claim the result. □

Theorem 4.2. $(1/2)[9(\alpha+\beta)^2\{\min(\mu)^2\} - 1] \leq N(|\Gamma\rangle_{AB}) \leq (1/2)[9(\alpha+\beta)^2\{\max(\mu)^2\} - 1]$, *where* $\min(\mu)$ *and* $\max(\mu)$ *denote, respectively, the least and greatest of the numbers* $\{\sqrt{a_i}, \sqrt{b_i}\}_{i=0}^{2}$.

Proof. $\min(\mu) \leq [\alpha(\sum_{i=0}^{2}\sqrt{a_i}) + \beta(\sum_{i=0}^{2}\sqrt{b_i})]/3(\alpha+\beta) \leq \max(\mu)$, where $\mu, \min(\mu), \max(\mu)$ are according to the definitions:

$$\frac{1}{2}\left[9(\alpha+\beta)^2\left\{\min(\mu)^2\right\}-1\right] \leq N(|\Gamma\rangle_{AB}) \leq \frac{1}{2}\left[9(\alpha+\beta)^2\left\{\max(\mu)^2\right\}-1\right]. \tag{4.7}$$

Now we have also observed the behaviour of bounds for another two measures, logarithmic negativity (LN) and Réyni entropy (S_δ). □

Theorem 4.3. *$LN(|\Gamma\rangle_{AB}) \geq (1/2)\{LN(|\psi\rangle_{AB}) + LN(|\phi\rangle_{AB})\} + 2 + \log\alpha\beta$.*

Proof. We have, $\alpha(\sum_{i=0}^{2}\sqrt{a_i}) + \beta(\sum_{i=0}^{2}\sqrt{b_i})/2 \geq \{\alpha(\sum_{i=0}^{2}\sqrt{a_i})\}^{1/2}\{\beta(\sum_{i=0}^{2}\sqrt{b_i})\}^{1/2}$, $\log_2\{\alpha(\sum_{i=0}^{2}\sqrt{a_i}) + \beta(\sum_{i=0}^{2}\sqrt{b_i})\} \geq 1 + (1/2)\{\log_2\alpha\beta + \log_2(\sum_{i=0}^{2}\sqrt{a_i}) + \log_2(\sum_{i=0}^{2}\sqrt{b_i})\}$, $2\log_2\{\alpha(\sum_{i=0}^{2}\sqrt{a_i}) + \beta(\sum_{i=0}^{2}\sqrt{b_i})\} \geq 2 + \log_2\alpha\beta + (1/2)(\mathrm{LN}(|\psi\rangle_{AB}) + \mathrm{LN}(|\phi\rangle_{AB})$.

Hence, we have $\mathrm{LN}(|\Gamma\rangle_{AB}) \geq (1/2)\{\mathrm{LN}(|\psi\rangle_{AB}) + \mathrm{LN}(|\phi\rangle_{AB})\} + 2 + \log\alpha\beta$. □

Theorem 4.4. *$2\log(3(\alpha+\beta)(\min(\xi))) \leq LN(|\Gamma\rangle_{AB}) \leq 2\log(3(\alpha+\beta)(\max(\xi)))$ where $\min(\xi) = \min\{\sqrt{a_i}, \sqrt{b_i}\}_{i=0}^{2}$ and $\max(\xi) = \max\{\sqrt{a_i}, \sqrt{b_i}\}_{i=0}^{2}$.*

Proof. According to the definition of $\min(\xi) = \min\{\sqrt{a_i}, \sqrt{b_i}\}_{i=0}^{2}$ and $\max(\xi) = \max\{\sqrt{a_i}, \sqrt{b_i}\}_{i=0}^{2}$, we have $\min(\xi) \leq \{\alpha(\sum_{i=0}^{2}\sqrt{a_i}) + \beta(\sum_{i=0}^{2}\sqrt{b_i})\}/3(\alpha+\beta) \leq \max(\xi)$.

Hence we have $2\log(3(\alpha+\beta)(\min(\xi))) \leq \mathrm{LN}(|\Gamma\rangle_{AB}) \leq 2\log(3(\alpha+\beta)(\max(\xi)))$. □

Corollary 4.5. *$LN(|\Gamma\rangle_{AB}) \geq 2\log(\alpha+\beta)$.*

Proof is same like the just above these theorems.

Theorem 4.6. *$S_\delta(|\Gamma\rangle_{AB}) \geq \ln\{3(\alpha\beta)^{2\delta}\}/(1-\delta) + S_\delta(|\psi\rangle_{AB}) + S_\delta(|\phi\rangle_{AB})$.*

Proof. By the definition of Réyni entropy (S_δ), we have $\sum_{i=0}^{2}(\alpha\sqrt{a_i}+\beta\sqrt{b_i})^{2\delta}/3 > (\sum_{i=0}^{2}(\alpha\sqrt{a_i}+\beta\sqrt{b_i}))^{2\delta}/3 \geq \alpha^{2\delta}(\sum_{i=0}^{2}(a_i)^{\delta})\beta^{2\delta}(\sum_{i=0}^{2}(b_i)^{\delta})$, $(1/(1-\delta))(\ln(\sum_{i=0}^{2}(\alpha\sqrt{a_i}+\beta\sqrt{b_i})^{2\delta}) - \ln 3) \geq (2\delta\ln\alpha\beta)/(1-\delta) + S_\delta(|\psi\rangle_{AB}) + S_\delta(|\phi\rangle_{AB})$, $S_\delta(|\Gamma\rangle_{AB}) \geq \ln\{3(\alpha\beta)^{2\delta}\}/(1-\delta) + S_\delta(|\psi\rangle_{AB}) + S_\delta(|\phi\rangle_{AB})$. □

Theorem 4.7. *$(2\delta/(1-\delta))\ln(\min(\eta)) \leq S_\delta(|\Gamma\rangle_{AB}) \leq (2\delta/(1-\delta))\ln(\max(\eta))$, where $\min(\eta) = \min\{\alpha\sqrt{a_i}+\beta\sqrt{b_i}\}_{i=0}^{2}$ and $\max(\eta) = \max\{\alpha\sqrt{a_i}+\beta\sqrt{b_i}\}_{i=0}^{2}$.*

Proof. We use the following inequality to prove the result,

$$\min(\eta) < \frac{\left\{\sum_{i=0}^{2}\left(\alpha\sqrt{a_i}+\beta\sqrt{b_i}\right)^{2\delta}\right\}}{3} < \max(\eta) \tag{4.8}$$

and using simple algebra we have the theorem.

In [31] Gour and Roy derive bounds on the entanglement of the superposed state as a function of the entanglement of the components and von-Neumann entropy (E) of the reduced state of either party is taken as the measure of entanglement. In their work, we find the following upper and lower bounds. □

Theorem 4.8. $E(|\Gamma\rangle_{AB}) \leq (\alpha\sqrt{E(|\psi\rangle_{AB})+1} + \beta\sqrt{E(|\phi\rangle_{AB})+1})^2$. *with* $E(|\psi\rangle_{AB}) = S(\mathrm{tr}_A(|\psi\rangle_{AB}\langle\psi|)) = S(\mathrm{tr}_B(|\psi\rangle_{AB}\langle\psi|))$.

In this context we have also found some upper bounds in two different forms; one is as a function of entanglement and other has a direct relation with the Schmidt coefficients of the states.

Theorem 4.9. $E(|\Gamma\rangle_{AB}) + \alpha\log_2\alpha + \beta\log_2\beta \leq \alpha E(|\psi\rangle_{AB}) + \beta E(|\phi\rangle_{AB})$.

Theorem 4.10. $E(|\Gamma\rangle_{AB}) \leq 2[\log_2 3(\alpha+\beta)]\max(\gamma)$ *where* $\max(\gamma) = \max\{\sqrt{a_i}, \sqrt{b_i}\}_{i=0}^{2}$.

We skip the proof of the above theorems. Now we concentrate ourselves on the motivations and physical interpretations of our results that is helpful for understanding the basic structure of the state space. In our paper, we have considered the pairs ($|\psi\rangle_{AB}$, $|\psi'\rangle_{AB}$) and ($|\phi\rangle_{AB}$, $|\phi'\rangle_{AB}$) almost all the possible combinations in the notion of comparability and incomparability under deterministic sense. The above theorems show that based on the upper and lower bounds of different entangle measures and these results we illustrate some counterintuitive examples which will be enough to establish the importance of the idea, comparability, and incomparability under deterministic LOCC that plays the crucial role in making the structure of the state space. Let the pairs ($|\psi\rangle_{AB}$, $|\psi'\rangle_{AB}$) and ($|\phi\rangle_{AB}$, $|\phi'\rangle_{AB}$) have same entanglement, that is, $E(\psi\rangle_{AB}) = E(\psi'\rangle_{AB})$ and $E(\phi\rangle_{AB}) = E(\phi'\rangle_{AB})$. This fact clearly indicates that both of the pairs ($|\psi\rangle_{AB}$, $|\psi'\rangle_{AB}$) and ($|\phi\rangle_{AB}$, $|\phi'\rangle_{AB}$) are incomparable to each other or in other word we can construct infinitely many incomparable pairs of ($|\psi\rangle_{AB}$, $|\psi'\rangle_{AB}$) and ($|\phi\rangle_{AB}$, $|\phi'\rangle_{AB}$). For specifically $\alpha = \alpha'$ and $\beta = \beta'$ *Theorem 4.8* establishes the fact that both $|\Gamma\rangle_{AB}$ and $|\Gamma'\rangle_{AB}$ have the same upper bounds, but in the same environment *Theorem 4.10* indicates some interesting features of the upper bounds. As the pairs ($|\psi\rangle_{AB}$, $|\psi'\rangle_{AB}$) and ($|\phi\rangle_{AB}$, $|\phi'\rangle_{AB}$) are incomparable to each other, so we have either $a_0 > \alpha_0$ and $b_0 > \beta_0$ or $a_0 < \alpha_0$ and $b_0 < \beta_0$ and for the first case we have upper bound ($|\Gamma\rangle_{AB}$) $\geq$ Upper Bound ($|\Gamma'\rangle_{AB}$) and for the latter case upper bound ($|\Gamma\rangle_{AB}$) $\leq$ Upper Bound ($|\Gamma'\rangle_{AB}$).

Same features would be found for considering the same entanglements of the pairs ($|\psi\rangle_{AB}$, $|\psi'\rangle_{AB}$) and ($|\phi\rangle_{AB}$, $|\phi'\rangle_{AB}$) and other combinations of choice of the pairs ($|\psi\rangle_{AB}$, $|\psi'\rangle_{AB}$) and ($|\phi\rangle_{AB}$, $|\phi'\rangle_{AB}$) with respect to the idea of comparability and incomparability under deterministic LOCC for any arbitrary choice of α, β, α', and β'.

Again we are employing the notion of incomparability in a different view point for constructing some new bounds of these different measures. Let the pairs ($|\psi\rangle_{AB}$, $|\psi'\rangle_{AB}$) and ($|\phi\rangle_{AB}$, $|\phi'\rangle_{AB}$) be incomparable to each other. So negativity of the both pairs have the following relations: $N(|\psi\rangle_{AB}) \gtreqless N(|\psi'\rangle_{AB})$ and $N(|\phi\rangle_{AB}) \gtreqless N(|\phi'\rangle_{AB})$. Now if we consider the following cases only $N(|\psi\rangle_{AB}) \geq N(|\psi'\rangle_{AB})$ and $N(|\phi\rangle_{AB}) \geq N(|\phi'\rangle_{AB})$ or $N(|\psi\rangle_{AB}) \leq N(|\psi'\rangle_{AB})$ and $N(|\phi\rangle_{AB}) \leq N(|\phi'\rangle_{AB})$, then using Theorem 4.1 we found some tight upper and lower bounds of $N(|\Gamma\rangle_{AB})$ and $N(|\Gamma'\rangle_{AB})$ for any arbitrary choice of α, β, α', and β' with assuming the other restrictions. In this environment with $\alpha = \alpha'$ and $\beta = \beta'$ we found the following relations:

$(1/2)[9(\alpha+\beta)^2\{\min(\alpha_2,\beta_2)^2\} - 1] \leq (1/2)[9(\alpha+\beta)^2\{\min(a_2,b_2)^2\} - 1] \leq \{N(|\Gamma'\rangle_{AB})$ or $N(|\Gamma\rangle_{AB})\} \leq \{N(|\Gamma\rangle_{AB})$ or $N(|\Gamma'\rangle_{AB})\} \leq (1/2)[9(\alpha+\beta)^2\{\max(\alpha_0,\beta_0)^2\} - 1] \leq (1/2)[9(\alpha+\beta)^2\{\min(a_0,b_0)^2\} - 1]$.

Some bounds like the previous can be observed for logarithmic negativity and Reyni entropy employing the above theorems and the comparability and incomparability relations with arbitrary choice of α, β, α', and β'.

5. Conclusion

In conclusion we have observed that superposition of states may lead to pairs of incomparable states to a pair of comparable states under deterministic LOCC. Therefore, through the superposition of states we have succeeded in making a connection between two classes of states, that is, comparable and incomparable. This technique would be useful in many aspects where we have some definite kind of states which are incomparable in nature; however, we could find a new pair that are comparable in nature. Since incomparability may be used as a detection of unphysical operations [32], therefore through the superposition we could form new classes of incomparable states and use them as detector of unphysical operations.

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