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Research Article

Existence of Almost-Periodic Solutions for Lotka-Volterra Cooperative Systems with Time Delay

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This paper considers the existence of positive almost-periodic solutions for almost-periodic Lotka-Volterra cooperative system with time delay. By using Mawhin's continuation theorem of coincidence degree theory, sufficient conditions for the existence of positive almost-periodic solutions are obtained. An example and its simulation figure are given to illustrate the effectiveness of our results.

1. Introduction

Lotka-Volterra system is one of the most celebrated models in mathematical biology and population dynamics. In recent years, it has also been found with successful and interesting applications in epidemiology, physics, chemistry, economics, biological science, and other areas (see [1–4]). Moreover, in [5], it was shown that the continuous-time recurrent neural networks can be embedded into Lotka-Volterra models by changing coordinates, which suggests that the existing techniques in the analysis of Lotak-Volterra systems can also be applied to recurrent neural networks.

Owing to its theoretical and practical significance, Lotka-Volterra system have been studied extensively (see [6–16] and the cites therein). Since biological and environmental parameters are naturally subject to fluctuation in time, the effects of a periodically varying environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.) are considered as important selective forces on systems in a fluctuating environment. Therefore, on the one hand, models should take into account both the periodically changing environment and the effects of time delays. However, on the other hand, in fact, it is more realistic and reasonable to study almost-periodic system than periodic system. Recently, there are two main approaches to obtain sufficient conditions for the existence and stability of the almost-periodic solutions of biological models: one is using the fixed point theorem, Lyapunov functional method, and differential inequality techniques (see [17–19]); the other is using functional hull theory and Lyapunov functional method (see [14–16]). However, to the best of our knowledge, there are very few published letters considering the almost-periodic solutions for nonautonomous Lotka-Volterra cooperative system with time delay by applying the method of coincidence degree theory. Motivated by this, in this letter, we apply the coincidence theory to study the existence of positive almost-periodic solutions for Lotka-Volterra cooperative system with time delay as follows:

$$\dot{u}_i(t) = u_i(t)\left(r_i(t) - b_i(t)u_i(t-\tau_i(t)) + \sum_{j=1, j\neq i}^{n} c_{ij}(t)u_j\left(t-\tau_{ij}(t)\right) \right), \quad i = 1, 2, \ldots, n, \tag{1.1}$$

where $u_i(t)$ stands for the ith species population density at time $t \in \mathbb{R}$, $r_i(t)$ is the natural reproduction rate, $b_i(t)$ represents the inner-specific competition, $c_{ij}(t)$ $(i \neq j)$ stands for the interspecific cooperation, $\tau_i(t) > 0$ and $\tau_{ij}(t) > 0$ are all continuous almost-periodic functions on $\mathbb{R}$. Throughout this paper, we always assume that $r_i(t)$, $b_i(t)$, and $c_{ij}(t)$ are all nonegative almost periodic functions with respect to $t \in \mathbb{R}$.

The initial condition of (1.1) is of the form

$$u_i(s) = \phi_i(s), \quad i = 1, 2, \ldots, n, \tag{1.2}$$

where $\phi_i(s)$ is positive bounded continuous function on $[-\tau, 0]$ and $\tau = \max_{1\leq i,j\leq n} \sup_{t\in R}\{|\tau_{ij}(t)|\}$.

The organization of the rest of this paper is as follows. In Section 2, we introduce some preliminary results which are needed in later sections. In Section 3, we establish our main results for the existence of almost-periodic solutions of (1.1). Finally, an example and its simulation figure are given to illustrate the effectiveness of our results in Section 4.

2. Preliminaries

To obtain the existence of an almost-periodic solution of system (1.1), we first make the following preparations.

Definition 2.1 (see [20]). Let $u(t) : \mathbb{R} \to \mathbb{R}$ be continuous in t. $u(t)$ is said to be almost-periodic on $\mathbb{R}$, if, for any $\epsilon > 0$, the set $K(u, \epsilon) = \{\delta : |u(t+\delta) - u(t)| < \epsilon,\ \text{for any } t \in \mathbb{R}\}$ is relatively dense, that is, for any $\epsilon > 0$, it is possible to find a real number $l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\delta = \delta(\epsilon)$ in this interval such that $|u(t+\delta) - u(t)| < \epsilon$, for any $t \in \mathbb{R}$.

Definition 2.2. A solution $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$ of (1.1) is called an almost periodic solution if and only if for each $i = 1, 2, \ldots, n$, $u_i(t)$ is almost periodic.

For convenience, we denote $AP(\mathbb{R})$ the set of all real-valued, almost-periodic functions on $\mathbb{R}$ and for each $j = 1, 2, \ldots, n$, let

$$\wedge(f_j) = \left\{ \tilde{\lambda} \in \mathbb{R} : \lim_{T\to\infty} \frac{1}{T} \int_0^T f_j(s) e^{-i\tilde{\lambda}s} \mathrm{d}s \neq 0 \right\},$$
$$\operatorname{mod}(f_j) = \left\{ \sum_{i=1}^{N} n_i \tilde{\lambda}_i : n_i \in Z,\ N \in N^+,\ \tilde{\lambda}_i \in \wedge(f_j) \right\} \tag{2.1}$$

be the set of Fourier exponents and the module of f_j, respectively, where $f_j(\cdot)$ is almost periodic. Suppose $f_j(t, \phi_j)$ is almost periodic in t, uniformly with respect to $\phi_j \in C([-\tau, 0], \mathbb{R})$. $K_j(f_j, \epsilon, \phi_j)$ denote the set of ϵ-almost periods for f_j uniformly with respect to $\phi_j \in C([-\tau, 0], \mathbb{R})$. $l_j(\epsilon)$ denote the length of inclusion interval. Let $m(f_j) = (1/T)\int_0^T f_j(s)\mathrm{d}s$ be the mean value of f_j on interval $[0, T]$, where $T > 0$ is a constant. clearly, $m(f_j)$ depends on T. $m[f_j] = \lim_{T\to\infty}(1/T)\int_0^T f_j(s)ds$.

Lemma 2.3 (see [20]). *Suppose that f and g are almost periodic. Then the following statements are equivalent.*

(i) $\operatorname{mod}(f) \supset \operatorname{mod}(g)$,

(ii) for any sequence $\{t_n^*\}$, if $\lim_{n\to\infty} f(t + t_n^*) = f(t)$ for each $t \in \mathbb{R}$, then there exists a subsequence $\{t_n\} \subseteq \{t_n^*\}$ such that $\lim_{n\to\infty} g(t + t_n) = f(t)$ for each $t \in \mathbb{R}$.

Lemma 2.4 (see [21]). *Let $u \in AP(\mathbb{R})$. Then $\int_{t-\tau}^t u(s)ds$ is almost periodic.*

Let X and Z be Banach spaces. A linear mapping $L : \operatorname{dom}(L) \subset X \to Z$ is called Fredholm if its kernel, denoted by $\ker(L) = \{X \in \operatorname{dom}(L) : Lx = 0\}$, has finite dimension and its range, denoted by $\operatorname{Im}(L) = \{Lx : x \in \operatorname{dom}(L)\}$, is closed and has finite codimension. The index of L is defined by the integer $\dim K(L) - \operatorname{codimdom}(L)$. If L is a Fredholm mapping with index 0, then there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im}(P) = \ker(L)$ and $\ker(Q) = \operatorname{Im}(L)$. Then $L|_{\operatorname{dom}(L)\cap\ker(P)} : \operatorname{Im}(L) \cap \ker(P) \to \operatorname{Im}(L)$ is bijective, and its inverse mapping is denoted by $K_P : \operatorname{Im}(L) \to \operatorname{dom}(L) \cap \ker(P)$. Since $\ker(L)$ is isomorphic to $\operatorname{Im}(Q)$, there exists a bijection $J : \ker(L) \to \operatorname{Im}(Q)$. Let Ω be a bounded open subset of X and let $N : X \to Z$ be a continuous mapping. If $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact, then N is called L-compact on Ω, where I is the identity.

Let L be a Fredholm linear mapping with index 0 and let N be a L-compact mapping on $\overline{\Omega}$. Define mapping $F : \operatorname{dom}(L) \cap \overline{\Omega} \to Z$ by $F = L - N$. If $Lx \neq Nx$ for all $x \in \operatorname{dom}(L) \cap \partial\Omega$, then by using P, Q, K_P, J defined above, the coincidence degree of F in Ω with respect to L is defined by

$$\deg_L(F, \Omega) = \deg\Big(I - P - \Big(J^{-1}Q + K_P(I - Q)\Big)N, \Omega, 0\Big), \tag{2.2}$$

where $\deg(g, \Gamma, p)$ is the Leray-Schauder degree of g at p relative to Γ.

Then The Mawhin's continuous theorem is given as follows.

Lemma 2.5 (see [22]). *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Z$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume*

(a) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \mathrm{dom}(L)$, $Lx \neq \lambda Nx$;*

(b) *for each $x \in \partial\Omega \cap L$, $QNx \neq 0$;*

(c) $\deg(JNQ, \Omega \cap \ker(L), 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \mathrm{dom}(L)$.

In this paper, since we need some related properties of M-matrix we introduce them as follows. In addition, A matrix $A = (a_{ij}) \geq 0$ means that each elements $a_{ij} \geq 0$.

Definition 2.6 (see [23]). If a real matrix $A = (a_{ij})_{n\times n}$ satisfies the following conditions (1) and (2):

(1) $a_{ii} > 0,\ i = 1,2,\ldots,n,\ a_{ij} \leq 0,\ i \neq j,\ i,j = 1,2,\ldots,n,$

(2) A is a positive-definite matrix,

then A is called a M-matrix.

Lemma 2.7 (see [23]). *If matrix $A = (a_{ij})_{n\times n}$ is a M-matrix, then A^{-1} exists and its every element is nonnegative.*

Lemma 2.8. *Suppose that matrix $A = (a_{ij})_{n\times n}$ is a M-matrix, then $AX \leq B$ implies $X \leq A^{-1}B$.*

Proof. In fact, there exists a nonnegative positive vector $\varepsilon_0 = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^T \in R^n$ such that $AX - B + \varepsilon_0 = (0,0,\ldots,0)^T$ which imply that $X - A^{-1}B + A^{-1}\varepsilon_0 = (0,0,\ldots,0)^T$. According to Lemma 2.4, there exists at least one positive element in the every row of A^{-1}, which imply $A^{-1}\varepsilon_0 \geq (0,0,\ldots,0)^T$. Thus, we obtain $X \leq A^{-1}B$. □

3. Main Result

In this section, we state and prove our main results of our this paper. By making the substitution

$$u_i(t) = e^{y_i(t)}, \quad i = 1,2,\ldots,n, \tag{3.1}$$

Equation (1.1) can be reformulated as

$$\dot{y}_i(t) = r_i(t) - b_i(t)e^{y_i(t-\tau_{ii}(t))} + \sum_{j=1, i\neq j}^{n} c_{ij}(t)e^{y_j(t-\tau_{ij}(t))}, \quad i = 1,2,\ldots,n. \tag{3.2}$$

The initial condition (1.2) can be rewritten as follows:

$$y_i(s) = \ln\phi_i(s) =: \varphi_i(s), \quad i = 1,2,\ldots,n. \tag{3.3}$$

Set $X = Z = V_1 \oplus V_2$, where

$$\begin{aligned} V_1 = \Big\{ & y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T \in C(\mathbb{R}, \mathbb{R}^n) : y_i(t) \in AP(\mathbb{R}), \\ & \operatorname{mod}(y_i(t)) \subset \operatorname{mod}(H_i(t)),\ \forall \tilde{\lambda}_i \in \wedge(y_i(t)) \text{ satisfies } \left|\tilde{\lambda}_i\right| > \beta,\ i = 1, 2, \dots, n \Big\}, \\ V_2 = \Big\{ & y(t) \equiv (h_1, h_2, \dots, h_n)^T \in \mathbb{R}^n \Big\}, \\ H_i(t) = {} & r_i(t) - b_i(t) e^{\varphi_i(-\tau_{ii}(0))} + \sum_{j=1, i\neq j}^{n} c_{ij}(t) e^{\varphi_j(-\tau_{ij}(0))} \end{aligned} \tag{3.4}$$

and $\varphi(\cdot)$ is defined as (3.3), $i = 1, 2, \dots, n$, $\beta > 0$ is a given constant. For $y = (y_1, y_2, \dots, y_n)^T \in Z$, define $\|y\| = \max_{1\le i\le n} \sup_{t\in\mathbb{R}} |y_i(t)|$.

Lemma 3.1. *Z is a Banach space equipped with the norm $\|\cdot\|$.*

Proof. If $y^{\{k\}} \subset V_1$ and $y^{\{k\}} = (y_1^{\{k\}}, y_2^{\{k\}}, \dots, y_n^{\{k\}})^T$ converges to $\overline{y} = (\overline{y}_1, \overline{y}_2, \dots, \overline{y}_n)^T$, that is, $y_j^{\{k\}} \to \overline{y}_j$, as $k \to \infty$, $j = 1, 2, \dots, n$. Then it is easy to show that $\overline{y}_j \in AP(\mathbb{R})$ and $\operatorname{mod}(\overline{y}_j) \in \operatorname{mod}(H_j)$. For any $|\tilde{\lambda}_j| \le \beta$, we have that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T y_j^{\{k\}}(t) e^{-i\tilde{\lambda}_j t} dt = 0, \quad j = 1, 2, \dots, n; \tag{3.5}$$

therefore,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \overline{y}_j(t) e^{-i\tilde{\lambda}_j t} dt = 0, \quad j = 1, 2, \dots, n, \tag{3.6}$$

which implies $\overline{y} \in V_1$. Then it is not difficult to see that V_1 is a Banach space equipped with the norm $\|\cdot\|$. Thus, we can easily verify that X and Z are Banach spaces equipped with the norm $\|\cdot\|$. The proof of Lemma 3.1 is complete. □

Lemma 3.2. *Let $L : X \to Z$, $Ly = \dot{y}$, then L is a Fredholm mapping of index 0.*

Proof. Clearly, L is a linear operator and $\ker(L) = V_2$. We claim that $\operatorname{Im}(L) = V_1$. Firstly, we suppose that $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T \in \operatorname{Im}(L) \subset Z$. Then there exist $z^{\{1\}}(t) = (z_1^{\{1\}}(t), z_2^{\{1\}}(t), \dots, z_n^{\{1\}}(t))^T \in V_1$ and constant vector $z^{\{2\}} = (z_1^{\{2\}}, z_2^{\{2\}}, \dots, z_n^{\{2\}})^T \in V_2$ such that

$$z(t) = z^{\{1\}}(t) + z^{\{2\}}, \tag{3.7}$$

that is,

$$z_i(t) = z_i^{\{1\}}(t) + z_i^{\{2\}}, \quad i = 1, 2, \dots, n. \tag{3.8}$$

From the definition of $z_i(t)$ and $z_i^{\{1\}}(t)$, we can easily see that $\int_{t-\tau}^{t} z_i(s)ds$ and $\int_{t-\tau}^{t} z_i^{\{1\}}(s)ds$ are almost-periodic functions. So we have $z_i^{\{2\}} \equiv 0$, $i = 1,2,\ldots,n$, then $z^{\{2\}} \equiv (0,0,\ldots,0)^T$, which implies $z(t) \in V_1$, that is $\mathrm{Im}(L) \subset V_1$.

On the other hand, if $u(t) = (u_1(t),u_2(t),\ldots,u_n(t))^T \in V_1 \setminus \{0\}$, then we have $\int_0^t u_j(s)ds \in AP(\mathbb{R})$, $j = 1,2,\ldots,n$. If $\widetilde{\lambda}_j \neq 0$, then we obtain

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\int_0^t u_j(s)ds\right)e^{-i\widetilde{\lambda}_j t}dt = \frac{1}{i\widetilde{\lambda}_j}\lim_{T\to\infty}\frac{1}{T}\int_0^T u_j(t)e^{-i\widetilde{\lambda}_j t}dt, \quad j = 1,2,\ldots,n. \tag{3.9}$$

It follows that

$$\wedge\left[\int_0^t u_j(s)ds - m\left(\int_0^t u_j(s)ds\right)\right] = \wedge\left(u_j(t)\right), \quad j = 1,2,\ldots,n, \tag{3.10}$$

hence

$$\int_0^t u(s)ds - m\left(\int_0^t u(s)ds\right) \in V_1 \subset X. \tag{3.11}$$

Note that $\int_0^t u(s)ds - m(\int_0^t u(s)ds)$ is the primitive of $u(t)$ in X, we have $u(t) \in \mathrm{Im}(L)$, that is, $V_1 \subset \mathrm{Im}(L)$. Therefore, $\mathrm{Im}(L) = V_1$.

Furthermore, one can easily show that $\mathrm{Im}(L)$ is closed in Z and

$$\dim\ker(L) = n = \mathrm{co}\dim\mathrm{Im}(L); \tag{3.12}$$

therefore, L is a Fredholm mapping of index 0. The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Let $N : X \to Z$, $Ny = (G_1^y, G_2^y, \ldots, G_n^y)^T$, where*

$$G_i^y = r_i(t) - b_i(t)^{y_i(t-\tau_i(t))} + \sum_{j=1, i\neq j}^{n} c_{ij}(t)e^{y_j(t-\tau_{ij}(t))}, \quad i = 1,2,\ldots,n. \tag{3.13}$$

Set

$$\begin{aligned} &P : X \longrightarrow Z, \quad Py = \left(m(y_1), m(y_2), \ldots, m(y_n)\right)^T, \\ &Q : Z \longrightarrow Z, \quad Qz = \left(m[z_1], m[z_2], \ldots, m[z_n]\right)^T. \end{aligned} \tag{3.14}$$

Then N is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X.

Proof. Obviously, P and Q are continuous projectors such that

$$\mathrm{Im}\, P = \ker(L), \qquad \mathrm{Im}(L) = \ker(Q). \tag{3.15}$$

It is clear that $(I-Q)V_2 = \{(0,0,\ldots,0)\}$, $(I-Q)V_1 = V_1$. Hence

$$\operatorname{Im}(I-Q) = V_1 = \operatorname{Im}(L). \tag{3.16}$$

Then in view of

$$\operatorname{Im}(P) = \ker(L), \qquad \operatorname{Im}(L) = \ker(Q) = \operatorname{Im}(I-Q), \tag{3.17}$$

we obtain that the inverse $K_P : \operatorname{Im}(L) \to \ker(P) \cap \operatorname{dom}(L)$ of L_P exists and is given by

$$K_P(z) = \int_0^t z(s)ds - m\left[\int_0^t z(s)ds\right]. \tag{3.18}$$

Thus,

$$\begin{gathered} QNy = \left(m\left[G_1^y\right], m\left[G_2^y\right], \ldots, m\left[G_n^y\right]\right)^T \\ K_P(I-Q)Ny = \left(f(y_1) - Q(f(y_1)), f(y_2) - Q(f(y_2)), \ldots, f(y_n) - Q(f(y_n))\right)^T, \end{gathered} \tag{3.19}$$

where

$$f(y_i) = \int_0^t \left(G_i^y - m\left[G_i^y\right]\right)ds, \quad i = 1,2,\ldots,n. \tag{3.20}$$

Clearly, QN and $(I-Q)N$ are continuous. Now we will show that K_P is also continuous. By assumptions, for any $0 < \epsilon < 1$ and any compact set $\phi_i \subset C([-\tau,0],\mathbb{R})$, let $l_i(\epsilon_i)$ be the length of the inclusion interval of $K_i(H_i,\epsilon_i,\phi_i)$, $i = 1,2,\ldots,n$. Suppose that $\{z^k(t)\} \subset \operatorname{Im}(L) = V_1$ and $z^k(t) = (z_1^k(t), z_2^k(t), \ldots, z_n^k(t))^T$ uniformly converges to $\overline{z}(t) = (\overline{z}_1(t), \overline{z}_2(t), \ldots, \overline{z}_n(t))^T$, that is $z_i^k \to \overline{z}_i$, as $k \to \infty$, $i = 1,2,\ldots,n$. Because of $\int_0^t z^k(s)ds \in Z$, $k = 1,2,\ldots,n$, there exists σ_i $(0 < \sigma_i < \epsilon_i)$ such that $K_i\ (H_i,\sigma_i,\phi_i) \subset K_i\ (\int_0^t z_i^k(s)ds, \sigma_i, \phi_i)$, $i = 1,2,\ldots,n$. Let $l_i(\sigma_i)$ be the length of the inclusion interval of $K_i(H_i,\sigma_i,\phi_i)$ and

$$l_i = \max\{l_i(\epsilon_i), l_i(\sigma_i)\}, \quad i = 1,2,\ldots,n. \tag{3.21}$$

It is easy to see that l_i is the length of the inclusion interval of $K_i(H_i,\sigma_i,\phi_i)$ and $K_i(H_i,\epsilon_i,\phi_i)$, $i = 1,2,\ldots,n$. Hence, for any $t \notin [0,l_i]$, there exists $\xi_t \in K_i(H_i,\sigma_i,\phi_i) \subset K_i(\int_0^t z_i^k(s)ds, \sigma_i, \phi_i)$

such that $t+\xi_t \in [0,l_i]$, $i=1,2,\ldots,n$. Hence, by the definition of almost periodic function we have

$$\begin{aligned}
\left\|\int_0^t z^k(s)ds\right\| &= \max_{1\le i\le n}\sup_{t\in R}\left|\int_0^t z_i^k(s)ds\right| \\
&\le \max_{1\le i\le n}\sup_{t\in[0,l_i]}\left|\int_0^t z_i^k(s)ds\right| + \max_{1\le i\le n}\sup_{t\notin[0,l_i]}\left|\int_0^t z_i^k(s)ds - \int_0^{t+\xi_t} z_i^k(s)ds + \int_0^{t+\xi_t} z_i^k(s)ds\right| \\
&\le 2\max_{1\le i\le n}\sup_{t\in[0,l_i]}\left|\int_0^t z_i^k(s)ds\right| + \max_{1\le i\le n}\sup_{t\notin[0,l_i]}\left|\int_0^t z_i^k(s)ds - \int_0^{t+\xi_t} z_i^k(s)ds\right| \\
&\le 2\max_{1\le i\le n}\left|\int_0^{l_i} z_i^k(s)ds\right| + \max_{1\le i\le n}\epsilon_i.
\end{aligned} \tag{3.22}$$

From this inequality, we can conclude that $\int_0^t z(s)ds$ is continuous, where $z(t)=(z_1(t),z_2(t),\ldots,z_n(t))^T \in \operatorname{Im}(L)$. Consequently, K_P and $K_P(I-Q)Ny$ are continuous.

From (3.22), we also have $\int_0^t z(s)ds$ and $K_P(I-Q)Ny$ also are uniformly bounded in $\overline{\Omega}$. Further, it is not difficult to verify that $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)Ny$ is equicontinuous in $\overline{\Omega}$. By the Arzela-Ascoli theorem, we have immediately concluded that $K_P(I-Q)N(\overline{\Omega})$ is compact. Thus N is L-compact on $\overline{\Omega}$. The proof of Lemma 3.3 is complete. □

Theorem 3.4. *Assume that the following conditions* (H_1) *and* (H_2) *hold:*

(H_1) $e_i := m[r_i(t)] > 0$, $m[b_i(t)] > 0$, $m[c_{ij}(t))] > 0$, $i,j=1,2,\ldots,n$;

(H_2) *D is a positive-definite matrix, where* $d_{ii} = m[b_i(t)]$, $d_{ij} = m[c_{ij}(t)]$, $i\neq j$, i, $j = 1,2,\ldots,n$,

$$D = \begin{pmatrix} d_{11} & -d_{12} & \cdots & -d_{1n} \\ -d_{21} & d_{22} & \cdots & -d_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -d_{n1} & -d_{n2} & \cdots & d_{nn} \end{pmatrix}. \tag{3.23}$$

Then (1.1) *has at least one positive almost periodic solution.*

Proof. To use the continuation theorem of coincidence degree theorem to establish the existence of a solution of (3.2), we set Banach space X and Z the same as those in Lemma 3.1 and set mappings L, N, P, Q the same as those in Lemmas 3.2 and 3.3, respectively. Then we can obtain that L is a Fredholm mapping of index 0 and N is a continuous operator which is L-compact on $\overline{\Omega}$.

Now, we are in the position of searching for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation

$$Ly = \lambda N y, \quad \lambda \in (0,1), \tag{3.24}$$

we obtain

$$\dot{y}_i(t) = \lambda\left[r_i(t) - b_i(t)e^{y_i(t-\tau_i(t))} + \sum_{j=1, i\neq j}^{n} c_{ij}(t)e^{y_j(t-\tau_{ij}(t))}\right], \quad i = 1, 2, \ldots, n. \tag{3.25}$$

Assume that $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \in X$ is a solution of (3.25) for some $\lambda \in (0,1)$. Denote $\overline{M}_i = \sup_{t\in R}\{y_i(t)\}$, $\underline{M}_i = \inf_{t\in R}\{y_i(t)\}$.

On the one hand, by (3.25), we derive

$$-|\dot{y}_i(t)| \leq \lambda\left[r_i(t) - b_i(t)e^{y_i(t-\tau_i(t))} + \sum_{j=1, i\neq j}^{n} c_{ij}(t)e^{y_j(t-\tau_{ij}(t))}\right], \quad i = 1, 2, \ldots, n. \tag{3.26}$$

On the both sides of (3.26), integrating from 0 to T and applying the mean value theorem of integral calculus, we have

$$\begin{aligned} 0 \leq \lambda &\left[m(r_i(t)) - m(b_i(t))e^{y_i(\xi_i-\tau_i(\xi_i))} + \sum_{j=1, i\neq j}^{n} m(c_{ij}(t))e^{y_j(\eta_{ij}-\tau_{ij}(\eta_{ij}))}\right] \\ &+ m(|\dot{y}_i(t)|), \quad i = 1, 2, \ldots, n, \end{aligned} \tag{3.27}$$

where $\xi_i \in [0,T]$, $\eta_{ij} \in [0,T]$, $i, j = 1, 2, \ldots, n$. In the light of (3.27), we get for $i = 1, 2, \ldots, n$,

$$\lambda\left[m(b_i(t))e^{y_i(\xi_i-\tau_i(\xi_i))}\right] \leq \lambda\left[m(r_i(t)) + \sum_{j=1, i\neq j}^{n} m(c_{ij}(t))e^{y_j(\eta_{ij}-\tau_{ij}(\eta_{ij}))}\right] + m(|\dot{y}_i(t)|). \tag{3.28}$$

On the both sides of (3.28), taking the supremum with respect to ξ_i, η_{ij} and letting $T \to +\infty$, we obtain

$$m[b_i(t)]e^{\overline{M}_i} \leq m[r_i(t)] + \sum_{j=1, j\neq i}^{n} + m[c_{ij}(t)]e^{\overline{M}_j}, \tag{3.29}$$

that is,

$$d_{ii}e^{\overline{M}_i} - \sum_{j=1, j\neq i}^{n} d_{ij}e^{\overline{M}_j} \leq e_i, \quad i = 1, 2, \ldots, n. \tag{3.30}$$

Equation (3.30) can be written by the following matrix form

$$\begin{pmatrix} d_{11} & -d_{12} & \cdots & -d_{1n} \\ -d_{21} & d_{22} & \cdots & -d_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -d_{n1} & -d_{n2} & \cdots & d_{nn} \end{pmatrix} \begin{pmatrix} e^{\overline{M}_1} \\ e^{\overline{M}_2} \\ \vdots \\ e^{\overline{M}_n} \end{pmatrix} \leq \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}. \tag{3.31}$$

By Lemma 2.8 and assumption, we obtain

$$\begin{pmatrix} e^{\overline{M}_1} \\ e^{\overline{M}_2} \\ \vdots \\ e^{\overline{M}_n} \end{pmatrix} \leq D^{-1} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} =: \begin{pmatrix} H_1^+ \\ H_2^+ \\ \vdots \\ H_n^+ \end{pmatrix}, \tag{3.32}$$

which imply that

$$\overline{M}_i \leq \ln H_i^+, \qquad i = 1, 2, \ldots, n. \tag{3.33}$$

On the two sides of (3.28), taking the infimum with respect to ξ_i, η_{ij}, and letting $T \to +\infty$, we obtain

$$\underline{M}_i \leq \ln H_i^+, \qquad i = 1, 2, \ldots, n. \tag{3.34}$$

On the other hand, according to (3.25), we derive

$$\lambda r_i(t) - \dot{y}_i(t) < \lambda b_i(t) e^{y_i(t-\tau_i(t))}, \qquad i = 1, 2, \ldots, n. \tag{3.35}$$

On the both sides of (3.35), integrating from 0 to T and using the mean value theorem of integral calculus, we get

$$\lambda m(r_i(t)) < m(\dot{y}_i(t)) + \lambda m(b_i(t)) e^{y_i(\zeta_i - \tau_i(\zeta_i))}, \qquad i = 1, 2, \ldots, n, \tag{3.36}$$

where $\zeta_i \in [0, T]$, $i = 1, 2, \ldots, n$. On the both sides of (3.36), take the supremum and infimum with respect to ζ_i, respectively, and let $T \to +\infty$, then we have for $i = 1, 2, \ldots, n$,

$$m[r_i(t)] < m[b_{ii}(t)] e^{\overline{M}_i}, \qquad m[r_i(t)] < m[b_{ii}(t)] e^{\underline{M}_i}, \tag{3.37}$$

namely,

$$e^{\overline{M}_i} > \frac{m[r_i(t)]}{m[b_i(t)]} = \frac{e_i}{d_{ii}}, \qquad e^{\underline{M}_i} > \frac{m[r_i(t)]}{m[b_i(t)]} = \frac{e_i}{d_{ii}} \tag{3.38}$$

which imply that

$$\overline{M}_i > \ln \frac{e_i}{d_{ii}}, \qquad \underline{M}_i > \ln \frac{e_i}{d_{ii}}. \tag{3.39}$$

Combining with (3.33), (3.34), and (3.39), we derive for all $t \in \mathbb{R}$, $i = 1, 2, \ldots, n$,

$$\min_{1 \le i \le n} \left\{ \ln \frac{e_i}{d_{ii}} \right\} \le \ln \frac{e_i}{d_{ii}} < \underline{M}_i \le y_i(t) \le \overline{M}_i \le \ln H_i^+ < \max_{1 \le i \le n} \{ \ln H_i^+ \} + 1. \tag{3.40}$$

Denote $M = \max\{|\min_{1\le i\le n}\{\ln(e_i/d_{ii})\}|,\ |\max_{1\le i\le n}\{\ln H_i^+\} + 1|\}$. Clearly, M is independent of λ. Take

$$\Omega = \left\{ y = (y_1, y_2, \ldots, y_n)^T \in X : \|y\| < M \right\}. \tag{3.41}$$

It is clear that Ω satisfies the requirement (a) in Lemma 2.5. When $y \in \partial\Omega \cap \ker(L)$, $y = (y_1, y_2, \ldots, y_n)^T$ is a constant vector in $\mathbb{R}^n$ with $\|y\| = M$. Then

$$QNy = (m[G_1], m[G_2], \ldots, m[G_n])^T, \quad y \in X, \tag{3.42}$$

where

$$\begin{gathered} G_i = r_i(t) - b_i(t)e^{y_i} + \sum_{j=1, j\ne i}^{n} c_{ij}(t) e^{y_j}, \quad i = 1, 2, \ldots, n, \\ m[G_i] = m[r_i(t)] - m[b_{ii}(t)]e^{y_i} + \sum_{j=1, j\ne i}^{n} m\left[c_{ij}(t)\right] e^{y_j} = e_i - d_{ii}e^{y_i} \\ + \sum_{j=1, j\ne i}^{n} d_{ij} e^{y_j}, \qquad i = 1, 2, \ldots, n. \end{gathered} \tag{3.43}$$

If $QNy = (0, 0, \ldots, 0])^T$, then we have

$$\begin{pmatrix} d_{11} & -d_{12} & \cdots & -d_{1n} \\ -d_{21} & d_{22} & \cdots & -d_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -d_{n1} & -d_{n2} & \cdots & d_{nn} \end{pmatrix} \begin{pmatrix} e^{y_1} \\ e^{y_2} \\ \vdots \\ e^{y_n} \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}, \tag{3.44}$$

which imply that $y_i = \ln H_i^+$, $i = 1, 2, \ldots, n$. Thus, $y = (y_1, y_2, \ldots, y_n)^T \in \Omega$, this contradicts the fact that $y \in \partial\Omega \cap \ker(L)$. Therefore, $QNy \ne (0, 0, \ldots, 0)^T$, which implies that the requirement (b) in Lemma 2.5 is satisfied. If necessary, we can let M be greater such that $y^T QNy < 0$, for any $y \in \partial\Omega \cap \ker(L)$. Furthermore, take the isomorphism $J : \mathrm{Im}(Q) \to \ker(L)$, $Jz \equiv z$ and let $\Phi(\gamma; y) = -\gamma y + (1-\gamma)JQNy$, then for any $y \in \partial\Omega \cap \ker(L)$, $y^T\Phi(\gamma; y) < 0$, we have

$$\deg\{JQN, \Omega \cap \ker(L), 0\} = \deg\{-y, \Omega \cap \ker(L), 0\} \neq 0. \tag{3.45}$$

So, the requirement (c) in Lemma 2.5 is satisfied. Hence, (3.2) has at least one almost-periodic solution in $\overline{\Omega}$, that is, (1.1) has at least one positive almost periodic solution. The proof is complete. □

4. An Example and Simulation

Consider the following two species cooperative system with time delay:

$$\begin{aligned}
\dot{x}(t) &= x(t)\left(r_1(t) - b_1(t)x(t-\tau_1(t)) + c_{12}(t)y(t-\tau_{12}(t))\right),\\
\dot{y}(t) &= y(t)\left(r_2(t) - b_2(t)y(t-\tau_2(t)) + c_{21}(t)x(t-\tau_{21}(t))\right),
\end{aligned} \tag{4.1}$$

where $r_1(t) = 2+\sin\sqrt{2}t+\sin\sqrt{3}t$, $b_1(t) = 2+\sin\sqrt{3}t+\sin\sqrt{5}t$, $c_{12}(t) = (2+\cos\sqrt{2}t+\cos\sqrt{3}t)/2$, $\tau_1(t) = e^{\sin\sqrt{2}t+\sin\sqrt{5}t}$, $\tau_{12}(t) = e^{\sin t+\cos\sqrt{2}t}$, $r_2(t) = 2-\sin\sqrt{2}t-\sin\sqrt{3}t$, $b_2(t) = 2+\sin\sqrt{3}t-\sin\sqrt{5}t$, $c_{21}(t) = (2-\cos\sqrt{2}t+\cos\sqrt{3}t)/2$, $\tau_2(t) = e^{\sin\sqrt{2}t+\cos\sqrt{5}t}$, $\tau_{21}(t) = e^{\sin t-\cos\sqrt{2}t}$. Since

$$\begin{gathered}
e_1 = m[r_1(t)] = 2, \qquad d_{11} = m[b_1(t)] = 2, \qquad d_{12} = m[c_{12}(t)] = 1,\\
e_2 = m[r_1(t)] = 2, \qquad d_{22} = m[b_1(t)] = 2, \qquad d_{21} = m[c_{12}(t)] = 1,\\
D = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad \det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3 > 0, \qquad D^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},
\end{gathered} \tag{4.2}$$

then, the matrix D is positive definite, and

$$\begin{gathered}
\ln\frac{e_1}{d_{11}} = \ln\frac{e_2}{d_{22}} = 0, \quad \begin{pmatrix} H_1^+ \\ H_2^+ \end{pmatrix} = D^{-1}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} \ln H_1^+ \\ \ln H_2^+ \end{pmatrix} = \begin{pmatrix} \ln 2 \\ \ln 2 \end{pmatrix},\\
M = \max\left\{\left|\min_{1\le i\le n}\left\{\ln\frac{e_i}{d_{ii}}\right\}\right|, \left|\max_{1\le i\le n}\{\ln H_i^+\} + 1\right|\right\} = 1 + \ln 2,\\
\Omega = \left\{y = (y_{1,}y_2, \ldots, y_n)^T \in X : \|y\| < 1 + \mathrm{In}\,2\right\}.
\end{gathered} \tag{4.3}$$

Therefore, all conditions of Theorem 3.4 are satisfied. By Theorem 3.4, system (4.1) has one positive almost-periodic solution. The resulting numerical simulation is depicted in Figure 1.

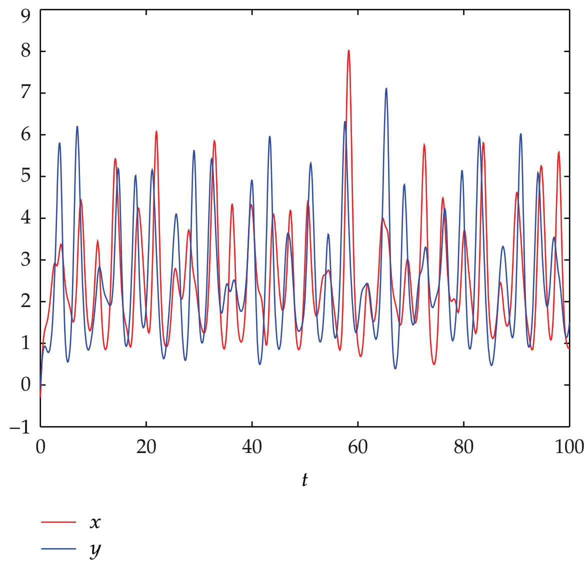

Figure 1

Acknowledgments

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