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*Research Article*

# Existence of Solutions of Nonlinear Mixed Two-Point Boundary Value Problems for Third-Order Nonlinear Differential Equation

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The authors use the upper and lower solution method to study the existence of solutions of nonlinear mixed two-point boundary value problems for third-order nonlinear differential equation $y''' = f(x, y, y', y'')$, $y'(b) = h(y'(a))$, $p(y(a), y(b), y'(a), y'(b)) = 0$, $g(y(a), y(b), y'(a), y'(b), y''(a), y''(b)) = 0$. Some new existence results are obtained by developing the upper and lower solution method. Some applications are also presented.

## 1. Introduction

It is well known that the upper and lower solution method is a powerful tool for proving existence results for boundary value problems. The upper and lower solution method has been used to deal with the multipoint boundary value problems for second-order ordinary differential equations [1–4] and for higher-order ordinary differential equations [5–11]. There are fewer results on nonlinear mixed two-point boundary value problems for higher-order equations in the literature of ordinary differential equations. For this reason, we consider the third-order nonlinear ordinary differential equation:

$$y''' = f(x, y, y', y'') \tag{1.1}$$

together with the nonlinear mixed two-point boundary conditions

$$\begin{gathered} y'(b) = h(y'(a)), \\ p(y(a), y(b), y'(a), y'(b)) = 0, \\ g(y(a), y(b), y'(a), y'(b), y''(a), y''(b)) = 0, \end{gathered} \tag{1.2}$$

where the functions $f$, $p$, and $g$ are continuous and monotonic, $h$ is a homeomorphic mapping.

We will develop the upper and lower solution method for the boundary value problem

$$\begin{gathered} y''' = f(x, y, y', y''), \\ y'(b) = h(y'(a)), \\ p(y(a), y(b), y'(a), y'(b)) = 0, \\ g(y(a), y(b), y'(a), y'(b), y''(a), y''(b)) = 0 \end{gathered} \tag{1.3}$$

and establish some new existence results. Furthermore, some applications are also presented.

## 2. Preliminaries

In this section, we will give some preliminary considerations and some lemmas which are essential to our main results.

*Definition 2.1.* Suppose the functions $\alpha(x)$ and $\beta(x) \in C^{(3)}[a,b]$ satisfy

$$\begin{gathered} \alpha^{(3)}(x) \geq f(x, \alpha(x), \alpha'(x), \alpha''(x)), \\ \beta^{(3)}(x) \leq f(x, \beta(x), \beta'(x), \beta''(x)), \\ \alpha'(x) \leq \beta'(x), \quad x \in [a,b], \\ \alpha(a) < \beta(a), \qquad \alpha'(a) < \beta'(a). \end{gathered} \tag{2.1}$$

Then $\alpha(x)$ and $\beta(x)$ are, respectively, called the lower and upper solutions of the BVP (1.3).

Because of Definition 2.1, it is clear that $\alpha(x) \leq \beta(x)$, $x \in [a,b]$. Let $D = [a,b] \times [\alpha(x), \beta(x)] \times [\alpha'(x), \beta'(x)]$.

*Definition 2.2.* Let $C[D \times \mathbb{R}, \mathbb{R}]$ denote the class of continuous functions from $D \times \mathbb{R}$ into $\mathbb{R}$, and let $f(x, y, y', y'') \in C[D \times \mathbb{R}, \mathbb{R}]$ and $\alpha(x), \beta(x) \in C^{(3)}[a,b]$ be lower and upper solutions of BVP (1.3). Suppose that there is a function $W(s) \in C[\mathbb{R}_+, (0, +\infty)]$ such that

$$\left| f(x, y, y', y'') \right| \leq W(|y''|), \tag{2.2}$$

for every $(x, y, y', y'') \in D \times \mathbb{R}$, where

$$\int_{\lambda}^{\infty} \frac{s}{W(s)} ds > \max_{x \in [a,b]} \beta'(x) - \min_{x \in [a,b]} \alpha'(x), \tag{2.3}$$

with

$$\lambda(b-a) = \max\{|\alpha'(a) - \beta'(b)|, |\alpha'(b) - \beta'(a)|\}. \tag{2.4}$$

Then we say that $f$ satisfies Nagumo's condition on the set $D$ relative to $\alpha(x),\ \beta(x)$.

We assume throughout this paper the following.

($H_1$) There are lower and upper solutions $\alpha(x)$ and $\beta(x)$ of BVP (1.3) as Definition 2.1.

($H_2$) Function $f(x,y,y',y'')$ satisfies Nagumo's condition on the set $D$ relative to $\alpha(x),\ \beta(x)$.

($H_3$) Function $f(x,y,y',y'') \in C[[a,b]\times\mathbb{R}^3,\mathbb{R}]$ is nonincreasing in $y$.

($H_4$) $h : [\alpha'(a), \beta'(a)] \to [\alpha'(b), \beta'(b)]$ is a homeomorphism with

$$h(\alpha'(a)) = \alpha'(b), \qquad h(\beta'(a)) = \beta'(b). \tag{2.5}$$

($H_5$) Function $p(s,t,u,v)$ is continuous on $\mathbb{R}^4$ and nondecreasing in $t,\ u,\ v$ and satisfies

$$\begin{gathered} p(\beta(a),\beta(b),\beta'(a),\beta'(b)) \le 0, \\ p(\alpha(a),\alpha(b),\alpha'(a),\alpha'(b)) \ge 0. \end{gathered} \tag{2.6}$$

($H_6$) Function $g(x,y,z,p,q,r)$ is continuous on $\mathbb{R}^6$ and nondecreasing in $x,\ y,\ q$ and nonincreasing in $r$, and it satisfies

$$\begin{gathered} g(\beta(a),\beta(b),\beta'(a),\beta'(b),\beta''(a),\beta''(b)) \le 0, \\ g(\alpha(a),\alpha(b),\alpha'(a),\alpha'(b),\alpha''(a),\alpha''(b)) \ge 0. \end{gathered} \tag{2.7}$$

It is not difficult to obtain the following lemma.

**Lemma 2.3.** *The boundary value problem*

$$\begin{gathered} y''' = f(x,y,y',y''), \\ y(a) = 0, \qquad y'(a) = 0, \qquad y'(b) = 0 \end{gathered} \tag{2.8}$$

*has a Green function*

$$G(x,s) = \begin{cases} \dfrac{(x-s)^2}{2} - \dfrac{(b-s)(a-x)^2}{2(b-a)}, & a \le s \le x \le b, \\ \dfrac{-(b-s)(a-x)^2}{2(b-a)}, & a \le x \le s \le b \end{cases} \tag{2.9}$$

*with*

$$\int_a^b |G(x,s)|dt \le \frac{(b-a)^3}{12},$$
$$\int_a^b |G_x(x,s)|dt \le \frac{(b-a)^2}{4}, \tag{2.10}$$
$$\int_a^b |G_{xx}(x,s)|dt \le b-a.$$

It is easy to prove the following lemma similarly to [12, page 25, Theorem 1.4.1].

**Lemma 2.4.** *Assume that* $(H_1)$, $(H_2)$ *hold. Then for any solution* $y$ *of* $y''' = f(x,y,y',y'')$ *with* $\alpha(x) \le y(x) \le \beta(x)$, $\alpha'(x) \le y'(x) \le \beta'(x)$ *on* $[a,b]$, *there exists a constant* $N > 0$ *depending only on* $\alpha$, $\beta$, $W$, *such that*

$$\left|y''(x)\right| \le N, \quad x \in [a,b] \tag{2.11}$$

*and one calls* $N$ *is Nagumo's constant.*

**Lemma 2.5.** *Assume that* $(H_1)$–$(H_3)$ *hold. Then for any constant* $A \in [\alpha(a),\beta(a)]$, $B \in [\alpha'(a),\beta'(a)]$, $C \in [\alpha'(b),\beta'(b)]$, *the boundary value problem*

$$\begin{gathered} y''' = f(x,y,y',y''), \\ y(a) = A, \\ y'(a) = B, \\ y'(b) = C \end{gathered} \tag{2.12}$$

*at least has a solution* $y \in C^{(3)}[a,b]$, *with*

$$\begin{gathered} \alpha(x) \le y(x) \le \beta(x), \\ \alpha'(x) \le y'(x) \le \beta'(x) \end{gathered} \tag{2.13}$$

*on* $[a,b]$.

*Proof.* By Lemma 2.3, it is clear that BVP (2.12) is equivalent to integral equation

$$y(x) = \int_a^b G(x,s) f(s,y(s),y'(s),y''(s))ds + W(x), \tag{2.14}$$

where $W(x)$ is a polynomial satisfying $y''' = 0$, $y(a) = A$, $y'(a) = B$, $y'(b) = C$.

Denote

$$m = \max\left\{ \max_{x\in[a,b]} \left|\alpha''(x)\right|, \max_{x\in[a,b]} \left|\beta''(x)\right|, N+1 \right\}, \tag{2.15}$$

where $N$ is Nagumo's constant.

Assume

$$f_m(x,y,y',y'') = \begin{cases} f(x,y,y',m), & y'' > m, \\ f(x,y,y',y''), & -m \le y'' \le m, \\ f(x,y,y',-m), & y'' < -m, \end{cases}$$

$$G(x,y,y',y'') = \begin{cases} f_m(x,\beta(x),y',y''), & y > \beta(x), \\ f_m(x,y,y',y''), & \alpha(x) \le y \le \beta(x), \\ f_m(x,\alpha(x),y',y''), & y < \alpha(x), \end{cases} \tag{2.16}$$

$$F(x,y,y',y'') = \begin{cases} G(x,y,\beta'(x),y'') + \dfrac{y'-\beta'(x)}{1+(y')^2}, & y' > \beta'(x), \\ G(x,y,y',y''), & \alpha'(x) \le y' \le \beta'(x), \\ G(x,y,\alpha'(x),y'') - \dfrac{\alpha'(x)-y'}{1+(y')^2}, & y' < \alpha'(x). \end{cases}$$

Then $F(x,y,y',y'')$ is bounded and continuous on $[a,b]\times\mathbb{R}^3$. Suppose $|F(x,y,y',y'')| \le M$; $|W^{(i)}(x)| \le K$ $(i=0,1,2)$, $x\in[a,b]$.

Now, define an operator $T$ on the set $E = C^{(2)}[[a,b],\mathbb{R}]$ by

$$Ty(x) = \int_a^b G(x,s)F(s,y(s),y'(s),y''(s))ds + W(x). \tag{2.17}$$

If $y \in E$, the norm is defined by

$$\|y\| = \max_{x\in[a,b]} \left[\left|y(x)\right| + \left|y'(x)\right| + \left|y''(x)\right|\right]. \tag{2.18}$$

It is clear that

$$\begin{aligned} \left|(Ty)(x)\right| &\le \frac{(b-a)^3}{12}M + K, \\ \left|(Ty)'(x)\right| &\le \frac{(b-a)^2}{4}M + K, \\ \left|(Ty)''(x)\right| &\le (b-a)M + K. \end{aligned} \tag{2.19}$$

This shows that $T$ maps the closed, bounded, and convex set

$$B = \left\{ y \mid y \in E,\ \|y\| \le \frac{(b-a)^3 M}{12} + \frac{(b-a)^2 M}{4} + (b-a)M + 3K \right\} \tag{2.20}$$

into itself. Also, $T$ is continuous and $(Ty)'$ is bounded. All of these considerations imply that $T$ is completely continuous by Ascoli's theorem. Schauder' fixed point theorem then yields the fixed point $y$ of $T$ on $B$. In other words, the following boundary value problem

$$\begin{aligned} y''' &= F(x, y, y', y''), \\ y(a) &= A, \\ y'(a) &= B, \\ y'(b) &= C \end{aligned} \tag{2.21}$$

has a solution $y \in C^{(3)}[a,b]$, and satisfying $\alpha'(x) \le y'(x) \le \beta'(x)$ and $\alpha(a) \le y(a) \le \beta(a)$, we have $\alpha(x) \le y(x) \le \beta(x)$, $x \in [a,b]$.

In the following we prove that

$$\alpha'(x) \le y'(x) \le \beta'(x), \quad x \in [a,b]. \tag{2.22}$$

In fact, if it is invalid, there is no harm in setting the right inequality to be not true (the case that the left inequality is not true can be proved in the same way). By the assumption, if $y'(x) > \beta'(x)$, for some $x \in [a,b]$, then there is a $x_0 \in (a,b)$ such that

$$y'(x_0) - \beta'(x_0) = \max_{x \in (a,b)} \left[ y'(x) - \beta'(x) \right] > 0, \tag{2.23}$$

$$y''(x_0) = \beta''(x_0), \qquad y'''(x_0) \le \beta'''(x_0). \tag{2.24}$$

Now, let

$$\omega(x_0) = \begin{cases} \beta(x_0), & y(x_0) > \beta(x_0), \\ y(x_0), & \alpha(x_0) \le y(x_0) \le \beta(x_0), \\ \alpha(x_0), & y(x_0) < \alpha(x_0). \end{cases} \tag{2.25}$$

They imply that $\alpha(x_0) \le \omega(x_0) \le \beta(x_0)$ and

$$G\left(x_0, y(x_0), \beta'(x_0), \beta''(x_0)\right) = f_m\left(x_0, \omega(x_0), \beta'(x_0), \beta''(x_0)\right). \tag{2.26}$$

We have

$$\begin{aligned}
&y'''(x_0) - \beta'''(x_0) \\
&\quad \geq F(x_0, y(x_0), y'(x_0), y''(x_0)) - f(x_0, \beta(x_0), \beta'(x_0), \beta''(x_0)) \\
&\quad = F(x_0, y(x_0), y'(x_0), \beta''(x_0)) - f(x_0, \beta(x_0), \beta'(x_0), \beta''(x_0)) \\
&\quad = G(x_0, y(x_0), \beta'(x_0), \beta''(x_0)) - f(x_0, \beta(x_0), \beta'(x_0), \beta''(x_0)) + \frac{y'(x_0) - \beta'(x_0)}{1 + (y'(x_0))^2} \\
&\quad > G(x_0, y(x_0), \beta'(x_0), \beta''(x_0)) - f(x_0, \beta(x_0), \beta'(x_0), \beta''(x_0)) \\
&\quad = f_m(x_0, \omega(x_0), \beta'(x_0), \beta''(x_0)) - f(x_0, \beta(x_0), \beta'(x_0), \beta''(x_0)) \\
&\quad = f(x_0, \omega(x_0), \beta'(x_0), \beta''(x_0)) - f(x_0, \beta(x_0), \beta'(x_0), \beta''(x_0)) \\
&\quad \geq 0
\end{aligned} \tag{2.27}$$

which contradicts (2.24); hence, (2.22) is true.

Further, by the definition of $F$, $y$ is a solution of the boundary value problem

$$\begin{gathered}
y''' = f_m(x, y, y', y''), \\
y(a) = A, \\
y'(a) = B, \\
y'(b) = C.
\end{gathered} \tag{2.28}$$

Because there is a $\xi \in [a, b]$, such that

$$|y''(\xi)| = \frac{|y'(b) - y'(a)|}{b - a} \leq \lambda < m, \tag{2.29}$$

so $[a, b]$ has a maximal subinterval $[c, d]$ with interior point $\xi$, for any $x \in [c, d]$, $|y''(x)| \leq m$. Hence, for $x \in [c, d]$, $y(x)$ is the solution of BVP (2.12). And by Lemma 2.4, we have $|y''(x)| \leq N < m$; this contracts that $[c, d]$ is the maximal subinterval, so we know $[c, d] = [a, b]$. Consequently, $y$ is a solution of BVP (2.12). □

## 3. Main Results

**Theorem 3.1.** *Assume* $(H_1)$–$(H_4)$, $(H_6)$ *hold, then BVP*

$$\begin{gathered}
y''' = f(x, y, y', y''), \\
y(a) = A, \\
y'(b) = h(y'(a)), \\
g(y(a), y(b), y'(a), y'(b), y''(a), y''(b)) = 0
\end{gathered} \tag{3.1}$$

*has a solution* $y \in C^{(3)}[a,b]$, *satisfying*

$$\begin{aligned} \alpha(x) \le y(x) \le \beta(x), \\ \alpha'(x) \le y'(x) \le \beta'(x) \end{aligned} \tag{3.2}$$

*on* $[a,b]$.

*Proof.* By Lemma 2.5, we know that the boundary value problem

$$\begin{gathered} y''' = f(x, y, y', y''), \\ y(a) = A, \\ y'(a) = B, \\ y'(b) = h(B) \end{gathered} \tag{3.3}$$

has a solution $y \in C^{(3)}[a,b]$, with

$$\begin{aligned} \alpha(x) \le y(x) \le \beta(x), \\ \alpha'(x) \le y'(x) \le \beta'(x) \end{aligned} \tag{3.4}$$

on $[a,b]$. For any $A \in [\alpha(a), \beta(a)]$, $B \in [\alpha'(a), \beta'(a)]$.

For fixed $A$, if $B = \alpha'(a)$, then $y''(a) \ge \alpha''(a)$, $y''(b) \le \alpha''(b)$. By $(H_6)$, we know

$$g(y(a), y(b), y'(a), y'(b), y''(a), y''(b)) \ge g(\alpha(a), \alpha(b), \alpha'(a), \alpha'(b), \alpha''(a), \alpha''(b)) \ge 0. \tag{3.5}$$

On the other hand, if $B = \beta'(a)$, then $y''(a) \le \beta''(a)$, $y''(b) \ge \beta''(b)$. By $(H_6)$, we have

$$g(y(a), y(b), y'(a), y'(b), y''(a), y''(b)) \le g(\beta(a), \beta(b), \beta'(a), \beta'(b), \beta''(a), \beta''(b)) \le 0. \tag{3.6}$$

Define the following sets:

$$\begin{aligned} \Pi(y) &= \{ y \mid y''' = f(x, y, y', y''), \\ &\qquad y(a) = A,\ y'(a) = B,\ y'(b) = h(B);\ A \in [\alpha(a), \beta(a)],\ B \in [\alpha'(a), \beta'(a)] \}, \\ M_1 &= \{ B \mid B \in [\alpha'(a), \beta'(a)],\ y(x) \in \Pi(y), \\ &\qquad g(y(a), y(b), y'(a), y'(b), y''(a), y''(b)) > 0 \}, \\ M_2 &= \{ B \mid B \in [\alpha'(a), \beta'(a)],\ y(x) \in \Pi(y), \\ &\qquad g(y(a), y(b), y'(a), y'(b), y''(a), y''(b)) < 0 \}. \end{aligned} \tag{3.7}$$

Obviously, $\Pi(y)$ is nonempty. If the theorem is not true, we know that $M_1$ and $M_1$ are all nonempty, and $M_1 \bigcup M_2 = [\alpha'(a), \beta'(a)]$. we claim that $M_1$ is closed. To see this, let $B_n \in M_1$, with $B_n \to B_0$ $(n \to \infty)$. Consider the following boundary value problem:

$$\begin{gathered} y''' = f(x, y, y', y''), \\ y(a) = A, \\ y'(a) = B_n, \\ y'(b) = h(B_n). \end{gathered} \tag{3.8}$$

By Lemma 2.5, it is known that, for every $n \in N$, BVP (3.8) has a solution $y_n(x) \in C^{(3)}[a, b]$, satisfying

$$\alpha(x) \le y_n(x) \le \beta(x), \quad \alpha'(x) \le y_n'(x) \le \beta'(x), \quad x \in [a, b] \tag{3.9}$$

and, by Lemma 2.4, we know $|y_n''(x)| \le N$.

Clearly, sequences $\{y_n(x)\}$, $\{y_n'(x)\}$, $\{y_n''(x)\}$ are uniformly bounded and equicontinuous on $[a, b]$. Consequently, there exists a subsequence of $y_n(x)$ which converges uniformly on $[a, b]$, to a solution $y_0(x)$ of the BVP:

$$\begin{gathered} y''' = f(x, y, y', y'', y'''), \\ y(a) = A, \\ y'(a) = B_0, \\ y'(b) = h(B_0) \end{gathered} \tag{3.10}$$

with

$$g\big(y_0(a), y_0(\mathrm{b}), y_0'(a), y_0'(b), y_0''(a), y_0''(b)\big) \ge 0. \tag{3.11}$$

By assumption, equality cannot occur, so that

$$g\big(y_0(a), y_0(b), y_0'(a), y_0'(b), y_0''(a), y_0''(b)\big) > 0, \tag{3.12}$$

and thus $B_0 \in M_1$. Consequently, $M_1$ is closed. Likewise, we may show $M_2$ is closed. This is a contradiction and proves the theorem. □

Similar to the proof of Theorem 3.1, we can obtain the following theorem.

**Theorem 3.2.** *Assume* $(H_1)$–$(H_6)$ *hold, then BVP*

$$
\begin{gathered}
y''' = f(x, y, y', y''), \\
y'(b) = h(y'(a)), \\
p(y(a), y(b), y'(a), y'(b)) = 0, \\
g(y(a), y(b), y'(a), y'(b), y''(a), y''(b)) = 0
\end{gathered}
\tag{3.13}
$$

*has a solution* $y \in C^{(3)}[a,b]$, *satisfying*

$$
\begin{gathered}
\alpha(x) \le y(x) \le \beta(x), \\
\alpha'(x) \le y'(x) \le \beta'(x)
\end{gathered}
\tag{3.14}
$$

*on* $[a,b]$.

## 4. Applications

We all know it is difficult to find a solution of some nonlinear ordinary differential equation. But according to Theorem 3.2, we can know whether a boundary value problem, especially a nonlinear boundary value problem, has a solution and we also can know the existence regions of the solution and its derivative.

*Example 4.1.* Consider the following linear boundary value problem

$$
\begin{gathered}
y'''(x) = -y(x) + 2y''(x), \quad x \in [0,1], \\
y'(1) - 2ey'(0) = 0, \\
(-3e - 2)y(0) + y(1) + y'(0) + y'(1) = 0, \\
y(0) + y(1) - 4y'(0) - 5y'(1) + y''(0) - y''(1) = 0.
\end{gathered}
\tag{4.1}
$$

It is easy to know that $\alpha(x) = -xe^x - 1$, $\beta(x) = xe^x + 1$ are lower and upper solutions of the linear boundary value problem, respectively, where

$$
\begin{gathered}
h(s) = 2es, \qquad p(s,t,u,v) = (-3e - 2)s + t + u + v, \\
g(x, y, z, p, q, r) = x + y - 4z - 5p + q - r,
\end{gathered}
\tag{4.2}
$$

and all assumptions of Theorem 3.2 hold. So the linear boundary value problem has a solution $y(x)$ satisfying

$$\begin{aligned} -xe^x - 1 &\leq y(x) \leq xe^x + 1, \\ -e^x - xe^x &\leq y'(x) \leq e^x + xe^x. \end{aligned} \tag{4.3}$$

Obviously, the trivial solution of the linear boundary value problem is one.

*Example 4.2.* Consider nonlinear boundary value problem

$$\begin{gathered} y'''(x) = -y(x) + \left[y''(x)\right]^2 - y''(x), \quad x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right], \\ y'\left(\frac{\pi}{2}\right) - \sin\left(\sqrt{2}\pi y'\left(\frac{\pi}{4}\right)\right) = 0, \\ y^2\left(\frac{\pi}{4}\right) + \frac{1}{4}y\left(\frac{\pi}{2}\right) + \frac{1}{8}y'\left(\frac{\pi}{4}\right) + y'\left(\frac{\pi}{2}\right) = 0, \\ y\left(\frac{\pi}{4}\right) + \frac{1}{8}y\left(\frac{\pi}{2}\right) + \left(y'\left(\frac{\pi}{4}\right)\right)^2 + \left(y'\left(\frac{\pi}{2}\right)\right)^2 + y''\left(\frac{\pi}{4}\right) - \frac{1}{8}y''\left(\frac{\pi}{2}\right) = 0. \end{gathered} \tag{4.4}$$

It is easy to verify that $\alpha(x) = -\sin(x)$, $\beta(x) = 0$ are lower and upper solutions of the nonlinear boundary value problem, respectively, where

$$\begin{gathered} h(s) = \sin\left(\sqrt{2}\pi s\right), \qquad p(s,t,u,v) = s^2 + \frac{1}{4}t + \frac{1}{8}u + v, \\ g(x,y,z,p,q,r) = x + \frac{1}{8}y + z^2 + p^2 + q - \frac{1}{8}r, \end{gathered} \tag{4.5}$$

and all assumptions of Theorem 3.2 hold, so the BVP has a solution $y(x)$ satisfying

$$-\sin(x) \leq y(x) \leq 0, \qquad -\cos(x) \leq y'(x) \leq 0. \tag{4.6}$$

## 5. Conclusion

In this paper, we study a nonlinear mixed two-point boundary value problem for a third-order nonlinear ordinary differential equation. Some new existence results are obtained by developing the upper and lower solution method. Furthermore, some applications are also presented.


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