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Research Article

A Boundary Integral Equation with the Generalized Neumann Kernel for a Certain Class of Mixed Boundary Value Problem

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We present a uniquely solvable boundary integral equation with the generalized Neumann kernel for solving two-dimensional Laplace's equation on multiply connected regions with mixed boundary condition. Two numerical examples are presented to verify the accuracy of the proposed method.

1. Introduction

The interplay of Riemann-Hilbert boundary value problem and integral equations with the generalized Neumann kernel on bounded multiply connected regions has been investigated in [1]. By reformulating the Dirichlet problem, the Neumann problem, and the conformal mapping as Riemann-Hilbert problems, boundary integral equations with the generalized Neumann kernel have been implemented successfully in [2] to solve the Dirichlet problem and the Neumann problem and in [3–5] to compute the conformal mapping of multiply connected regions onto the classical canonical slit domains. The mixed boundary value problem also can be reformulated as a Riemann-Hilbert problem (see [6–8]). Based on this reformulation, we present in this paper a new boundary integral equation method for two-dimensional Laplace's equation with mixed boundary condition in bounded multiply connected regions. However, in this paper, we will consider only a certain class of mixed boundary condition where the boundary condition in each boundary component is either the Dirichlet condition or the Neumann condition but not both. This class of mixed boundary value problem has been considered in [9] for Laplace's equation and in [10, page 78] for biharmonic equation on multiply connected regions. A similar class of mixed boundary value problem has been considered in [11, page 317] for doubly connected regions.

The presented method is an extension of the method presented in [2] for Laplace's equation with the Dirichlet boundary conditions and Laplace's equation with the Neumann boundary conditions on multiply connected regions. The method is based on a uniquely solvable boundary integral equation with the generalized Neumann kernel. The generalized Neumann kernel which will be considered in this paper is slightly different from the generalized Neumann kernel for integral equation associated with the Dirichlet problem and the Neumann problem which has been considered in [2]. Thus, not all of the properties of the generalized Neumann kernel which have been studied [2] are valid for the generalized Neumann kernel which will be used in this paper. For example, it is still true that $\lambda = 1$ is not an eigenvalue of the generalized Neumann kernel which means that the presented integral equation is uniquely solvable. However, it is no longer true that all eigenvalues of the generalized Neumann kernel are real (see Figure 3 and [2, Theorem 8]).

2. Notations and Auxiliary Material

In this section we will review the definition and some properties of the generalized Neumann kernel. For further details we refer the reader to [1–3, 5].

Let G be a bounded multiply connected region of connectivity $m + 1 \geq 1$ with $0 \in G$. The boundary $\Gamma := \partial G = \bigcup_{j=0}^{m} \Gamma_j$ consists of $m+1$ smooth closed Jordan curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ where Γ_0 contains the other curves $\Gamma_1, \ldots, \Gamma_m$ (see Figure 1). The complement $G^- := \overline{\mathbb{C}} \setminus \overline{G}$ of G with respect to $\overline{\mathbb{C}}$ consists of $m+1$ simply connected components $G_0, G_1, \ldots, G_m$. The components $G_1, \ldots, G_m$ are bounded and the component G_0 is unbounded where $\infty \in G_0$. We assume that the orientation of the boundary Γ is such that G is always on the left of Γ. Thus, the curves $\Gamma_1, \ldots, \Gamma_m$ always have clockwise orientations and the curve Γ_0 has a counterclockwise orientation. The curve Γ_j is parametrized by a 2π-periodic twice continuously differentiable complex function $\eta_j(t)$ with nonvanishing first derivative

$$\dot{\eta}_j(t) = \frac{d\eta_j(t)}{dt} \neq 0, \quad t \in J_j := [0, 2\pi],\ j = 0, 1, \ldots, m. \tag{2.1}$$

The total parameter domain J is the disjoint union of the intervals J_j. We define a parametrization of the whole boundary Γ as the complex function η defined on J by

$$\eta(t) := \begin{cases} \eta_0(t), & t \in J_0, \\ \vdots \\ \eta_m(t), & t \in J_m. \end{cases} \tag{2.2}$$

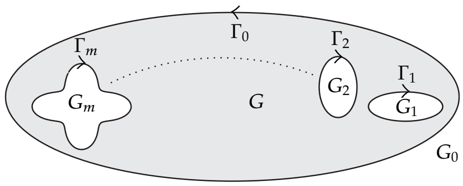


Figure 1: A bounded multiply connected region G of connectivity $m+1$.

Let H be the space of all real Hölder continuous functions on the boundary Γ. In view of the smoothness of η, a function $\phi \in H$ can be interpreted via $\widehat{\phi}(t) := \phi(\eta(t))$, $t \in J$, as a real Hölder continuous 2π-periodic functions $\widehat{\phi}(t)$ of the parameter $t \in J$, that is,

$$\widehat{\phi}(t) := \begin{cases} \widehat{\phi}_0(t), & t \in J_0, \\ \vdots & \\ \widehat{\phi}_m(t), & t \in J_m, \end{cases} \tag{2.3}$$

with real Hölder continuous 2π-periodic functions $\widehat{\phi}_j$ defined on J_j, and vice versa. So, here and in what follows, we will not distinguish between $\phi(\eta(t))$ and $\phi(t)$.

The subspace of H which consists of all piecewise constant functions defined on Γ will be denoted by S, that is, a function $h \in S$ has the representation

$$h(t) := \begin{cases} h_0, & t \in J_0, \\ \vdots & \\ h_m, & t \in J_m, \end{cases} \tag{2.4}$$

where $h_0, \ldots, h_m$ are real constants. For simplicity, the function h will be denoted by

$$h(t) = (h_0, \ldots, h_m). \tag{2.5}$$

In this paper we will assume that the function A is the continuously differentiable complex function

$$A(t) := e^{-\mathrm{i}\theta(t)}\ \eta(t), \tag{2.6}$$

where θ is the real piecewise constant function

$$\theta(t) = (\theta_0, \ldots, \theta_m), \tag{2.7}$$

with either $\theta_j = 0$ or $\theta_j = \pi/2$, $j = 0, 1, \ldots, m$. This function, $A(t)$, is a special case of the function $A(t)$ in [5, Equation (4)] in connection with numerical conformal mapping of multiply connected regions.

The generalized Neumann kernel formed with A is defined by

$$N(s,t) := \frac{1}{\pi}\operatorname{Im}\left(\frac{A(s)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right). \tag{2.8}$$

We define also a real kernel M by

$$M(s,t) := \frac{1}{\pi}\operatorname{Re}\left(\frac{A(s)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right). \tag{2.9}$$

The kernel N is continuous and the kernel M has a cotangent singularity type (see [1] for more details). Hence, the operator

$$\mathbf{N}\mu(s) := \int_J N(s,t)\mu(t)dt, \quad s \in J, \tag{2.10}$$

is a Fredholm integral operator and the operator,

$$\mathbf{M}\mu(s) := \int_J M(s,t)\mu(t)dt, \quad s \in J, \tag{2.11}$$

is a singular integral operator.

The solvability of boundary integral equations with the generalized Neumann kernel is determined by the *index* (winding number in other terminology) of the function A (see [1]). For the function A given by (2.6), the index κ_j of A on the curve Γ_j and the index $\kappa = \sum_{j=0}^{m}\kappa_j$ of A on the whole boundary Γ are given by

$$\kappa_0 = 1, \quad \kappa_j = 0, \quad j = 1,\ldots,m, \quad \kappa = 1. \tag{2.12}$$

Although the function $A(t)$ in (2.6) is different from the function $A(t)$ in [2, Equation (11)], both functions have the same indices. So by using the same approach used in [2], we can prove that the properties of the generalized Neumann kernel proved in [2], except Theorem 8, Theorem 10, and Corollary 2, are still valid for the generalized Neumann kernel formed with the function $A(t)$ in (2.6) (see [5]). Numerical evidence show that [2, Theorem 8], which claims that the eigenvalues of the generalized Neumann kernel lies in $[-1, 1)$, is no longer true for the function $A(t)$ in (2.6) (see Figure 3).

Theorem 2.1 (see [5]). *For a given function $\gamma \in H$, there exist unique functions $h \in S$ and $\mu \in H$ such that*

$$Af = \gamma + h + \mathrm{i}\,\mu \tag{2.13}$$

is a boundary value of a unique analytic function $f(z)$ in G. The function μ is the unique solution of the integral equation

$$(\mathbf{I}-\mathbf{N})\mu = -\mathbf{M}\gamma \tag{2.14}$$

and the function h is given by

$$h = \frac{[\mathbf{M}\mu - (\mathbf{I} - \mathbf{N})\gamma]}{2}. \tag{2.15}$$

3. The Mixed Boundary Value Problem

Let $S_{\mathbf{d}}$ and $S_{\mathbf{n}}$ be two subsets of the set $\{0, 1, \ldots, m\}$ such that

$$S_{\mathbf{d}} \neq \emptyset, \quad S_{\mathbf{n}} \neq \emptyset, \quad S_{\mathbf{d}} \cup S_{\mathbf{n}} = \{0, 1, \ldots, m\}, \quad S_{\mathbf{d}} \cap S_{\mathbf{n}} = \emptyset. \tag{3.1}$$

We will consider the following class of mixed boundary value problems. Find a real function u in G such that

$$\Delta u = 0, \quad \text{on } G, \tag{3.2a}$$

$$u = \phi_j, \quad \text{on } \Gamma_j \quad \text{for } j \in S_{\mathbf{d}}, \tag{3.2b}$$

$$\frac{\partial u}{\partial \mathbf{n}} = \phi_j, \quad \text{on } \Gamma_j \quad \text{for } j \in S_{\mathbf{n}}, \tag{3.2c}$$

where n is the unit exterior normal to Γ.

The problem (3.2a), (3.2b), and (3.2c) reduces to the Dirichlet problem for $S_{\mathbf{n}} = \emptyset$ and to the Neumann problem for $S_{\mathbf{d}} = \emptyset$. Both problems have been considered in [2]. So we have assumed in this paper that $S_{\mathbf{n}} \neq \emptyset$ and $S_{\mathbf{d}} \neq \emptyset$. The mixed boundary value problem (3.2a), (3.2b), and (3.2c) is uniquely solvable. Its unique solution u can be regarded as a real part of an analytic function F in G which is not necessairly single-valued. However, the function F can be written as

$$F(z) = f(z) - \sum_{j=1}^{m} a_j \log(z - z_j), \tag{3.3}$$

where f is a single-valued analytic function in G, each z_j is a fixed point in G_j, $j = 1, 2, \ldots, m$; and $a_1, \ldots, a_m$ are real constants uniquely determined by ϕ (see [12, page 145] and [13, page 174]). Without los of generality, we assume that $\operatorname{Im} f(0) = 0$. The constants $a_1, \ldots, a_m$ are chosen to ensure that (see [14, page 222] and [15, page 88])

$$\int_{\Gamma_k} f'(\eta) d\eta = 0, \quad k = 1, 2, \ldots, m, \tag{3.4}$$

that is, a_j are given by (see [2])

$$a_j = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_j} F'(\eta) d\eta, \quad j = 1, 2, \ldots, m. \tag{3.5}$$

We define a real constant a_0 by

$$a_0 := -\sum_{j=1}^{m} a_j. \tag{3.6}$$

Hence, we have

$$a_0 = -\sum_{j=1}^{m} a_j = -\sum_{j=1}^{m} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_j} F'(\eta)\, d\eta = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma} F'(\eta)\, d\eta + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_0} F'(\eta)\, d\eta. \tag{3.7}$$

Since $(1/2\pi\mathrm{i}) \int_\Gamma F'(\eta) d\eta = 0$, we obtain

$$a_0 = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_0} F'(\eta)\, d\eta. \tag{3.8}$$

The Dirichlet boundary condition is a special case of the Riemann-Hilbert boundary condition. The Neumann boundary condition also can be reduced to a Riemann-Hilbert boundary condition by using the Cauchy-Riemann equations and integrating along the boundary Γ_j for $j \in S_{\mathbf{n}}$. Let $\mathbf{T}(\zeta)$ be the unit tangent vector and $\mathbf{n}(\zeta)$ be the unit external normal vector to Γ at $\zeta \in \Gamma$. Let also $\nu(\zeta)$ be the angle between the normal vector $\mathbf{n}(\zeta)$ and the positive real axis, that is, $\mathbf{n}(\zeta) = e^{\mathrm{i}\nu(\zeta)}$. Then,

$$e^{\mathrm{i}\nu(\eta(t))} = -\mathrm{i}\mathbf{T}(\eta(t)) = -\mathrm{i}\frac{\dot{\eta}(t)}{|\dot{\eta}(t)|}. \tag{3.9}$$

Thus

$$\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n} = \cos\nu \frac{\partial u}{\partial x} + \sin\nu \frac{\partial u}{\partial y} = \operatorname{Re}\left[e^{\mathrm{i}\nu}\left(\frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y}\right)\right]. \tag{3.10}$$

Since $u(z) = \operatorname{Re} F(z)$, then by the Cauchy-Riemann equation, we have

$$F'(z) = \frac{\partial u(z)}{\partial x} - \mathrm{i}\frac{\partial u(z)}{\partial y}. \tag{3.11}$$

Thus (3.10) becomes

$$\operatorname{Re}\left[-\mathrm{i}\dot{\eta}F'\right] = |\dot{\eta}| \frac{\partial u}{\partial \mathbf{n}}. \tag{3.12}$$

Let the boundary value of the multivalued analytic function F be given by

$$F = \psi + \mathrm{i}\varphi. \tag{3.13}$$

Then the function $F'(z)$ is a single-valued analytic function and has the boundary value

$$\dot{\eta}F' = \psi' + \mathrm{i}\varphi'. \tag{3.14}$$

Let $\theta(t)$ be the real piecewise constant function

$$\theta(t) = \begin{cases} 0, & t \in J_j,\ j \in S_{\mathbf{d}}, \\ \dfrac{\pi}{2}, & t \in J_j,\ j \in S_{\mathbf{n}}. \end{cases} \tag{3.15}$$

Hence, the boundary values of the function $F(z)$ satisfy on the boundary Γ

$$\mathrm{Re}\left[e^{-\mathrm{i}\theta(t)} F(\eta(t))\right] = \widehat{\phi}(t), \tag{3.16}$$

where

$$\widehat{\phi}(t) = \begin{cases} \phi_j(t), & t \in J_j,\ j \in S_{\mathbf{d}}, \\ \varphi_j(t), & t \in J_j,\ j \in S_{\mathbf{n}}, \end{cases} \tag{3.17}$$

and for $t \in J_j$ and $j \in S_{\mathbf{n}}$, the function $\varphi_j'(t)$ is known and is given by

$$\varphi_j'(t) = \mathrm{Re}\left[-\mathrm{i}\dot{\eta}_j(t) F'(\eta_j(t))\right] = \phi_j(t)\left|\dot{\eta}_j(t)\right|, \quad t \in J_j,\ j \in S_{\mathbf{n}}. \tag{3.18}$$

The functions $\phi_j(t)$ for $j \in S_{\mathbf{d}} \cup S_{\mathbf{n}}$ are given by (3.2b) and (3.2c). The function $\varphi_j(t)$ can be then computed for $t \in J_j$ and $j \in S_{\mathbf{n}}$ by integrating the function $\varphi_j'(t)$. Then, it follows from (3.3), (3.16), and (3.17) that the the function $f(z)$ is a solution of the Riemann-Hilbert problem

$$\mathrm{Re}\left[e^{-\mathrm{i}\theta(t)} f(\eta(t))\right] = \widehat{\phi}(t) + \sum_{k=1}^{m} a_k \mathrm{Re}\left[e^{-\mathrm{i}\theta(t)} \log(\eta(t) - z_k)\right]. \tag{3.19}$$

Since

$$\log(\eta(t) - z_k) = \log \eta(t) + \log\left(1 - \frac{z_k}{\eta(t)}\right), \tag{3.20}$$

then, in view of (3.6), the boundary condition (3.19) can be written as

$$\mathrm{Re}\left[e^{-\mathrm{i}\theta(t)} f(\eta(t))\right] = \widehat{\phi}(t) + \sum_{k=0}^{m} a_k \gamma^{[k]}(t), \tag{3.21}$$

where

$$\gamma^{[0]}(t) = -\mathrm{Re}\left[e^{-\mathrm{i}\theta(t)} \log \eta(t)\right], \qquad \gamma^{[k]}(t) = \mathrm{Re}\left[e^{-\mathrm{i}\theta(t)} \log\left(1 - \frac{z_k}{\eta(t)}\right)\right], \tag{3.22}$$

for $k = 1, 2, \ldots, m$.

In view of (3.5), (3.8), and (3.18), the constants a_k are known for $k \in S_{\mathbf{n}}$ and are given by

$$a_k = \frac{1}{2\pi} \int_{J_k} \phi_k(t) \left|\dot{\eta}_k(t)\right| dt, \quad k \in S_{\mathbf{n}}. \tag{3.23}$$

For $k \in S_{\mathbf{d}}$, the real constants a_k are unknown. Thus, the boundary condition (3.19) can be written as

$$\mathrm{Re}\left[e^{-\mathrm{i}\theta(t)} f(\eta(t))\right] = \widehat{\varphi}(t) + \sum_{k \in S_{\mathbf{d}}} a_k \gamma^{[k]}(t), \quad t \in J, \tag{3.24}$$

where the function $\widehat{\varphi}(t)$ is known and is given by

$$\widehat{\varphi}(t) = \begin{cases} \phi_j(t) + \sum_{k \in S_{\mathbf{n}}} a_k \gamma_j^{[k]}(t), & t \in J_j,\ j \in S_{\mathbf{d}}, \\ \varphi_j(t) + \sum_{k \in S_{\mathbf{n}}} a_k \gamma_j^{[k]}(t), & t \in J_j,\ j \in S_{\mathbf{n}}. \end{cases} \tag{3.25}$$

It is clear that the function $\widehat{\varphi}_j(t)$ is known explicitly for $t \in J_j$ with $j \in S_{\mathbf{d}}$. However, for $t \in J_j$ with $j \in S_{\mathbf{n}}$ it is required to integrate $\varphi'_j(t)$ to obtain $\varphi_j(t)$.

The function $\varphi_j(t)$ is not necessary 2π-periodic. Numerically, we prefer to deal with periodic functions. So we will not compute $\varphi_j(t)$ directly by integrating the function $\varphi'_j(t)$. Instead, we will integrate the function

$$\widehat{\varphi}'_j(t) = \phi_j(t) \left|\dot{\eta}_j(t)\right| + \sum_{k \in S_{\mathbf{n}}} a_k \frac{d}{dt} \gamma_j^{[k]}(t). \tag{3.26}$$

By the definitions of the constants a_k and the functions $\gamma^{[k]}$, we have

$$\int_0^{2\pi} \widehat{\varphi}'_j(t) dt = 0, \tag{3.27}$$

which implies that the function $\widehat{\varphi}_j(t) = \varphi_j(t) + \sum_{k \in S_{\mathbf{n}}} a_k \gamma_j^{[k]}(t)$ is always 2π-periodic. So, for $t \in J_j$ with $j \in S_{\mathbf{n}}$, we write the function $\widehat{\varphi}'_j(t)$ using the Fourier series as

$$\widehat{\varphi}'_j(t) = \sum_{i=1}^{\infty} a_i^{[j]} \cos it + \sum_{i=1}^{\infty} b_i^{[j]} \sin it. \tag{3.28}$$

Then the function $\widehat{\varphi}_j(t)$ is given for $t \in J_j$ with $j \in S_{\mathbf{n}}$ by

$$\widehat{\varphi}_j(t) = \widetilde{\varphi}_j(t) + c_j, \tag{3.29}$$

where c_j is undetermined real constant and the function $\widetilde{\varphi}_j(t)$ is given by

$$\widetilde{\varphi}_j(t) = \sum_{i=1}^{\infty} \frac{a_i^{[j]}}{i} \sin it - \sum_{i=1}^{\infty} \frac{b_i^{[j]}}{i} \cos it, \quad t \in J_j,\ j \in S_{\mathbf{n}}. \tag{3.30}$$

Hence, the boundary condition (3.24) can be then written as

$$\operatorname{Re}\left[e^{-\mathrm{i}\theta(t)} f(\eta(t))\right] = \widehat{\gamma}(t) + \widetilde{h}(t) + \sum_{k\in S_{\mathbf{d}}} a_k \gamma^{[k]}(t), \quad t \in J, \tag{3.31}$$

where $\widetilde{h}(t)$ is the real piecewise constant function

$$\widetilde{h}(t) = \begin{cases} 0, & t \in J_j,\ j \in S_{\mathbf{d}}, \\ c_j, & t \in J_j,\ j \in S_{\mathbf{n}}, \end{cases} \tag{3.32}$$

and the function $\widehat{\gamma}(t)$ is given by

$$\widehat{\gamma}(t) = \begin{cases} \phi_j(t) + \sum_{k\in S_{\mathbf{n}}} a_k \gamma_j^{[k]}(t), & t \in J_j,\ j \in S_{\mathbf{d}}, \\ \widetilde{\varphi}_j(t), & t \in J_j,\ j \in S_{\mathbf{n}}. \end{cases} \tag{3.33}$$

Let $c := f(0)$ (unknown real constant) and $g(z)$ be the analytic function defined on G by

$$g(z) := \frac{f(z) - c}{z}, \quad z \in G. \tag{3.34}$$

Thus the function $g(z)$ is a solution of the Riemann-Hilbert problem

$$\operatorname{Re}\left[A(t) g(\eta(t))\right] = \widehat{\gamma}(t) + h(t) + \sum_{j\in S_{\mathbf{d}}} a_j \gamma^{[j]}(t), \quad t \in J, \tag{3.35}$$

where the function $A(t)$ is given by (2.6) and the function $h(t)$ is defined by

$$h(t) = \widetilde{h}(t) - c\cos\theta(t), \quad t \in J. \tag{3.36}$$

Let $\mu(t) := \operatorname{Im}[A(t)g(\eta(t))]$, then the boundary value of the function $g(z)$ is given on the boundary Γ by

$$A(t)g(\eta(t)) = \hat{\gamma}(t) + h(t) + \sum_{j\in S_{\mathbf{d}}} a_j\gamma^{[j]}(t) + \mathrm{i}\mu(t), \quad t \in J, \tag{3.37}$$

where $\hat{\gamma}$, $\gamma^{[j]}$ are knowns and h, μ are unknowns.

4. The Solution of the Mixed Boundary Value Problem

Let $\hat{\mu}$ and let $\mu^{[j]}$ for $j \in S_{\mathbf{d}}$ be the unique solutions of the integral equations

$$(\mathbf{I}-\mathbf{N})\hat{\mu} = -\mathbf{M}\hat{\gamma}, \qquad (\mathbf{I}-\mathbf{N})\mu^{[j]} = -\mathbf{M}\gamma^{[j]}, \tag{4.1}$$

and $\hat{h}$, $h^{[j]}$ be given by

$$\hat{h} = \frac{[\mathbf{M}\hat{\mu} - (\mathbf{I}-\mathbf{N})\hat{\gamma}]}{2}, \qquad h^{[j]} = \frac{[\mathbf{M}\mu^{[j]} - (\mathbf{I}-\mathbf{N})\gamma^{[j]}]}{2}. \tag{4.2}$$

Then, it follows from Theorem 2.1 that

$$A(t)\hat{g}(\eta(t)) = \hat{\gamma} + \hat{h} + \mathrm{i}\hat{\mu} + \sum_{j\in S_{\mathbf{d}}} a_j\left(\gamma^{[j]} + h^{[j]} + \mathrm{i}\mu^{[j]}\right) \tag{4.3}$$

is the boundary value of an analytic function $\hat{g}(z)$. Since the unknown functions μ and h in (3.37) are uniquely determined by the known functions $\hat{\gamma}$ and $\gamma^{[j]}$ for $j \in S_{\mathbf{d}}$, it follows from (3.37) and (4.3) that

$$\mu(t) = \hat{\mu}(t) + \sum_{j\in S_{\mathbf{d}}} a_j\mu^{[j]}(t), \tag{4.4}$$

$$h(t) = \hat{h}(t) + \sum_{j\in S_{\mathbf{d}}} a_j h^{[j]}(t). \tag{4.5}$$

In view of (3.15), (3.32), and (3.36), the function $h(t)$ is given by

$$h(t) = \begin{cases} -c, & t \in J_j,\ j \in S_{\mathbf{d}}, \\ c_j, & t \in J_j,\ j \in S_{\mathbf{n}}. \end{cases} \tag{4.6}$$

Thus, we have from (4.5), (3.23), and (3.6) the linear system

$$h(t) - \sum_{j \in S_{\mathbf{d}}} a_j h^{[j]}(t) = \widehat{h}(t), \tag{4.7a}$$

$$\sum_{j \in S_{\mathbf{d}}} a_j = -\sum_{j \in S_{\mathbf{n}}} a_j, \tag{4.7b}$$

of $m+2$ equations in $m+2$ unknowns c, a_j for $j \in S_{\mathbf{d}}$ and c_j for $j \in S_{\mathbf{n}}$.

By solving the linear system (4.7a) and (4.7b), we obtain the values of the constants a_j and the function $h(t)$. Then, we obtain the function μ from (4.4). Consequently, the boundary value of the function g is given by

$$A(t)g(\eta(t)) = \gamma(t) + h(t) + \mathrm{i}\mu(t), \quad t \in J, \tag{4.8}$$

where

$$\gamma(t) = \widehat{\gamma}(t) + \sum_{j \in S_{\mathbf{d}}} a_j \gamma^{[j]}(t), \quad t \in J. \tag{4.9}$$

The function $g(z)$ can be computed for $z \in G$ by the Cauchy integral formula. Then the function $f(z)$ is given by

$$f(z) = c + zg(z). \tag{4.10}$$

Finally, the solution of the mixed boundary value problem can be computed from $u(z) = \operatorname{Re} F(z)$ where $F(z)$ is given by (3.3).

5. Numerical Implementations

Since the functions A_j and η_j are 2π-periodic, the integrands in the integral equations (4.1) are 2π-periodic. Hence, the most efficient numerical method for solving (4.1) is generally the Nyström method with the trapezoidal rule (see e.g., [16, page 321]). We will use the trapezoidal rule with n (an even positive integer) equidistant node points on each boundary component to discretize the integrals in (4.1). If the integrands in (4.1) are k times continuously differentiable, then the rate of convergence of the trapezoidal rule is $O(1/n^k)$. For analytic integrands, the rate of convergence is better than $1/n^k$ for any positive integer k (see e.g., [17, page 83]). The obtained approximate solutions of the integral equations converge to the exact solutions with a similarly rapid rate of convergence (see e.g., [16, page 322]). Since the smoothness of the integrands in (4.1) depends on the smoothness of the function $\eta(t)$, results of high accuracy can be obtained for very smooth boundaries.

By using the Nyström method with the trapezoidal rule, the integral equations (4.1) reduce to $(m+1)n$ by $(m+1)n$ linear systems. Since the integral equations (4.1) are uniquely solvable, then for sufficiently large values of n the obtained linear systems are also uniquely solvable [16]. The linear systems are solved using the Gauss elimination method. The computational details are similar to previous works in [2–5].

By solving the linear systems, we obtain approximations to $\widehat{\mu}$ and $\mu^{[j]}$ for $j \in S_{\mathbf{d}}$ which give approximations to $\widehat{h}$ and $h^{[j]}$ for $j \in S_{\mathbf{d}}$ through (4.2). By solving (4.7a) and (4.7b) we get approximations to the constants c, a_j for $j \in S_{\mathbf{d}}$ and c_j for $j \in S_{\mathbf{n}}$. These approximations allow us to obtain approximations to the boundary value of the function $g(z)$ from (4.8). The values of $g(z)$ for $z \in G$ will be calculated by the Cauchy integral formula. The approximate values of the function $f(z)$ are then computed from (4.10). Finally, in view of (3.3), the solution of the mixed boundary value problem can be computed from

$$u(z) = \operatorname{Re} F(z) = \operatorname{Re} f(z) - \sum_{j=1}^{m} a_j \log\left|z - z_j\right|. \tag{5.1}$$

In this paper, we have considered regions with smooth boundaries. For some ideas on how to solve numerically boundary integral equations with the generalized Neumann kernel on regions with piecewise smooth boundaries, see [18].

6. Numerical Examples

In this section, the proposed method will be used to solve two mixed boundary value problems in bounded multiply connected regions with smooth boundaries.

Example 6.1. In this example we consider a bounded multiply connected region of connectivity 4 bounded by the four circles (see Figure 2(a))

$$\begin{aligned}
\Gamma_0 &: \eta_0(t) = 1 + 2e^{\mathrm{i}t}, \\
\Gamma_1 &: \eta_1(t) = 1 + 0.25e^{-\mathrm{i}t}, \\
\Gamma_2 &: \eta_2(t) = 1 + \mathrm{i} + 0.25e^{-\mathrm{i}t}, \\
\Gamma_3 &: \eta_2(t) = 1 - \mathrm{i} + 0.25e^{-\mathrm{i}t},
\end{aligned} \tag{6.1}$$

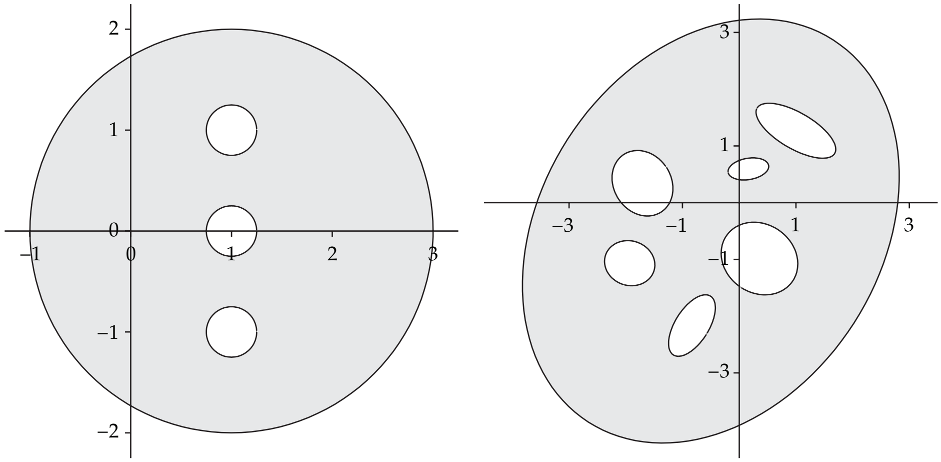
where $0 \le t \le 2\pi$.

We assume that the condition on the boundaries Γ_0, Γ_1 is the Neumann condition and the condition on the boundaries Γ_2, Γ_3 is the Dirichlet condition. The functions ϕ_j in (3.2a), (3.2b), and (3.2c) are obtained based on choosing an exact solution of the form

$$u(z) = \operatorname{Re}(z-1)^3. \tag{6.2}$$

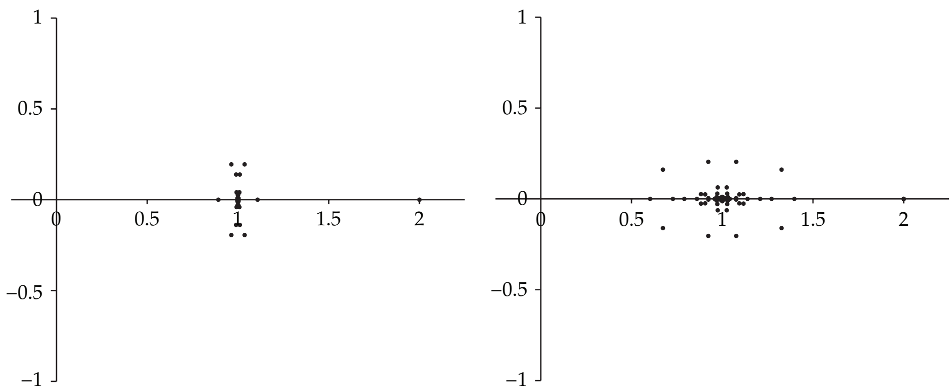
This example has been considered in [9, page 894] using the regularized meshless method. The domain here is shifted by one unit to the right to ensure that $0 \in G$. To compare our method with the method presented in [9], we use the same error norm used in [9], namely,

$$\int_0^{2\pi} \left|u\left(1 + 0.5e^{\mathrm{i}s}\right) - u_n\left(1 + 0.5e^{\mathrm{i}s}\right)\right|^2 ds, \tag{6.3}$$



(a) (b)

Figure 2: The regions of Example 6.1 (a) and Example 2 (b).



(a) (b)

Figure 3: The eigenvalues of the coefficient matrix of the linear systems obtained by discretizing the integral equations (4.1) with $n = 128$ for Example 6.1 (a) and with $n = 256$ for Example 6.2 (b).

where $u(z)$ is the exact solution of the mixed boundary value problem and $u_n(z)$ is the approximate solution obtained with n node points on each boundary component. The error norm versus the total number of calculation points $4n$ by using the presented method is shown in Figure 4(a) where the integral in (6.3) is discretized by the trapezoidal rule. By using only $n = 16$ (64 calculation points on the whole boundary), we obtain error norm less than 10^{-9}. In [9], the error norm is only less than 10^{-3} when 200 boundary points are used. The absolute errors $|u(z)-u_n(z)|$ for selected points in the entire domain are plotted in Figure 5(a) (compare the results with [9, Figure 10]).

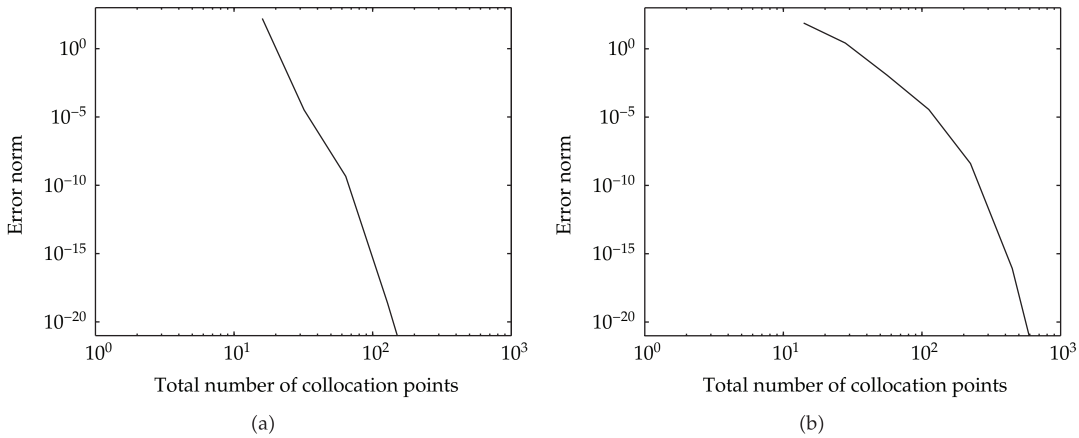


Figure 4: The error norm (6.3) for Example 1 (a) and the error norm (6.6) for Example 6.2 (b) versus the total number of node points.

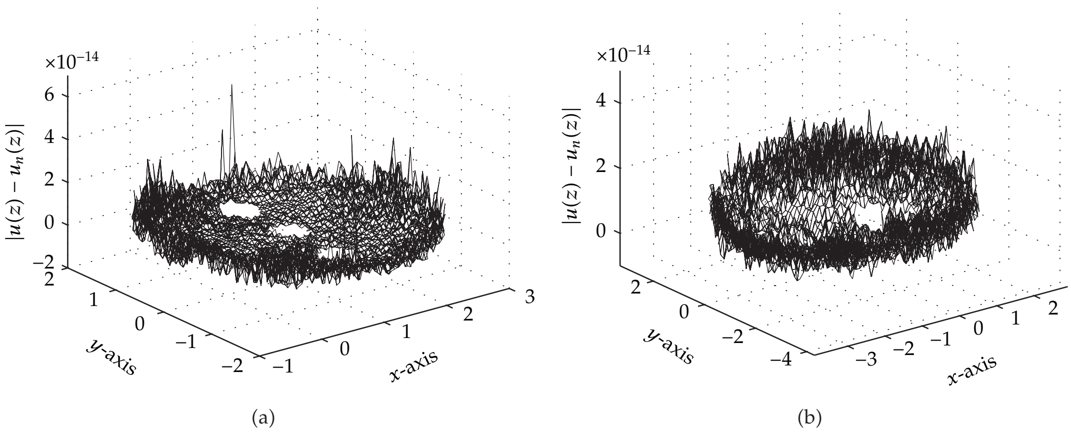


Figure 5: The absolute error $|u(z) - u_n(z)|$ for the entire domain obtained with $n = 64$ for Example 6.1 (a) and with $n = 256$ for Example 6.2 (b).

Example 6.2. In this example we consider a bounded multiply connected region of connectivity 7 (see Figure 2(b)). The boundary $\Gamma = \partial G$ is parametrized by

$$\eta_j(t) = z_j + e^{\mathrm{i}\sigma_j}\left(\alpha_j \cos t + \mathrm{i}\beta_j \sin t\right), \quad j = 0, 1, \ldots, 6, \tag{6.4}$$

where the values of the complex constants z_j and the real constants α_j, β_j, σ_j are as in Table 1.

The region in this example has been considered in [2, 19, 20] for the Dirichlet problem and the Neumann problem. In this example, we will consider a mixed boundary value problem where we assume that the condition on the boundaries Γ_0, Γ_1, Γ_2, and Γ_3 is the Dirichlet condition and the condition on the boundaries Γ_4, Γ_5, and Γ_6 is the Neumann

Table 1: The values of constants $\alpha_j, \beta_j, z_j, \sigma_j$, and ζ_j in (6.4).

j	α_j	β_j	z_j	σ_j	ζ_j
0	4.0000	3.0000	−0.5000 − 0.5000i	1.0000	5.00 + 5.00i
1	0.3626	−0.1881	0.1621 + 0.5940i	3.3108	0.10 + 0.50i
2	0.5061	−0.6053	−1.7059 + 0.3423i	0.5778	−1.60 + 0.40i
3	0.6051	−0.7078	0.3577 − 0.9846i	4.1087	0.30 − 0.90i
4	0.7928	−0.3182	1.0000 + 1.2668i	2.6138	0.95 + 1.20i
5	0.3923	−0.4491	−1.9306 − 1.0663i	4.4057	−1.85 − 1.00i
6	0.2976	−0.6132	−0.8330 − 2.1650i	5.7197	−0.80 − 2.10i

condition. The functions ϕ_j in (3.2a), (3.2b), and (3.2c) are obtained based on choosing an exact solution of the form

$$u(z) = 1 + 2\mathrm{Re}\left(\frac{1}{z-\zeta_0}\right) + \sum_{j=1}^{6}(j-7/2)\log\left(\left|z-\zeta_j\right|^2\right), \tag{6.5}$$

where the values of the complex constants ζ_j are as in Table 1. For the error, we use the error norm (see Figure 4(b)):

$$\int_0^{2\pi}\left|u\left(-0.35-0.35\mathrm{i}+2.7e^{\mathrm{i}s}\right)-u_n\left(-0.35-0.35\mathrm{i}+2.7e^{\mathrm{i}s}\right)\right|^2 ds. \tag{6.6}$$

The absolute errors $|u(z) - u_n(z)|$ for selected points in the entire domain are plotted in Figure 5(b).

7. Conclusions

A new uniquely solvable boundary integral equation with the generalized Neumann kernel has been presented in this paper for solving a certain class of mixed boundary value problem on multiply connected regions. Two numerical examples are presented to illustrate the accuracy of the presented method.

The presented method can be applied to mixed boundary value problem with both the Dirichlet condition and the Neumann condition on the same boundary component Γ_k. For this case, the function $A(t)$ is discontinuous on J_k, where $A(t) = \eta_k(t)$ on the part of Γ_k corresponding to the Dirichlet condition and $A(t) = -\mathrm{i}\eta_k(t)$ on the part of Γ_k corresponding to the Neumann condition. Hence, the Riemann-Hilbert problem (3.35) will be a problem with discontinuous coefficient $A(t)$. The solvability of Riemann-Hilbert problems with discontinuous coefficients is different from the ones with continuous coefficients (see e.g., [6, page 449] and [13, page 271]). Furthermore, the discontinuity of the function $A(t)$ implies that we cannot apply the theory of the generalized Neumann kernel developed in [21] for simply connected regions and in [1] for multiply connected regions. Hence, further investigations are required. This will be considered in future work.

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