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Research Article

Global Error Bound Estimation for the Generalized Nonlinear Complementarity Problem over a Closed Convex Cone

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The global error bound estimation for the generalized nonlinear complementarity problem over a closed convex cone (GNCP) is considered. To obtain a global error bound for the GNCP, we first develop an equivalent reformulation of the problem. Based on this, a global error bound for the GNCP is established. The results obtained in this paper can be taken as an extension of previously known results.

1. Introduction

Let mappings $F, G : R^n \to R^m$, $H : R^n \to R^l$, and the generalized nonlinear complementarity problem, abbreviated as GNCP, is to find vector $x^* \in R^n$ such that

$$F(x^*) \in \mathcal{K}, \qquad G(x^*) \in \mathcal{K}^0, \qquad F(x^*)^{\top} G(x^*) = 0, \qquad H(x^*) = 0, \tag{1.1}$$

where $\mathcal{K}$ is a nonempty closed convex cone in R^m and $\mathcal{K}^\circ$ is its dual cone, that is, $\mathcal{K}^\circ = \{u \in R^m \mid u^{\top} v \geq 0, \text{for all } v \in \mathcal{K}\}$. We denote the solution set of the GNCP by X^*, and assume that it is nonempty throughout this paper.

The GNCP is a direct generalization of the classical nonlinear complementarity problem which finds applications in engineering, economics, finance, and robust optimization operations research [1–3]. For example, the balance of supply and demand is central to all economic systems; mathematically, this fundamental equation in economics is often described by a complementarity relation between two sets of decision variables.

Furthermore, the classical Walrasian law of competitive equilibria of exchange economies can be formulated as a generalized nonlinear complementarity problem in the price and excess demand variables [2]. Up to now, the issues of numerical methods and existence of the solution for the problem were discussed in the literature [4].

Among all the useful tools for theoretical and numerical treatment to variational inequalities, nonlinear complementarity problems, and other related optimization problems, the global error bound, that is, an upper bound estimation of the distance from a given point in R^n to the solution set of the problem in terms of some residual functions, is an important one [5, 6]. The error bound estimation for the generalized linear complementarity problems over a polyhedral cone was analyzed by Sun et al. [7]. Using the natural residual function, Pang [8] obtained a global error bound for the strongly monotone and Lipschitz continuous classical nonlinear complementarity problem with a linear constraint set. Xiu and Zhang [9] also presented a global error bound for general variational inequalities with the mapping being strongly monotone and Lipschitz continuous in terms of the natural residual function. If $F(x) = x$, $G(x)$ is γ-strongly monotone and $H\ddot{o}lder$ continuous, the local error bound for classical variational inequality problems was given by Solodov [6].

To our knowledge, the global error bound for the problem (1.1) with the mapping being γ-strongly monotone and $H\ddot{o}lder$-continuous hasn't been investigated. Motivated by this fact, The main contribution of this paper is to establish a global error bound for the GNCP via the natural residual function under milder conditions than those needed in [6, 8, 9]. The results obtained in this paper can be taken as an extension of the previously known results in [6, 8, 9].

We give some notations used in this paper. Vectors considered in this paper are all taken in Euclidean space equipped with the standard inner product. The Euclidean norm of vector in the space is denoted by $\|\cdot\|$. The inner product of vector in the space is denoted by $\langle\cdot,\cdot\rangle$.

2. The Global Error Bound for GNCP

In this section, we would give error bound for GNCP, which can be viewed as extensions of previously known results. To this end, we will in the following establish an equivalent reformulation of the GNCP and state some well-known properties of the projection operator which is crucial to our results.

In the following, we first give the equivalent reformulation of the GNCP.

Theorem 2.1. *A point x^* is a solution of* (1.1) *if and only if x^* is a solution of the following problem:*

$$\begin{gathered} G(x^*)^{\top}(F(x) - F(x^*)) \geq 0, \quad \forall F(x) \in \mathcal{K}, \\ H(x^*) = 0. \end{gathered} \tag{2.1}$$

Proof. Suppose that x^* is a solution of (2.1). Since vector $0 \in \mathcal{K}$, by substituting $F(x) = 0$ into (2.1), we have $G(x^*)^{\top}F(x^*) \leq 0$. On the other hand, since $F(x^*) \in \mathcal{K}$, then $2F(x^*) \in \mathcal{K}$. By substituting $F(x) = 2F(x^*)$ into (2.1), we obtain $G(x^*)^{\top}F(x^*) \geq 0$. Consequently, $G(x^*)^{\top}F(x^*) = 0$. For any $F(x) \in \mathcal{K}$, we have $G(x^*)^{\top}F(x) = G(x^*)^{\top}[F(x) - F(x^*)] \geq 0$, that is, $G(x^*) \in \mathcal{K}^{\circ}$. Combining $H(x^*) = 0$, thus, x^* is a solution of (1.1).

On the contrary, suppose that x^* is a solution of (1.1), since $G(x^*) \in \mathcal{K}^\circ$, for any $F(x) \in \mathcal{K}$, we have $G(x^*)^\top F(x) \geq 0$, and from $G(x^*)^\top F(x^*) = 0$, we have $G(x^*)^\top [F(x) - F(x^*)] \geq 0$, combining $H(x^*) = 0$. Therefor, x^* is a solution of (2.1). □

Now, we give the definition of projection operator and some related properties [10]. For nonempty closed convex set $\mathcal{K} \subset R^m$ and any vector $x \in R^m$, the orthogonal projection of x onto $\mathcal{K}$, that is, $\operatorname{argmin}\{\|y - x\| \mid y \in \mathcal{K}\}$, is denoted by $P_{\mathcal{K}}(x)$.

Lemma 2.2. *For any $u \in R^m$, $v \in \mathcal{K}$, then*

(i) $\langle P_{\mathcal{K}}(u) - u, v - P_{\mathcal{K}}(u) \rangle \geq 0$,

(ii) $\|P_K(u) - P_K(v)\| \leq \|u - v\|$.

For (2.1), $\beta > 0$ is a constant, $e(x) := F(x) - P_{\mathcal{K}}[F(x) - \beta G(x)]$ is called projection-type residual function, and let $r(x) := \|e(x)\|$. The following conclusion provides the relationship between the solution set of (2.1) and that of projection-type residual function [11], which is due to Noor [11].

Lemma 2.3. *x is a solution of* (2.1) *if and only if $e(x) = 0$, $H(x) = 0$.*

To establish the global error bound of GNCP, we also need the following definition.

Definition 2.4. The mapping $F : R^n \to R^m$ is said to be

(1) γ-strongly monotone with respect to $G : R^n \to R^m$ if there are constants $\mu > 0$, $\gamma > 1$ such that

$$\langle F(x) - F(y), G(x) - G(y) \rangle \geq \mu \|x - y\|^{1+\gamma}, \quad \forall x, y \in R^n; \tag{2.2}$$

(2) *Hölder*-continuous if there are constants $L > 0$, $v \in (0, 1]$ such that

$$\|F(x) - F(y)\| \leq L \|x - y\|^{v}, \quad \forall x, y \in R^n. \tag{2.3}$$

In this following, based on Lemmas 2.2 and 2.3, we establish error bound for GNCP in the set $\Omega := \{x \in R^n \mid H(x) = 0\}$.

Theorem 2.5. *Suppose that F is γ-strongly monotone with respect to G and with positive constants μ, γ, both F and G are Hölder continuous with positive constants $L_1 > 0$, $L_2 > 0$, v_1, $v_2 \in (0, 1]$,* *respectively, and* $\beta\mu \leq (L_1 + L_2\beta)(2L_1 + L_2\beta)$ *holds. Then for any* $x \in \Omega := \{x \in R^n \mid H(x) = 0\}$, *there exists a solution* x^* *of* (1.1) *such that*

$$\left(\frac{r(x)}{2L_1 + L_2\beta}\right)^{1/\min\{v_1,v_2\}} \leq \|x - x^*\| \leq \left(\frac{L_1 + L_2\beta}{\beta\mu} r(x)\right)^{1/(1+\gamma-\min\{v_1,v_2\})}, \quad \text{if } r(x) \leq \frac{\beta\mu}{L_1 + L_2\beta}. \tag{2.4}$$

$$\left(\frac{r(x)}{2L_1 + L_2\beta}\right)^{1/\max\{v_1,v_2\}} \leq \|x - x^*\| \leq \left(\frac{L_1 + L_2\beta}{\beta\mu} r(x)\right)^{1/(1+\gamma-\max\{v_1,v_2\})}, \quad \text{if } r(x) \geq 2L_1 + L_2\beta. \tag{2.5}$$

$$\left(\frac{r(x)}{2L_1 + L_2\beta}\right)^{1/\min\{v_1,v_2\}} \leq \|x - x^*\| \leq \left(\frac{L_1 + L_2\beta}{\beta\mu} r(x)\right)^{1/(1+\gamma-\max\{v_1,v_2\})}, \quad \text{if } \frac{\beta\mu}{L_1 + L_2\beta} < r(x) < 2L_1 + L_2\beta. \tag{2.6}$$

Proof. Since

$$F(x) - e(x) = P_{\mathcal{K}}\left[F(x) - \beta G(x)\right] \in \mathcal{K}, \tag{2.7}$$

by the first inequality of (2.1),

$$\left(F(x) - e(x) - F(x^*)\right)^{\top} \beta G(x^*) \geq 0. \tag{2.8}$$

Combining $F(x^*) \in \mathcal{K}$ with Lemma 2.2(i), we have

$$\left\langle F(x^*) - P_{\mathcal{K}}\left[F(x) - \beta G(x)\right], P_{\mathcal{K}}\left[F(x) - \beta G(x)\right] - \left[F(x) - \beta G(x)\right]\right\rangle \geq 0. \tag{2.9}$$

Substituting $P_{\mathcal{K}}[F(x) - \beta G(x)]$ in (2.9) by $F(x) - e(x)$ leads to that

$$\left(F(x) - F(x^*) - e(x)\right)^{\top}\left[e(x) - \beta G(x)\right] \geq 0. \tag{2.10}$$

Using (2.8) and (2.10), we obtain

$$\left[(F(x) - F(x^*)) - e(x)\right]^{\top}\left[e(x) + \beta(G(x^*) - G(x))\right] \geq 0, \tag{2.11}$$

that is,

$$\beta[F(x)-F(x^*)]^{\top}[G(x^*)-G(x)]+e(x)^{\top}\left[(F(x)-F(x^*))-\beta(G(x^*)-G(x))\right]-e(x)^{\top}e(x)\geq 0. \tag{2.12}$$

Base on Definition 2.4, a direct computation yields that

$$\begin{aligned}
\|x-x^*\|^{1+\gamma} &\leq \frac{1}{\mu}\left[(F(x)-F(x^*))^{\top}(G(x)-G(x^*))\right] \\
&\leq \frac{1}{\beta\mu}\left\{e(x)^{\top}\left[(F(x)-F(x^*))+\beta(G(x)-G(x^*))\right]-e(x)^{\top}e(x)\right\} \\
&\leq \frac{1}{\beta\mu}\left\{\|e(x)\|\left(\|F(x)-F(x^*)\|+\beta\|(G(x)-G(x^*))\|\right)\right\} \\
&\leq \frac{1}{\beta\mu}\left\{r(x)\left[L_1\|x-x^*\|^{v_1}+\beta L_2\|x-x^*\|^{v_2}\right]\right\}.
\end{aligned} \tag{2.13}$$

Combining this, we have

$$\|x-x^*\|\leq\begin{cases}\left(\dfrac{L_1+L_2\beta}{\beta\mu}r(x)\right)^{1/(1+\gamma-\min\{v_1,v_2\})}, & \text{if } \|x-x^*\|\leq 1,\\[2ex] \left(\dfrac{L_1+L_2\beta}{\beta\mu}r(x)\right)^{1/(1+\gamma-\max\{v_1,v_2\})}, & \text{if } \|x-x^*\|\geq 1.\end{cases} \tag{2.14}$$

On the other hand, for the first inequality of (2.1), by Lemmas 2.3 and 2.2(ii), we have

$$\begin{aligned}
r(x) &= \|e(x)-e(x^*)\| \\
&= \left\|F(x)-P_{\mathcal{K}}[F(x)-\beta G(x)]-F(x^*)+P_{\mathcal{K}}[F(x^*)-\beta G(x^*)]\right\| \\
&= \|F(x)-F(x^*)\|+\left\|P_{\mathcal{K}}[F(x)-\beta G(x)]-P_{\mathcal{K}}[F(x^*)-\beta G(x^*)]\right\| \\
&\leq \|F(x)-F(x^*)\|+\left\|[F(x)-\beta G(x)]-[F(x^*)-\beta G(x^*)]\right\| \\
&= 2\|F(x)-F(x^*)\|+\beta\|G(x)-G(x^*)\| \\
&\leq 2L_1\|x-x^*\|^{v_1}+L_2\beta\|x-x^*\|^{v_2}.
\end{aligned} \tag{2.15}$$

Thus,

$$\|x-x^*\|\geq\begin{cases}\left(\dfrac{r(x)}{2L_1+L_2\beta}\right)^{1/\min\{v_1,v_2\}}, & \text{if } \|x-x^*\|\leq 1,\\[2ex] \left(\dfrac{r(x)}{2L_1+L_2\beta}\right)^{1/\max\{v_1,v_2\}}, & \text{if } \|x-x^*\|\geq 1.\end{cases} \tag{2.16}$$

Combining (2.14) with (2.16) for any $x \in R^n$, if $\|x - x^*\| \le 1$, then

$$\left(\frac{r(x)}{2L_1 + L_2\beta}\right)^{1/\min\{v_1,v_2\}} \le \|x - x^*\| \le \min\left\{\left(\frac{L_1 + L_2\beta}{\beta\mu} r(x)\right)^{1/(1+\gamma-\min\{v_1,v_2\})}, 1\right\}. \tag{2.17}$$

For any $x \in R^n$, if $\|x - x^*\| \ge 1$, then

$$\max\left\{\left(\frac{r(x)}{2L_1 + L_2\beta}\right)^{1/\max\{v_1,v_2\}}, 1\right\} \le \|x - x^*\| \le \left(\frac{L_1 + L_2\beta}{\beta\mu} r(x)\right)^{1/(1+\gamma-\max\{v_1,v_2\})}. \tag{2.18}$$

If $r(x) \le \beta\mu/(L_1 + L_2\beta)$, then $((L_1 + L_2\beta)/\beta\mu)r(x) \le 1$, by (2.17), we have $\|x - x^*\| \le 1$, and using (2.17) again, we obtain that (2.4) holds.

If $r(x) \ge L_2\beta + 2L_1$, then $r(x)/(2L_1 + L_2\beta) \ge 1$, combining this with (2.18), we have $\|x - x^*\| \ge 1$, and using (2.18) again, we conclude that (2.5) holds.

If $\beta\mu/(L_1 + L_2\beta) < r(x) < 2L_1 + L_2\beta$, then

$$\frac{L_1 + L_2\beta}{\beta\mu} r(x) > 1, \qquad \frac{r(x)}{L_2\beta + 2L_1} < 1. \tag{2.19}$$

Combining (2.17) with (2.18), we conclude that (2.6) holds. □

Definition 2.6. The mapping H involved in the GNCP is said to be α-strongly monotone in R^n if there are positive constants $\sigma > 0$, $\alpha > 1$ such that

$$\langle x - y, H(x) - H(y)\rangle \ge \sigma \|x - y\|^{1+\alpha}, \quad \forall x, y \in R^n. \tag{2.20}$$

Base on Theorem 2.5, we are at the position to state our main results in the following.

Theorem 2.7. *Suppose that the hypotheses of Theorem 2.5 hold, H is α-strongly monotone, and the set $\Omega := \{x \in R^n \mid H(x) = 0\}$ is convex. Then, there exists a constant $\rho > 0$, such that, for any $x \in R^n$, there exists $x^* \in X^*$ such that*

$$\|x - x^*\| \le \rho\left\{\|H(x)\|^{1/\alpha} + R(x)\right\}, \quad \forall x \in R^n, \tag{2.21}$$

where

$$\rho = \max\left\{\begin{array}{l} \dfrac{1}{\sigma}, \left(\max\left\{\dfrac{L_1 + L_2\beta}{\beta\mu}, \dfrac{(2L_1 + \beta L_2)(L_1 + L_2\beta)}{\beta\mu}\sigma^{-\min\{v_1,v_2\}}\right\}\right)^{1/(1+\gamma-\min\{v_1,v_2\})}, \\ \left(\max\left\{\dfrac{L_1 + L_2\beta}{\beta\mu}, \dfrac{(2L_1 + \beta L_2)(L_1 + L_2\beta)}{\beta\mu}\sigma^{-\max\{v_1,v_2\}}\right\}\right)^{1/(1+\gamma-\min\{v_1,v_2\})}, \\ \left(\max\left\{\dfrac{L_1 + L_2\beta}{\beta\mu}, \dfrac{(2L_1 + \beta L_2)(L_1 + L_2\beta)}{\beta\mu}\sigma^{-\min\{v_1,v_2\}}\right\}\right)^{1/(1+\gamma-\max\{v_1,v_2\})}, \\ \left(\max\left\{\dfrac{L_1 + L_2\beta}{\beta\mu}, \dfrac{(2L_1 + \beta L_2)(L_1 + L_2\beta)}{\beta\mu}\sigma^{-\max\{v_1,v_2\}}\right\}\right)^{1/(1+\gamma-\max\{v_1,v_2\})}, \end{array}\right\}, \tag{2.22}$$

$$R(x) = \max\left\{\begin{array}{l} \left(r(x) + \|H(x)\|^{\min\{v_1,v_2\}/\alpha}\right)^{1/(1+\gamma-\min\{v_1,v_2\})} \\ \left(r(x) + \|H(x)\|^{\max\{v_1,v_2\}/\alpha}\right)^{1/(1+\gamma-\min\{v_1,v_2\})} \\ \left(r(x) + \|H(x)\|^{\min\{v_1,v_2\}/\alpha}\right)^{1/(1+\gamma-\max\{v_1,v_2\})} \\ \left(r(x) + \|H(x)\|^{\max\{v_1,v_2\}/\alpha}\right)^{1/(1+\gamma-\max\{v_1,v_2\})} \end{array}\right\}. \tag{2.23}$$

Proof. For given $x \in R^n$, we only need to first project x to $\Omega := \{x \in R^n \mid H(x) = 0\}$, that is, there exists a vector $\overline{x} \in \Omega$ such that $\|x - \overline{x}\| = \operatorname{dist}(x, \Omega)$. By Definition 2.6, there exist constants $\sigma > 0$, $\alpha > 0$ such that

$$\begin{aligned} \|x - \overline{x}\|^{1+\alpha} &\le \frac{1}{\sigma}\langle x - \overline{x}, H(x) - H(\overline{x})\rangle \\ &\le \frac{1}{\sigma}\|x - \overline{x}\|\|H(x) - H(\overline{x})\| \\ &= \frac{1}{\sigma}\|x - \overline{x}\|\|H(x)\|, \end{aligned} \tag{2.24}$$

that is, $\operatorname{dist}(x, \Omega) = \|x - \overline{x}\| \le (1/\sigma)\|H(x)\|^{1/\alpha}$.

Since

$$\begin{aligned} r(\overline{x}) - r(x) &\le \|e(x) - e(\overline{x})\| \\ &= \left\|\{F(x) - P_{\mathcal{K}}[F(x) - \beta G(x)]\} - \{F(\overline{x}) - P_{\mathcal{K}}[F(\overline{x}) - \beta G(\overline{x})]\}\right\| \\ &\le \|F(x) - F(\overline{x})\| + \left\|P_{\mathcal{K}}(F(x) - \beta G(x)) - P_{\mathcal{K}}(F(\overline{x}) - \beta G(\overline{x}))\right\| \\ &\le \|F(x) - F(\overline{x})\| + \left\|(F(x) - \beta G(x)) - (F(\overline{x}) - \beta G(\overline{x}))\right\| \\ &\le 2\|F(x) - F(\overline{x})\| + \beta\|G(x) - G(\overline{x})\| \end{aligned}$$

$$
\begin{aligned}
&\leq 2L_1\|x-\overline{x}\|^{\nu_1}+\beta L_2\|x-\overline{x}\|^{\nu_2}\\
&\leq \begin{cases}(2L_1+\beta L_2)\|x-\overline{x}\|^{\min\{v_1,v_2\}}, & \text{if } \|x-\overline{x}\|<1\\ (2L_1+\beta L_2)\|x-\overline{x}\|^{\max\{v_1,v_2\}}, & \text{if } \|x-\overline{x}\|\geq 1\end{cases}\\
&= \begin{cases}(2L_1+\beta L_2)\operatorname{dist}(x,\Omega)^{\min\{v_1,v_2\}}, & \text{if } \|x-\overline{x}\|\leq 1\\ (2L_1+\beta L_2)\operatorname{dist}(x,\Omega)^{\max\{v_1,v_2\}}, & \text{if } \|x-\overline{x}\|\geq 1,\end{cases}
\end{aligned}
\tag{2.25}
$$

Combining this, we have

$$
r(\overline{x}) \leq r(x) + \begin{cases}(2L_1+\beta L_2)\operatorname{dist}(x,\Omega)^{\min\{v_1,v_2\}}, & \text{if } \|x-\overline{x}\|<1\\ (2L_1+\beta L_2)\operatorname{dist}(x,\Omega)^{\max\{v_1,v_2\}}, & \text{if } \|x-\overline{x}\|\geq 1.\end{cases}
\tag{2.26}
$$

Combining (2.26) with Theorem 2.5, we have the following results.

Case 1 (*if* $r(\overline{x}) \leq \beta\mu/(L_1+L_2\beta)$ *and* $\|x-\overline{x}\|\leq 1$). Combining (2.4) with the first inequality in (2.26), we can obtain that

$$
\begin{aligned}
\|x-x^*\| \leq\ & \operatorname{dist}(x,\Omega)+\|\overline{x}-x^*\|\\
\leq\ & \operatorname{dist}(x,\Omega)+\left(\frac{L_1+L_2\beta}{\beta\mu}r(\overline{x})\right)^{1/(1+\gamma-\min\{v_1,v_2\})}\\
\leq\ & \operatorname{dist}(x,\Omega)\\
& +\left(\frac{L_1+L_2\beta}{\beta\mu}r(x)+\frac{(2L_1+\beta L_2)(L_1+L_2\beta)}{\beta\mu}\operatorname{dist}(x,\Omega)^{\min\{v_1,v_2\}}\right)^{1/(1+\gamma-\min\{v_1,v_2\})}\\
\leq\ & \frac{1}{\sigma}\|H(x)\|^{1/\alpha}+\eta_1\left(r(x)+\|H(x)\|^{\min\{v_1,v_2\}/\alpha}\right)^{1/(1+\gamma-\min\{v_1,v_2\})}\\
\leq\ & \rho_1\left\{\|H(x)\|^{1/\alpha}+\left(r(x)+\|H(x)\|^{\min\{v_1,v_2\}/\alpha}\right)^{1/(1+\gamma-\min\{v_1,v_2\})}\right\},
\end{aligned}
\tag{2.27}
$$

where $\eta_1=(\max\{(L_1+L_2\beta)/\beta\mu,((2L_1+\beta L_2)(L_1+L_2\beta)/\beta\mu)\sigma^{-\min\{v_1,v_2\}}\})^{1/(1+\gamma-\min\{v_1,v_2\})}$, $\rho_1=\max\{1/\sigma,\eta_1\}$.

Case 2 (*If* $r(\overline{x}) \leq \beta\mu/(L_1 + L_2\beta)$ *and* $\|x - \overline{x}\| \geq 1$). Combining (2.4) with the second inequality in (2.26), we can also obtain that

$$\begin{aligned}
\|x - x^*\| &\leq \operatorname{dist}(x, \Omega) + \|\overline{x} - x^*\| \\
&\leq \operatorname{dist}(x, \Omega) + \left(\frac{L_1 + L_2\beta}{\beta\mu} r(\overline{x})\right)^{1/(1+\gamma-\min\{v_1,v_2\})} \\
&\leq \operatorname{dist}(x, \Omega) \\
&\quad + \left(\frac{L_1 + L_2\beta}{\beta\mu} r(x) + \frac{(2L_1 + \beta L_2)(L_1 + L_2\beta)}{\beta\mu} \operatorname{dist}(x, \Omega)^{\max\{v_1,v_2\}}\right)^{1/(1+\gamma-\min\{v_1,v_2\})} \\
&\leq \frac{1}{\sigma}\|H(x)\|^{1/\alpha} + \eta_2\left(r(x) + \|H(x)\|^{\max\{v_1,v_2\}/\alpha}\right)^{1/(1+\gamma-\min\{v_1,v_2\})} \\
&\leq \rho_2\left\{\|H(x)\|^{1/\alpha} + \left(r(x) + \|H(x)\|^{\max\{v_1,v_2\}/\alpha}\right)^{1/(1+\gamma-\min\{v_1,v_2\})}\right\},
\end{aligned} \tag{2.28}$$

where $\eta_2 = (\max\{(L_1 + L_2\beta)/\beta\mu, ((2L_1 + \beta L_2)(L_1 + L_2\beta)/\beta\mu)\sigma^{-\max\{v_1,v_2\}}\})^{1/(1+\gamma-\min\{v_1,v_2\})}$, $\rho_2 = \max\{1/\sigma, \eta_2\}$.

Case 3 (*if* $r(\overline{x}) > \beta\mu/(L_1+L_2\beta)$ *and* $\|x-\overline{x}\| \leq 1$). Combining (2.5)-(2.6) with the first inequality in (2.26), we can obtain that

$$\begin{aligned}
\|x - x^*\| &\leq \operatorname{dist}(x, \Omega) + \|\overline{x} - x^*\| \\
&\leq \operatorname{dist}(x, \Omega) + \left(\frac{L_1 + L_2\beta}{\beta\mu} r(\overline{x})\right)^{1/(1+\gamma-\max\{v_1,v_2\})} \\
&\leq \operatorname{dist}(x, \Omega) \\
&\quad + \left(\frac{L_1 + L_2\beta}{\beta\mu} r(x) + \frac{(2L_1 + \beta L_2)(L_1 + L_2\beta)}{\beta\mu} \operatorname{dist}(x, \Omega)^{\min\{v_1,v_2\}}\right)^{1/(1+\gamma-\max\{v_1,v_2\})} \\
&\leq \frac{1}{\sigma}\|H(x)\|^{1/\alpha} + \eta_3\left(r(x) + \|H(x)\|^{\min\{v_1,v_2\}/\alpha}\right)^{1/(1+\gamma-\max\{v_1,v_2\})} \\
&\leq \rho_1\left\{\|H(x)\|^{1/\alpha} + \left(r(x) + \|H(x)\|^{\min\{v_1,v_2\}/\alpha}\right)^{1/(1+\gamma-\max\{v_1,v_2\})}\right\},
\end{aligned} \tag{2.29}$$

where $\eta_3 = (\max\{(L_1 + L_2\beta)/\beta\mu, ((2L_1 + \beta L_2)(L_1 + L_2\beta)/\beta\mu)\sigma^{-\min\{v_1,v_2\}}\})^{1/(1+\gamma-\max\{v_1,v_2\})}$, $\rho_3 = \max\{1/\sigma, \eta_3\}$.

Case 4 (*If* $r(\overline{x}) > \beta\mu/(L_1 + L_2\beta)$ *and* $\|x - \overline{x}\| \geq 1$). Combining (2.5)-(2.6) with the second inequality in (2.26), we can also obtain that

$$\begin{aligned}
\|x - x^*\| &\leq \operatorname{dist}(x, \Omega) + \|\overline{x} - x^*\| \\
&\leq \operatorname{dist}(x, \Omega) + \left(\frac{L_1 + L_2\beta}{\beta\mu} r(\overline{x})\right)^{1/(1+\gamma-\max\{v_1,v_2\})} \\
&\leq \operatorname{dist}(x, \Omega) \\
&\quad + \left(\frac{L_1 + L_2\beta}{\beta\mu} r(x) + \frac{(2L_1 + \beta L_2)(L_1 + L_2\beta)}{\beta\mu} \operatorname{dist}(x, \Omega)^{\max\{v_1,v_2\}}\right)^{1/(1+\gamma-\max\{v_1,v_2\})} \\
&\leq \frac{1}{\sigma}\|H(x)\|^{1/\alpha} + \eta_4\left(r(x) + \|H(x)\|^{\max\{v_1,v_2\}/\alpha}\right)^{1/(1+\gamma-\max\{v_1,v_2\})} \\
&\leq \rho_4\left\{\|H(x)\|^{1/\alpha} + \left(r(x) + \|H(x)\|^{\max\{v_1,v_2\}/\alpha}\right)^{1/(1+\gamma-\max\{v_1,v_2\})}\right\},
\end{aligned} \tag{2.30}$$

where $\eta_4 = (\max\{(L_1 + L_2\beta)/\beta\mu, ((2L_1 + \beta L_2)(L_1 + L_2\beta)/\beta\mu)\sigma^{-\max\{v_1,v_2\}}\})^{1/(1+\gamma-\max\{v_1,v_2\})}$, $\rho_4 = \max\{1/\sigma, \eta_4\}$.

By (2.27)–(2.30), we can deduce that (2.21) holds. □

Based on Theorem 2.7, we can further establish a global error bound for the GNCP. First, we give that the needed result from [12] mainly discusses the error bound for a polyhedral cone to reach our claims.

Lemma 2.8. *For polyhedral cone* $P = \{x \in R^n \mid D_1 x = d_1, D_2 x \leq d_2\}$ *with* $D_1 \in R^{l\times n}$, $D_2 \in R^{m\times n}$, $d_1 \in R^l$, *and* $d_2 \in R^m$, *there exists a constant* $c_1 > 0$ *such that*

$$\operatorname{dist}(x, P) \leq c_1[\|D_1 x - d_1\| + \|(D_2 x - d_2)_+\|] \quad \forall x \in R^n. \tag{2.31}$$

Theorem 2.9. *Suppose that the hypotheses of Theorem 2.5 hold, and* H *is linear mapping. Then, there exists a constant* $\mu > 0$, *such that, for any* $x \in R^n$, *there exists* $x^* \in X^*$ *such that*

$$\|x - x^*\| \leq \mu\{\|H(x)\| + R(x)\}, \quad \forall x \in R^n, \tag{2.32}$$

where

$$R(x) = \max\left\{\begin{array}{l}
\left(r(x) + \|H(x)\|^{\min\{v_1,v_2\}}\right)^{1/(1+\gamma-\min\{v_1,v_2\})} \\
\left(r(x) + \|H(x)\|^{\max\{v_1,v_2\}}\right)^{1/(1+\gamma-\min\{v_1,v_2\})} \\
\left(r(x) + \|H(x)\|^{\min\{v_1,v_2\}}\right)^{1/(1+\gamma-\max\{v_1,v_2\})} \\
\left(r(x) + \|H(x)\|^{\max\{v_1,v_2\}}\right)^{1/(1+\gamma-\max\{v_1,v_2\})}
\end{array}\right\}. \tag{2.33}$$

Proof. For given $x \in R^n$, we only need to first project x to Ω, that is, there exists a vector $\overline{x} \in \Omega$ such that $\|x - \overline{x}\| = \text{dist}(x, \Omega)$. By Lemma 2.8, there exists a constant $\tau > 0$ such that $\text{dist}(x, \Omega) = \|x - \overline{x}\| \le \tau \|H(x)\|$. In the following, the proof is similar to that of Theorem 2.7, and we can deduce that (2.32) holds. □

Remark 2.10. If the constraint condition $H(x) = 0$ is removed in (1.1), F is strongly monotone with respect to G (i.e., $\gamma = 1$), and both F and G are Lipschitz continuous (i.e., $v_1 = v_2 = 1$), the error bound in Theorems 2.5, 2.7, and 2.9 reduces to result of Theorem 3.1 in [9].

If the constraint condition $H(x) = 0$ is removed in (1.1) and $F(x) = x$, $G(x)$ is strongly monotone (i.e., $\gamma = 1$) and Lipschitz continuous (i.e., $v_1 = v_2 = 1$), the error bound in Theorems 2.5, 2.7, and 2.9 reduces to result of Theorem 3.1 in [8].

If the constraint condition $H(x) = 0$ is removed in (1.1) and $F(x) = x$, $G(x)$ is γ-strongly monotone in set $\{x \in R^n \mid \|x - x^*\| \le 1\}$ and $H\ddot{o}lder$ continuous, the error bound in Theorems 2.5, 2.7, and 2.9 reduces to result of Theorem 2 in [6].

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