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Research Article

Extended Extragradient Methods for Generalized Variational Inequalities

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We suggest a modified extragradient method for solving the generalized variational inequalities in a Banach space. We prove some strong convergence results under some mild conditions on parameters. Some special cases are also discussed.

1. Introduction

The well-known variational inequality problem is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H and $A : C \to H$ is a nonlinear operator. This problem has been researched extensively due to its applications in industry, finance, economics, optimization, medical sciences, and pure and applied sciences; see, for instance, [1–19] and the reference contained therein. For solving the above variational inequality, Korpelevičh [20] introduced the following so-called extragradient method:

$$\begin{gathered} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n) \end{gathered} \tag{1.2}$$

for every $n = 0, 1, 2, \ldots$, where P_C is the metric projection from R^n onto C and $\lambda \in (0, 1/k)$. He showed that the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.2) converge to the same point $z \in \mathrm{VI}(C, A)$. Since some methods related to extragradient methods have been considered in Hilbert spaces by many authors, please see, for example, [3, 5, 7, 14].

This naturally brings us to the following questions.

Question 1. Could we extend variational inequality from Hilbert spaces to Banach spaces?

Question 2. Could we extend the extragradient methods from Hilbert spaces to Banach spaces?

For solving Question 1, very recently, Aoyama et al. [21] first considered the following generalized variational inequality problem in a Banach space.

Problem 1. Let X be a smooth Banach space and C a nonempty closed convex subset of X. Let A be an accretive operator of C into X. Find a point $x^* \in C$ such that

$$\langle Ax^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

This problem is connected with the fixed point problem for nonlinear mapping, the problem of finding a zero point of an accretive operator, and so on. For the problem of finding a zero point of an accretive operator by the proximal point algorithm, please consult [22]. In order to find a solution of Problem 1, Aoyama et al. [21] introduced the following iterative scheme for an accretive operator A in a Banach space X:

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= Q_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) y_n, \end{aligned} \tag{1.4}$$

for every $n = 1, 2, \ldots$, where Q_C is a sunny nonexpansive retraction from X onto C. Then, they proved a weak convergence theorem in a Banach space which is generalized simultaneously by theorems of [4, 23] as follows.

Theorem 1.1. *Let X be a uniformly convex and 2-uniformly smooth Banach space, and let C be a nonempty closed convex subset of X. Let Q_C be a sunny nonexpansive retraction from X onto C, let $\alpha > 0$, and let A be an α-inverse-strongly accretive operator of C into X with $S(C, A) \neq \emptyset$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in [a, \alpha/K^2]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then $\{x_n\}$ defined by (1.4) converges weakly to some element z of $S(C, A) := \{x^* \in C : \langle Ax^*, J(x - x^*) \rangle \geq 0,$ for all $x \in C\}$, where K is the 2-uniformly smoothness constant of X.*

In this paper, motivated by the ideas in the literature, we first introduce a new iterative method in a Banach space as follows.

For fixed $u \in C$ and arbitrarily given $x_0 \in C$, define a sequence $\{x_n\}$ iteratively by

$$\begin{aligned} y_n &= Q_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - \lambda_n A y_n), \end{aligned} \tag{1.5}$$

for every $n = 1, 2, \ldots$, where Q_C is a sunny nonexpansive retraction from X onto C, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$, and $\{\lambda_n\}$ is a sequence of real numbers. We prove some strong convergence results under some mild conditions on parameters.

2. Preliminaries

Let X be a real Banach space, and let X^* denote the dual of X. Let C be a nonempty closed convex subset of X. A mapping A of C into X is said to be accretive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq 0, \tag{2.1}$$

for all $x, y \in C$, where J is called the duality mapping. A mapping A of C into X is said to be α-strongly accretive if, for $\alpha > 0$,

$$\langle Ax - Ay, j(x-y) \rangle \geq \alpha \|x-y\|^2, \tag{2.2}$$

for all $x, y \in C$. A mapping A of C into X is said to be α-inverse-strongly accretive if, for $\alpha > 0$,

$$\langle Ax - Ay, j(x-y) \rangle \geq \alpha \|Ax - Ay\|^2, \tag{2.3}$$

for all $x, y \in C$.

Remark 2.1. (1) Evidently, the definition of the inverse strongly accretive mapping is based on that of the inverse strongly monotone mapping.

(2) If A is an α-strongly accretive and L-Lipschitz continuous mapping of C into X, then

$$\langle Ax - Ay, j(x-y) \rangle \geq \alpha \|x-y\|^2 \geq \frac{\alpha}{L^2} \|Ax - Ay\|^2, \quad \forall x, y \in C, \tag{2.4}$$

from which it follows that A must be (α/L^2)-inverse-strongly accretive mapping.

Let $U = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be uniformly convex if, for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x-y\| \geq \epsilon \text{ implies } \left\| \frac{x+y}{2} \right\| \leq 1 - \delta. \tag{2.5}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.6}$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit (2.6) is attained uniformly for $x, y \in U$. The norm of X is said to be Frechet differentiable if, for each $x \in U$, the limit (2.6) is attained uniformly for $y \in \mathrm{U}$. And we define a function $\rho : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup\left\{ \frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \right\}. \tag{2.7}$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

Remark 2.2. Takahashi et al. [24] remind us of the following fact: no Banach space is q-uniformly smooth for $q > 2$. So, in this paper, we study a strong convergence theorem in a 2-uniformly smooth Banach space.

We need the following lemmas for the proof of our main results.

Lemma 2.3 (see [25]). *Let q be a given real number with $1 < q \leq 2$, and let X be a q-uniformly smooth Banach space. Then,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q, \tag{2.8}$$

for all $x, y \in X$, where K is the q-uniformly smoothness constant of X and J_q is the generalized duality mapping from X into 2^{X^*} defined by

$$J_q(x) = \left\{ f \in X^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \right\}, \tag{2.9}$$

for all $x \in X$.

Let D be a subset of C, and let Q be a mapping of C into D. Then, Q is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx, \tag{2.10}$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. We know the following lemma concerning sunny nonexpansive retraction.

Lemma 2.4 (see [26]). *Let C be a closed convex subset of a smooth Banach space X, D a nonempty subset of C, and Q a retraction from C onto D. Then, Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu)\rangle \leq 0, \tag{2.11}$$

for all $u \in C$ and $y \in D$.

Remark 2.5. (1) It is well known that, if X is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from X onto C.

(2) Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space X, and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then, the set $F(T)$ is a sunny nonexpansive retract of C.

The following lemma is characterized by the set of solution Problem AIT by using sunny nonexpansive retractions.

Lemma 2.6 (see [21]). *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C, and let A be an accretive operator of C into X. Then, for all $\lambda > 0$,*

$$S(C, A) = F(Q_C(I - \lambda A)), \tag{2.12}$$

where $S(C, A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0, \text{ for all } x \in C\}$.

Lemma 2.7 (see [27]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X, and let T be nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then x is a fixed point of T.*

Lemma 2.8 (see [28]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X, and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1. \tag{2.13}$$

Suppose that

$$\begin{gathered} x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n, \quad n \geq 0, \\ \limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \end{gathered} \tag{2.14}$$

Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.9 (see [26]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \quad n \geq 0, \tag{2.15}$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n / \gamma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we obtain a strong convergence theorem for finding a solution of Problem AIT for an α-strongly accretive and L-Lipschitz continuous mapping in a uniformly convex and 2-uniformly smooth Banach space. First, we assume that $\alpha > 0$ is a constant, $L > 0$ a Lipschitz constant of A, and $K > 0$ the 2-uniformly smoothness constant of X appearing in the following.

In order to obtain our main result, we need the following lemma concerning (α/L^2)-inverse-strongly accretive mapping.

Lemma 3.1. *Let X be a uniformly convex and 2-uniformly smooth Banach space, and let C be a nonempty closed convex subset of X. Let Q_C be a sunny nonexpansive retraction from X onto C, and let A be an (α/L^2)-inverse-strongly accretive mapping of C into X with $S(C,A) \neq \emptyset$. For given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by (1.5), where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$ and $\{\lambda_n\}$ is a real number sequence in $[a,b]$ for some a,b with $0 < a < b < \alpha/K^2L^2$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ *and* $\lim_{n\to\infty}\|Ay_n - Ax_n\| = 0$.

Proof. First, we observe that $I-\lambda_n A$ is nonexpansive. Indeed, for all $x, y \in C$, from Lemma 2.3, we have

$$\begin{aligned}
&\|(I-\lambda_n A)x - (I-\lambda_n A)y\|^2 \\
&\quad = \|(x-y) - \lambda_n(Ax - Ay)\|^2 \\
&\quad \leq \|x-y\|^2 - 2\lambda_n\langle Ax - Ay, j(x-y)\rangle + 2K^2\lambda_n^2\|Ax - Ay\|^2 \\
&\quad \leq \|x-y\|^2 - 2\lambda_n\frac{\alpha}{L^2}\|Ax - Ay\|^2 + 2K^2\lambda_n^2\|Ax - Ay\|^2 \\
&\quad = \|x-y\|^2 + 2\lambda_n\left(K^2\lambda_n - \frac{\alpha}{L^2}\right)\|Ax - Ay\|^2.
\end{aligned} \tag{3.1}$$

If $0 < a < \lambda_n < b < \alpha/(K^2L^2)$, then $I - \lambda_n A$ is a nonexpansive mapping.

Letting $p \in S(C,A)$, it follows from Lemma 2.6 that $p = Q_C(p - \lambda_n Ap)$. Setting $z_n = Q_C(y_n - \lambda_n Ay_n)$, from (3.1), we have

$$\begin{aligned}
\|z_n - p\| &= \|Q_C(y_n - \lambda_n Ay_n) - Q_C(p - \lambda_n Ap)\| \\
&\leq \|(y_n - \lambda_n Ay_n) - (p - \lambda_n Ap)\| \\
&\leq \|y_n - p\| \\
&= \|Q_C(x_n - \lambda_n Ax_n) - Q_C(p - \lambda_n Ap)\| \\
&\leq \|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\| \\
&\leq \|x_n - p\|.
\end{aligned} \tag{3.2}$$

By (1.5) and (3.2), we have

$$\begin{aligned}\left\|x_{n+1}-p\right\| &= \left\|\alpha_n u+\beta_n x_n+\gamma_n z_n-p\right\| \\ &\le \alpha_n\left\|u-p\right\|+\beta_n\left\|x_n-p\right\|+\gamma_n\left\|z_n-p\right\| \\ &= \alpha_n\left\|u-p\right\|+(1-\alpha_n)\left\|x_n-p\right\| \\ &\le \max\{\left\|u-p\right\|,\left\|x_0-p\right\|\}.\end{aligned} \tag{3.3}$$

Therefore, $\{x_n\}$ is bounded. Hence $\{z_n\}$, $\{Ax_n\}$, and $\{Ay_n\}$ are also bounded. We observe that

$$\begin{aligned}\|z_{n+1}-z_n\| &= \left\|Q_C(y_{n+1}-\lambda_{n+1}Ay_{n+1})-Q_C(y_n-\lambda_n Ay_n)\right\| \\ &\le \left\|(y_{n+1}-\lambda_{n+1}Ay_{n+1})-(y_n-\lambda_n Ay_n)\right\| \\ &= \left\|(y_{n+1}-\lambda_{n+1}Ay_{n+1})-(y_n-\lambda_{n+1}Ay_n)+(\lambda_n-\lambda_{n+1})Ay_n\right\| \\ &\le \left\|(y_{n+1}-\lambda_{n+1}Ay_{n+1})-(y_n-\lambda_{n+1}Ay_n)\right\|+|\lambda_n-\lambda_{n+1}|\left\|Ay_n\right\| \\ &\le \left\|y_{n+1}-y_n\right\|+|\lambda_n-\lambda_{n+1}|\left\|Ay_n\right\| \\ &= \|Q_C(x_{n+1}-\lambda_{n+1}Ax_{n+1})-Q_C(x_n-\lambda_n Ax_n)\|+|\lambda_n-\lambda_{n+1}|\left\|Ay_n\right\| \\ &\le \|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|(\|Ax_n\|+\left\|Ay_n\right\|).\end{aligned} \tag{3.4}$$

Setting $x_{n+1}=\beta_n x_n+(1-\beta_n)w_n$ for all $n\ge 0$ we obtain

$$\begin{aligned}w_{n+1}-w_n &= \frac{\alpha_{n+1}u+\gamma_{n+1}z_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n u+\gamma_n z_n}{1-\beta_n} \\ &= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)u+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(z_{n+1}-z_n)+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)z_n.\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we have

$$\begin{aligned}\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\| &\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|u\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| \\ &\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_{n+1}-\lambda_n|(\|Ax_n\|+\left\|Ay_n\right\|) \\ &\quad+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|\|z_n\|-\|x_{n+1}-x_n\| \\ &\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|(\|u\|+\|z_n\|) \\ &\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_{n+1}-\lambda_n|(\|Ax_n\|+\left\|Ay_n\right\|);\end{aligned} \tag{3.6}$$

this together with (ii) and (iv) implies that

$$\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.7}$$

Hence, by Lemma 2.8, we obtain $\|w_n-x_n\|\to 0$ as $n\to\infty$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|w_n-x_n\|=0. \tag{3.8}$$

From (1.5), we can write $x_{n+1}-x_n=\alpha_n(u-x_n)+\gamma_n(z_n-x_n)$ and note that $0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<1$ and $\lim_{n\to\infty}\alpha_n=0$. It follows from (3.8) that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.9}$$

For $p\in S(C,A)$, from (3.1) and (3.2), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2&=\|\alpha_n u+\beta_n x_n+\gamma_n z_n-p\|^2\\
&\le\alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|z_n-p\|^2\\
&\le\alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\
&\le\alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\left\{\|(x_n-\lambda_nAx_n)-(p-\lambda_nAp)\|^2\right\}\\
&\le\alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2\\
&\quad+\gamma_n\left\{\|x_n-p\|^2+2\lambda_n\left(K^2\lambda_n-\frac{\alpha}{L^2}\right)\|Ax_n-Ap\|^2\right\}\\
&\le\alpha_n\|u-p\|^2+\|x_n-p\|^2+2\gamma_n a\left(K^2b-\frac{\alpha}{L^2}\right)\|Ax_n-Ap\|^2.
\end{aligned} \tag{3.10}$$

Therefore, we have

$$\begin{aligned}
0&\le-2\gamma_n a\left(K^2b-\frac{\alpha}{L^2}\right)\|Ax_n-A\mathrm{p}\|^2\\
&\le\alpha_n\|u-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&=\alpha_n\|u-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)(\|x_n-p\|-\|x_{n+1}-p\|)\\
&\le\alpha_n\|u-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|.
\end{aligned} \tag{3.11}$$

Since $\alpha_n\to 0$ and $\|x_n-x_{n+1}\|\to 0$ as $n\to\infty$, from (3.11), we obtain

$$\|Ax_n-Ap\|\longrightarrow 0\quad\text{as } n\longrightarrow\infty. \tag{3.12}$$

From the definition of z_n and (3.1), we also have

$$\begin{aligned} \|z_n - p\|^2 &= \|Q_C(y_n - \lambda_n A y_n) - Q_C(p - \lambda_n A p)\|^2 \\ &\le \|(y_n - \lambda_n A y_n) - (p - \lambda_n A p)\|^2 \\ &\le \|y_n - p\|^2 + 2\lambda_n\left(K^2\lambda_n - \frac{\alpha}{L^2}\right)\|Ay_n - Ap\|^2. \end{aligned} \tag{3.13}$$

From the above results and assumptions, we note that $\|y_n - p\| \le \|x_n - p\|$, $0 < a < b < \alpha/(K^2L^2)$, $\|x_n - p\|$, $\|z_n - p\|$ are bounded, and $\|x_n - z_n\| \to 0$ as $n \to \infty$. Therefore, from (3.13), we have

$$\begin{aligned} 0 &\le -2a\left(K^2 b - \frac{\alpha}{L^2}\right)\|Ay_n - Ap\|^2 \\ &\le \|y_n - p\|^2 - \|z_n - p\|^2 \\ &\le \|x_n - p\|^2 - \|z_n - p\|^2 \\ &= (\|x_n - p\| + \|z_n - p\|)(\|x_n - p\| - \|z_n - p\|) \\ &\le (\|x_n - p\| + \|z_n - p\|)\|x_n - z_n\| \longrightarrow 0, \end{aligned} \tag{3.14}$$

which implies that

$$\|Ay_n - Ap\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.15}$$

It follows from (3.12) and (3.15) that

$$\|Ay_n - Ax_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.16}$$

This completes the proof. □

Now we state and study our main result.

Theorem 3.2. *Let X be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping, and let C be a nonempty closed convex subset of X. Let Q_C be a sunny nonexpansive retraction from X onto C, and let A be an α-strongly accretive and L-Lipschitz continuous mapping of C into X with $S(C, A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $(0,1)$ and $\{\lambda_n\}$ a real number sequence in $[a, b]$ for some a, b with $0 < a < b < \alpha/(K^2L^2)$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \ge 0$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ defined by (1.5) *converges strongly to $Q'u$, where Q' is a sunny nonexpansive retraction of C onto $S(C, A)$.*

Proof. From Remark 2.1(2), we have that A is an (α/L^2)-inverse-strongly accretive mapping. Then, from Lemma 3.1, we have

$$\lim_{n\to\infty} \|Ay_n - Ax_n\| = 0. \tag{3.17}$$

On the other hand, we note that

$$\|Ay_n - Ax_n\| \geq \alpha \|y_n - x_n\|, \tag{3.18}$$

which implies that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0, \tag{3.19}$$

that is,

$$\lim_{n\to\infty} \|Q_C(x_n - \lambda_n Ax_n) - x_n\| = 0. \tag{3.20}$$

Next, we show that

$$\limsup_{n\to\infty} \langle u - Q'u, j(x_n - Q'u) \rangle \leq 0. \tag{3.21}$$

To show (3.21), since $\{x_n\}$ is bounded, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to z such that

$$\limsup_{n\to\infty} \langle u - Q'u, j(x_n - Q'u) \rangle = \limsup_{i\to\infty} \langle u - Q'u, j(x_{n_i} - Q'u) \rangle. \tag{3.22}$$

We first prove $z \in S(C, A)$. Since λ_n is in $[a, b]$, it follows that $\{\lambda_{n_i}\}$ is bounded, and so there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ which converges to $\lambda_0 \in [a, b]$. We may assume, without loss of generality, that $\lambda_{n_i} \to \lambda_0$ as $i \to \infty$. Since Q_C is nonexpansive, it follows that

$$\begin{aligned}
\|Q_C(x_{n_i} - \lambda_0 Ax_{n_i}) - x_{n_i}\| &\leq \|Q_C(x_{n_i} - \lambda_0 Ax_{n_i}) - Q_C(x_{n_i} - \lambda_{n_i} Ax_{n_i})\| \\
&\quad + \|Q_C(x_{n_i} - \lambda_{n_i} Ax_{n_i}) - x_{n_i}\| \\
&\leq \|(x_{n_i} - \lambda_0 Ax_{n_i}) - (x_{n_i} - \lambda_{n_i} Ax_{n_i})\| \\
&\quad + \|Q_C(x_{n_i} - \lambda_{n_i} Ax_{n_i}) - x_{n_i}\| \\
&\leq |\lambda_{n_i} - \lambda_0| \|Ax_{n_i}\| + \|Q_C(x_{n_i} - \lambda_{n_i} Ax_{n_i}) - x_{n_i}\|,
\end{aligned} \tag{3.23}$$

which implies that (noting that (3.20))

$$\lim_{i\to\infty}\|Q_C(I-\lambda_0 A)x_{n_i}-x_{n_i}\|=0. \tag{3.24}$$

By Lemma 2.7 and (3.24), we have $z\in F(Q_C(I-\lambda_0 A))$, and it follows from Lemma 2.6 that $z\in S(C,A)$.

Now, from (3.22) and Lemma 2.4, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u-Q'u, j(x_n-Q'u)\rangle &= \limsup_{i\to\infty}\langle u-Q'u, j(x_{n_i}-Q'u)\rangle\\ &= \langle u-Q'u, j(z-Q'u)\rangle\le 0.\end{aligned} \tag{3.25}$$

Finally, from (1.5) and (3.2), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \langle \alpha_n u+\beta_n x_n+\gamma_n z_n - z, j(x_{n+1}-z)\rangle\\ &= \alpha_n\langle u-z, j(x_{n+1}-z)\rangle+\beta_n\langle x_n-z, j(x_{n+1}-z)\rangle\\ &\quad+\gamma_n\langle z_n-z, j(x_{n+1}-z)\rangle\\ &\le \frac{1}{2}\beta_n\left(\|x_n-z\|^2+\|x_{n+1}-z\|^2\right)+\alpha_n\langle u-z, j(x_{n+1}-z)\rangle\\ &\quad+\frac{1}{2}\gamma_n\left(\|z_n-z\|^2+\|x_{n+1}-z\|^2\right)\\ &\le \frac{1}{2}(1-\alpha_n)\left(\|x_n-z\|^2+\|x_{n+1}-z\|^2\right)\\ &\quad+\alpha_n\langle u-z, j(x_{n+1}-z)\rangle,\end{aligned} \tag{3.26}$$

which implies that

$$\|x_{n+1}-z\|^2\le(1-\alpha_n)\|x_n-z\|^2+2\alpha_n\langle u-z, j(x_{n+1}-z)\rangle. \tag{3.27}$$

Finally, by Lemma 2.9 and (3.27), we conclude that x_n converges strongly to $Q'u$. This completes the proof. □

Remark 3.3. From (3.1), we know that $Q(I-\lambda_n A)$ is nonexpansive. If $S(C,A)\neq\emptyset$, it follows that there exists a sunny nonexpansive retraction Q' of C onto $F(Q(I-\lambda_n A))=S(C,A)$.

4. Application

In this section, we prove a strong convergence theorem in a uniformly convex and 2-uniformly smooth Banach space by using Theorem 3.2. We study the problem of finding a fixed point of a strictly pseudocontractive mapping.

A mapping T of C into itself is said to be strictly pseudocontractive if there exists $0 \leq \sigma < 1$ such that for all $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \sigma \|x-y\|^2. \tag{4.1}$$

This inequality can be written in the following form

$$\langle (I-T)x - (I-T)y, j(x-y) \rangle \geq (1-\sigma)\|x-y\|^2. \tag{4.2}$$

Now we give an application concerning a strictly pseudocontractive mapping.

Theorem 4.1. *Let X be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping, and let C be a nonempty closed convex subset and a sunny nonexpansive retract of X. Let T be a strictly pseudocontractive and L-Lipschitz continuous mapping of C into itself with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $(0,1)$ and $\{\lambda_n\}$ a real number sequence in $[a,b]$ for some a, b with $0 < a < b < 1 - \sigma / K^2(L+1)^2$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

For fixed $u \in C$ and arbitrarily given $x_0 \in C$, define a sequence $\{x_n\}$ iteratively by

$$\begin{aligned} y_n &= (1-\lambda_n)x_n + \lambda_n T x_n, \\ z_n &= (1-\lambda_n)y_n + \lambda_n T y_n, \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n z_n, \end{aligned} \tag{4.3}$$

for every $n = 1, 2, \ldots$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Putting $A = I - T$, we have from (4.2) that A is $(1-\sigma)$-strongly accretive. At the same time, since T is L-Lipschitz continuous, then we have

$$\|Ax - Ay\| = \|(I-T)x - (I-T)y\| \leq (L+1)\|x-y\|, \tag{4.4}$$

for all $x, y \in C$, that is, A is $(L+1)$-Lipschitz continuous mapping. It follows from Remark 2.1 (2) that A is $(1-\sigma)/(L+1)^2$-inverse-strongly accretive mapping. It is easy to show that $S(C, A) = S(C, I-T) = F(T) \neq \emptyset$. Therefore, using Theorem 3.2, we can obtain the desired conclusion. This completes the proof. □

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