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Research Article

Viscosity Approximations by the Shrinking Projection Method of Quasi-Nonexpansive Mappings for Generalized Equilibrium Problems

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We introduce viscosity approximations by using the shrinking projection method established by Takahashi, Takeuchi, and Kubota, for finding a common element of the set of solutions of the generalized equilibrium problem and the set of fixed points of a quasi-nonexpansive mapping. Furthermore, we also consider the viscosity shrinking projection method for finding a common element of the set of solutions of the generalized equilibrium problem and the set of fixed points of the super hybrid mappings in Hilbert spaces.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and C a nonempty closed convex subset of H and let T be a mapping of C into H. Then, $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ is said to be quasi-nonexpansive if $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T) := \{x \in C : Tx = x\}$. Recall that a mapping $\Psi : C \to H$ is said to be δ-inverse strongly monotone if there exists a positive real number δ such that

$$\langle \Psi x - \Psi y, x - y \rangle \geq \delta \|\Psi x - \Psi y\|^2, \quad \forall x, y \in C. \tag{1.1}$$

If Ψ is an δ-inverse strongly monotone mapping of C into H, then it is obvious that Ψ is $1/\delta$-Lipschitz continuous.

Let $F : C \times C \to \mathbb{R}$ be a bifunction and $\Psi : C \to H$ be δ-inverse strongly monotone mapping. The generalized equilibrium problem (for short, GEP) for F and Ψ is to find $z \in C$ such that

$$F(z, y) + \langle \Psi z, y - z \rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The problem (1.2) was studied by Moudafi [1]. The set of solutions for the problem (1.2) is denoted by $\mathrm{GEP}(F, \Psi)$, that is,

$$\mathrm{GEP}(F, \Psi) = \{z \in C : F(z, y) + \langle \Psi z, y - z \rangle \geq 0, \forall y \in C\}. \tag{1.3}$$

If $\Psi \equiv 0$ in (1.2), then GEP reduces to the classical equilibrium problem and $\mathrm{GEP}(F, 0)$ is denoted by $\mathrm{EP}(F)$, that is,

$$\mathrm{EP}(F) = \{z \in C : F(z, y) \geq 0, \forall y \in C\}. \tag{1.4}$$

If $F \equiv 0$ in (1.2), then GEP reduces to the classical variational inequality and $\mathrm{GEP}(0, \Psi)$ is denoted by $\mathrm{VI}(\Psi, C)$, that is,

$$\mathrm{VI}(\Psi, C) = \{z \in C : \langle \Psi z, y - z \rangle \geq 0, \forall y \in C\}. \tag{1.5}$$

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, min-max problems, and the Nash equilibrium problems in noncooperative games, see, for example, Blum and Oettli [2] and Moudafi [3].

In 2005, Combettes and Hirstoaga [4] introduced an iterative algorithm of finding the best approximation to the initial data and proved a strong convergence theorem. In 2007, by using the viscosity approximation method, S. Takahashi and W. Takahashi [5] introduced another iterative scheme for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of a nonexpansive mapping. Subsequently, algorithms constructed for solving the equilibrium problems and fixed point problems have further developed by some authors. In particular, Ceng and Yao [6] introduced an iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings. Maingé and Moudafi [7] introduced an iterative algorithm for equilibrium problems and fixed point problems. Wangkeeree [8] introduced a new iterative scheme for finding the common element of the set of common fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem, and the set of solutions of the variational inequality. Wangkeeree and Kamraksa [9] introduced an iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings, and the set of solutions of a general system of variational inequalities for a cocoercive mapping in a real Hilbert space. Their results extend and improve many results in the literature.

In 1953, Mann [10] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N}, \tag{1.6}$$

where the initial point x_1 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in $[0,1]$. Wittmann [11] obtained the strong convergence results of the sequence $\{x_n\}$ defined by (1.6) to $P_F x_1$ under the following assumptions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$,

where $P_{F(T)}$ is the metric projection of H onto $F(T)$. In 2000, Moudafi [12] introduced the viscosity approximation method for nonexpansive mappings (see [13] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_1 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 1, \tag{1.7}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. It is proved [12, 13] that under conditions (C1), (C2), and (C3) imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.7) strongly converges to the unique fixed point x^* of $P_{F(T)}f$ which is a unique solution of the variational inequality

$$\langle (I-f)x^*, x-x^*\rangle \geq 0, \quad x \in C. \tag{1.8}$$

Suzuki [14] considered the Meir-Keeler contractions, which is extended notion of contractions and studied equivalency of convergence of these approximation methods.

Using the viscosity approximation method, in 2007, S. Takahashi and W. Takahashi [5] introduced an iterative scheme for finding a common element of the solution set of the classical equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Let $T : C \to H$ be a nonexpansive mapping. Starting with arbitrary initial $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tu_n, \quad \forall n \in \mathbb{N}. \end{gathered} \tag{1.9}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(T)\cap \mathrm{EP}(F)$, where $z = P_{F(T)\cap \mathrm{EP}(F)}f(z)$.

On the other hand, in 2008, Takahashi et al. [15] has adapted Nakajo and Takahashi's [16] idea to modify the process (1.6) so that strong convergence has been guaranteed. They proposed the following modification for a family of nonexpansive mappings in a Hilbert space: $x_0 \in H$, $C_1 = C$, $u_1 = P_{C_1}x_0$ and

$$\begin{gathered} y_n = \alpha_n u_n + (1-\alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{gathered} \tag{1.10}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. They proved that if $\{T_n\}$ satisfies the appropriate conditions, then $\{u_n\}$ generated by (1.10) converges strongly to a common fixed point of T_n.

Very recently, Kimura and Nakajo [17] considered viscosity approximations by using the shrinking projection method established by Takahashi et al. [15] and the modified shrinking projection method proposed by Qin et al. [18], for finding a common fixed point of countably many nonlinear mappings, and they obtained some strong convergence theorems.

Motivated by these results, we introduce the viscosity shrinking projection method for finding a common element of the set of solutions of the generalized equilibrium problem and the set of fixed points of a quasi-nonexpansive mapping. Furthermore, we also consider the viscosity shrinking projection method for finding a common element of the set of solutions of the generalized equilibrium problem and the set of fixed points of the super hybrid mappings in Hilbert spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [19], we know the following basic properties. For $x, y \in H$ and $\lambda \in \mathbb{R}$ we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

We also know that for $u, v, x, y \in H$, we have

$$2\langle u-v, x-y\rangle = \|u-y\|^2 + \|v-x\|^2 - \|u-x\|^2 - \|v-y\|^2. \tag{2.2}$$

For every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection from H onto C. It is well known that $z = P_C x \Leftrightarrow \langle x - z, z - y\rangle \geq 0$, for all $x \in H$ and $z, y \in C$. We also know that P_C is firmly nonexpansive mapping from H onto C, that is,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H, \tag{2.3}$$

and so is nonexpansive mapping.

For solving the generalized equilibrium problem, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C, \lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 (see [2]). *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3), and (A4). Then, for any $r > 0$ and $x \in H$, there exists a unique $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.4}$$

Lemma 2.2 (see [4]). *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3), and (A4). Then, for any $r > 0$ and $x \in H$, define a mapping $T_r x : H \to C$ as follows:*

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\} \quad \forall x \in H,\, r \in \mathbb{R}. \tag{2.5}$$

Then the following hold:

(i) T_r *is single-valued;*

(ii) T_r *is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H; \tag{2.6}$$

(iii) $F(T_r) = \mathrm{EP}(F)$;

(iv) $\mathrm{EP}(F)$ *is closed and convex.*

Remark 2.3 (see [20]). Using (ii) in Lemma 2.2 and (2.2), we have

$$\begin{aligned} 2\|T_r x - T_r y\|^2 &\leq 2\langle T_r x - T_r y, x - y\rangle \\ &= \|T_r x - y\|^2 + \|T_r y - x\|^2 - \|T_r x - x\|^2 - \|T_r y - y\|^2. \end{aligned} \tag{2.7}$$

So, for $y \in F(T_r)$ and $x \in H$, we have

$$\|T_r x - u\|^2 + \|T_r x - x\|^2 \leq \|x - u\|^2. \tag{2.8}$$

Remark 2.4. For any $x \in H$ and $r > 0$, by Lemma 2.1, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.9}$$

Replacing x with $x - r\Psi x \in H$ in (2.9), we have

$$F(z, y) + \langle \Psi x, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C, \tag{2.10}$$

where $\Psi : C \to H$ is an inverse-strongly monotone mapping.

For a sequence $\{C_n\}$ of nonempty closed convex subsets of a Hilbert space H, define $s-\mathrm{Li}_n C_n$ and $w-\mathrm{Ls}_n C_n$ as follows.

$x \in s-\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset H$ such that $x_n \to x$ and that $x_n \in C_n$ for all $n \in \mathbb{N}$.

$x \in w-\mathrm{Ls}_n C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a subsequence $\{y_i\} \subset H$ such that $y_i \rightharpoonup y$ and that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$.

If C_0 satisfies

$$C_0 = s-\mathrm{Li}_n C_n = w-\mathrm{Ls}_n C_n, \tag{2.11}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [21] and we write $C_0 = M-\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [21]. Tsukada [22] proved the following theorem for the metric projection.

Theorem 2.5 (see Tsukada [22]). *Let H be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. If $C_0 = M-\lim_{n\to\infty} C_n$ exists and is nonempty, then for each $x \in H$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the metric projections of H onto C_n and C_0, respectively.*

On the other hand, a mapping f of a complete metric space (X, d) into itself is said to be a contraction with coefficient $r \in (0,1)$ if $d(f(x), f(y)) \le r d(x,y)$ for all $x, y \in C$. It is well known that f has a unique fixed point [23]. Meir-Keeler [24] defined the following mapping called Meir-Keeler contraction. Let (X, d) be a complete metric space. A mapping $f : X \to X$ is called a Meir-Keeler contraction if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \le d(x,y) < \varepsilon + \delta$ implies $d(f(x), f(y)) < \varepsilon$ for all $x, y \in X$. It is well known that Meir-Keeler contraction is a generalization of contraction and the following result is proved in [24].

Theorem 2.6 (see Meir-Keeler [24]). *A Meir-Keeler contraction defined on a complete metric space has a unique fixed point.*

We have the following results for Meir-Keeler contractions defined on a Banach space by Suzuki [14].

Theorem 2.7 (see Suzuki [14]). *Let f be a Meir-Keeler contraction on a convex subset C of a Banach space E. Then, for every $\varepsilon > 0$, there exists $r \in (0,1)$ such that $\|x-y\| \ge \varepsilon$ implies $\|f(x) - f(y)\| \le r\|x-y\|$ for all $x, y \in C$.*

Lemma 2.8 (see Suzuki [14]). *Let C be a convex subset of a Banach space E. Let T be a nonexpansive mapping on C, and let f be a Meir-Keeler contraction on C. Then the following hold.*

(i) *$T \circ f$ is a Meir-Keeler contraction on C.*

(ii) *For each $\alpha \in (0,1)$, a mapping $x \mapsto (1-\alpha)Tx + \alpha f x$ is a Meir-Keeler contraction on C.*

3. Main Results

In this section, using the shrinking projection method by Takahashi et al. [15], we prove a strong convergence theorem for a quasi-nonexpansive mapping with a generalized equilibrium problem in a Hilbert space. Before proving it, we need the following lemmas.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $\delta > 0$ and let $\Psi : C \to H$ be δ-inverse strongly monotone. If $0 < \lambda \leq 2\delta$, then $I - \lambda\Psi$ is a nonexpansive mapping.*

Proof. For $x, y \in C$, we can calculate

$$\begin{aligned} \left\|(I-\lambda\Psi)x-(I-\lambda\Psi)y\right\|^2 &= \left\|x-y-\lambda(\Psi x-\Psi y)\right\|^2 \\ &= \left\|x-y\right\|^2 - 2\lambda\langle x-y, \Psi x-\Psi y\rangle + \lambda^2\left\|\Psi x-\Psi y\right\|^2 \\ &\leq \left\|x-y\right\|^2 - 2\lambda\delta\left\|\Psi x-\Psi y\right\|^2 + \lambda^2\left\|\Psi x-\Psi y\right\|^2 \\ &= \left\|x-y\right\|^2 + \lambda(\lambda-2\delta)\left\|\Psi x-\Psi y\right\|^2 \\ &\leq \left\|x-y\right\|^2. \end{aligned} \tag{3.1}$$

Therefore $I - \lambda\Psi$ is nonexpansive. This completes the proof. □

Lemma 3.2. *Let C be a nonempty closed convex subset of H, and let T be a quasi-nonexpansive mapping of C into H. Then, $F(T)$ is closed and convex.*

Proof. We first show that $F(T)$ is closed. Let $\{z_n\}$ be any sequence in $F(T)$ with $z_n \to z$. We claim that $z \in F(T)$. Since C is closed, we have $z \in C$. We observe that

$$\begin{aligned} \|z-Tz\| &\leq \|z-z_n\| + \|z_n-Tz\| \\ &\leq \|z-z_n\| + \|z-z_n\| \\ &= 2\|z-z_n\|. \end{aligned} \tag{3.2}$$

Since $z_n \to z$, we obtain that $\|z-Tz\| \leq 0$ and hence $z = Tz$. This show that $z \in F(T)$.

Next, we show that $F(T)$ is convex. Let $x, y \in F(T)$ and $\alpha \in [0,1]$. We claim that $\alpha x + (1-\alpha)y \in F(T)$. Putting $z = \alpha x + (1-\alpha)y$, we have

$$\begin{aligned} \|z-Tz\|^2 &= \left\|\alpha x + (1-\alpha)y - Tz\right\|^2 \\ &= \left\|\alpha(x-Tz) + (1-\alpha)(y-Tz)\right\|^2 \\ &= \alpha\|x-Tz\|^2 + (1-\alpha)\left\|y-Tz\right\|^2 - \alpha(1-\alpha)\left\|x-y\right\|^2 \\ &\leq \alpha\|x-z\|^2 + (1-\alpha)\left\|y-z\right\|^2 - \alpha(1-\alpha)\left\|x-y\right\|^2 \end{aligned}$$

$$= \alpha \left\| (1-\alpha)(x-y) \right\|^2 + (1-\alpha) \left\| \alpha(y-x) \right\|^2 - \alpha(1-\alpha) \left\| x-y \right\|^2$$
$$= \left\{ \alpha(1-\alpha)^2 + \alpha^2(1-\alpha) - \alpha(1-\alpha) \right\} \left\| x-y \right\|^2$$
$$= \alpha(1-\alpha)(1-\alpha+\alpha-1) \left\| x-y \right\|^2 = 0. \tag{3.3}$$

Hence $F(T)$ is convex. This completes the proof. □

Theorem 3.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4) and let Ψ be a δ-inverse strongly monotone mapping from C into H. Let $T : C \to C$ be a quasi-nonexpansive mapping which is demiclosed on C, that is, if $\{w_k\} \subset C, w_k \rightharpoonup w$ and $(I-T)w_k \to 0$, then $w \in F(T)$. Assume that $\Omega := \mathrm{GEP}(F,\Psi) \cap F(T) \neq \emptyset$ and f is a Meir-Keeler contraction of C into itself. Let the sequence $\{x_n\} \subset C$ be defined by*

$$\begin{gathered} C_1 = C, \qquad x_1 = x \in C, \\ F(z_n, y) + \langle \Psi x_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle z_n - x_n, y - z_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) T z_n, \\ C_{n+1} = \left\{ z \in C_n : \left\| y_n - z \right\| \leq \left\| x_n - z \right\| \right\}, \\ x_{n+1} = P_{C_{n+1}} f(x_n), \quad \forall n \in \mathbb{N}, \end{gathered} \tag{3.4}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0, 2\delta)$ are real sequences satisfying

$$\liminf_{n \to \infty} \alpha_n < 1, \qquad 0 < a \leq \lambda_n \leq b < 2\delta, \tag{3.5}$$

for some $a, b \in \mathbb{R}$. Then, $\{x_n\}$ converges strongly to $z_0 \in \Omega$, which satisfies $z_0 = P_\Omega f(z_0)$.

Proof. Since Ω is a closed convex subset of C, we have that P_Ω is well defined and nonexpansive. Furthermore, we know that f is Meir-Keeler contraction and we know from Lemma 2.8 (i) that $P_\Omega f$ of C onto Ω is a Meir-Keeler contraction on C. By Theorem 2.6, there exists a unique fixed point $z_0 \in C$ such that $z_0 = P_\Omega f(z_0)$. Next, we observe that $z_n = T_{\lambda_n}(x_n - \lambda_n \Psi x_n)$ for each $n \in \mathbb{N}$ and take $z \in F(T) \cap \mathrm{GEP}(F,\Psi)$. From $z = T_{\lambda_n}(z - \lambda_n \Psi z)$ and Lemma 2.2, we have that for any $n \in \mathbb{N}$,

$$\begin{aligned} \|z_n - z\|^2 &= \|T_{\lambda_n}(x_n - \lambda_n \Psi x_n) - T_{\lambda_n}(z - \lambda_n \Psi z)\|^2 \\ &\leq \|(x_n - \lambda_n \Psi x_n) - (z - \lambda_n \Psi z)\|^2 \\ &\leq \|x_n - z\|^2. \end{aligned} \tag{3.6}$$

Next, we divide the proof into several steps.

Step 1. C_n is closed convex and $\{x_n\}$ is well defined for every $n \in \mathbb{N}$.

It is obvious from the assumption that $C_1 := C$ is closed convex and $\Omega \subset C_1$. For any $k \in \mathbb{N}$, suppose that C_k is closed and convex, and $\Omega \subset C_k$. Note that for all $z \in C_k$,

$$\begin{aligned} \|y_k - z\|^2 \leq \|x_k - z\|^2 &\Longleftrightarrow \|y_k - z\|^2 - \|x_k - z\|^2 \leq 0 \\ &\Longleftrightarrow \|y_k\|^2 + \|z\|^2 - 2\langle y_k, z\rangle - \|x_k\|^2 - \|z\|^2 + 2\langle x_k, z\rangle \leq 0 \\ &\Longleftrightarrow \|y_k\|^2 - 2\langle y_k - x_k, z\rangle - \|x_k\|^2 \leq 0. \end{aligned} \tag{3.7}$$

It is easy to see that C_{k+1} is closed. Next, we prove that C_{k+1} is convex. For any $u, v \in C_{k+1}$ and $\alpha \in [0,1]$, we claim that $z := \alpha u + (1-\alpha)v \in C_{k+1}$. Since $u \in C_{k+1}$, we have $\|y_k - u\| \leq \|x_k - u\|$ and so $\|y_k - u\|^2 \leq \|x_k - u\|^2$, that is, $\|y_k\|^2 - 2\langle y_k - x_k, u\rangle - \|x_k\|^2 \leq 0$. Similarly, $v \in C_{k+1}$, we get $\|y_k\|^2 - 2\langle y_k - x_k, v\rangle - \|x_k\|^2 \leq 0$.

Thus,

$$\begin{gathered} \alpha\|y_k\|^2 - 2\langle y_k - x_k, \alpha u\rangle - \alpha\|x_k\|^2 \leq 0, \\ (1-\alpha)\|y_k\|^2 - 2\langle y_k - x_k, (1-\alpha)v\rangle - (1-\alpha)\|x_k\|^2 \leq 0. \end{gathered} \tag{3.8}$$

Combining the above inequalities, we obtain

$$\|y_k\|^2 - 2\langle y_k - x_k, \alpha u + (1-\alpha)v\rangle - \|x_k\|^2 \leq 0. \tag{3.9}$$

Therefore $\|y_k - z\| \leq \|x_k - z\|$. This shows that $z \in C_{k+1}$ and hence C_{k+1} is convex. Therefore C_n is closed and convex for all $n \in \mathbb{N}$.

Next, we show that $\Omega \subset C_n$, for all $n \in \mathbb{N}$. For any $k \in \mathbb{N}$, suppose that $v \in \Omega \subset C_k$. Since T is quasi-nonexpansive and from (3.6), we have

$$\begin{aligned} \|y_k - v\|^2 &= \|\alpha_k x_k + (1-\alpha_k)Tz_k - v\|^2 \\ &= \|\alpha_k(x_k - v) + (1-\alpha_k)(Tz_k - v)\|^2 \\ &\leq \alpha_k\|x_k - v\|^2 + (1-\alpha_k)\|Tz_k - v\|^2 \\ &\leq \alpha_k\|x_k - v\|^2 + (1-\alpha_k)\|z_k - v\|^2 \\ &\leq \alpha_k\|x_k - v\|^2 + (1-\alpha_k)\|x_k - v\|^2 \\ &= \|x_k - v\|^2. \end{aligned} \tag{3.10}$$

So, we have $v \in C_{k+1}$. By principle of mathematical induction, we can conclude that C_n is closed and convex, and $\Omega \subset C_n$, for all $n \in \mathbb{N}$. Hence, we have

$$\emptyset \neq \Omega \subset C_{n+1} \subset C_n, \tag{3.11}$$

for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ is well defined.

Step 2. $\lim_{n\to\infty}\|x_n - u\| = 0$ for some $u \in \bigcap_{n=1}^{\infty} C_n$ and $\langle f(u) - u, u - y\rangle \geq 0$ for all $y \in \Omega$.

Since $\bigcap_{n=1}^{\infty} C_n$ is closed convex, we also have that $P_{\bigcap_{n=1}^{\infty} C_n}$ is well defined and so $P_{\bigcap_{n=1}^{\infty} C_n} f$ is a Meir-Keeler contraction on C. By Theorem 2.6, there exists a unique fixed point $u \in \bigcap_{n=1}^{\infty} C_n$ of $P_{\bigcap_{n=1}^{\infty} C_n} f$. Since C_n is a nonincreasing sequence of nonempty closed convex subsets of H with respect to inclusion, it follows that

$$\emptyset \neq \Omega \subset \bigcap_{n=1}^{\infty} C_n = M - \lim_{n\to\infty} C_n. \tag{3.12}$$

Setting $u_n := P_{C_n} f(u)$ and applying Theorem 2.5, we can conclude that

$$\lim_{n\to\infty} u_n = P_{\bigcap_{n=1}^{\infty} C_n} f(u) = u. \tag{3.13}$$

Next, we will prove that $\lim_{n\to\infty}\|x_n - u\| = 0$. Assume to contrary that $\limsup_{n\to\infty}\|x_n - u\| \neq 0$, there exists $\varepsilon > 0$ and a subsequence $\{\|x_{n_j} - u\|\}$ of $\{\|x_n - u\|\}$ such that

$$\left\|x_{n_j} - u\right\| \geq \varepsilon, \quad \forall j \in \mathbb{N}, \tag{3.14}$$

which gives that

$$\limsup_{j\to\infty}\left\|x_{n_j} - u\right\| \geq \varepsilon > 0. \tag{3.15}$$

We choose a positive number $\varepsilon' > 0$ such that

$$\limsup_{j\to\infty}\left\|x_{n_j} - u\right\| > \varepsilon' > 0. \tag{3.16}$$

For such ε', by the definition of Meir-Keeler contraction, there exists $\delta_{\varepsilon'} > 0$ with

$$\varepsilon' + \delta_{\varepsilon'} < \limsup_{j\to\infty}\left\|x_{n_j} - u\right\|, \tag{3.17}$$

such that

$$\|x - y\| < \varepsilon' + \delta_{\varepsilon'} \quad \text{implies} \quad \|f(x) - f(y)\| < \varepsilon', \tag{3.18}$$

for all $x, y \in C$. Again for such ε', by Theorem 2.7, there exists $r_{\varepsilon'} \in (0,1)$ such that

$$\|x - y\| \geq \varepsilon' + \delta_{\varepsilon'} \quad \text{implies} \quad \|f(x) - f(y)\| < r_{\varepsilon'}\|x - y\|. \tag{3.19}$$

Since $u_n \to u$, there exists $n_0 \in \mathbb{N}$ such that

$$\|u_n - u\| < \delta_{\varepsilon'}, \quad \forall n \geq n_0. \tag{3.20}$$

By the idea of Suzuki [14] and Kimura and Nakajo [17], we consider the following two cases.

Case I. Assume that there exists $n_1 \geq n_0$ such that

$$\|x_{n_1} - u\| < \varepsilon' + \delta_{\varepsilon'}. \tag{3.21}$$

Thus, we get

$$\begin{aligned}
\|x_{n_1+1} - u\| &\leq \|x_{n_1+1} - u_{n_1+1}\| + \|u_{n_1+1} - u\| \\
&= \left\|P_{C_{n_1+1}} f(x_{n_1}) - P_{C_{n_1+1}} f(u)\right\| + \|u_{n_1+1} - u\| \\
&\leq \left\|f(x_{n_1}) - f(u)\right\| + \|u_{n_1+1} - u\| \\
&< \varepsilon' + \delta_{\varepsilon'}.
\end{aligned} \tag{3.22}$$

By induction on $\{n\}$, we can obtain that

$$\|x_n - u\| < \varepsilon' + \delta_{\varepsilon'}, \tag{3.23}$$

for all $n \geq n_0$. In particular, for all $j \geq n_0$, we have $n_j \geq j \geq n_0$ and

$$\left\|x_{n_j} - u\right\| < \varepsilon' + \delta_{\varepsilon'}. \tag{3.24}$$

This implies that

$$\limsup_{j \to \infty} \left\|x_{n_j} - u\right\| \leq \varepsilon' + \delta_{\varepsilon'} < \limsup_{j \to \infty} \left\|x_{n_j} - u\right\|, \tag{3.25}$$

which is a contradiction. Therefore, we conclude that $\|x_n - u\| \to 0$ as $n \to \infty$.

Case II. Assume that

$$\|x_n - u\| \geq \varepsilon' + \delta_{\varepsilon'}, \quad \forall n \geq n_0. \tag{3.26}$$

By (3.19), we have

$$\left\|f(x_n) - f(u)\right\| < r_{\varepsilon'} \|x_n - u\|, \quad \forall n \geq n_0. \tag{3.27}$$

Thus, we have

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &= \left\|P_{C_{n+1}} f(x_n) - P_{C_{n+1}} f(u)\right\| \\
&\leq \left\|f(x_n) - f(u)\right\| \\
&\leq r_{\varepsilon'} \|x_n - u\| \\
&\leq r_{\varepsilon'} (\|x_n - u_n\| + \|u_n - u\|),
\end{aligned} \tag{3.28}$$

for every $n \geq n_0$. In particular, we have

$$\left\|x_{n_j+1} - u_{n_j+1}\right\| \leq r_{\varepsilon'}\left(\left\|x_{n_j} - u_{n_j}\right\| + \left\|u_{n_j} - u\right\|\right), \tag{3.29}$$

for every $j \geq n_0$ $(n_j \geq j \geq n_0)$. Let us consider

$$\begin{aligned}
\limsup_{n\to\infty}\left\|x_{n_j} - u_{n_j}\right\| &= \limsup_{j\to\infty}\left\|x_{n_j+1} - u_{n_j+1}\right\| \\
&\leq r_{\varepsilon'}\limsup_{j\to\infty}\left(\left\|x_{n_j} - u_{n_j}\right\| + \left\|u_{n_j} - u\right\|\right) \\
&\leq r_{\varepsilon'}\limsup_{j\to\infty}\left\|x_{n_j} - u_{n_j}\right\| + r_{\varepsilon'}\limsup_{j\to\infty}\left\|u_{n_j} - u\right\| \\
&= r_{\varepsilon'}\limsup_{j\to\infty}\left\|x_{n_j} - u_{n_j}\right\| \\
&< \limsup_{j\to\infty}\left\|x_{n_j} - u_{n_j}\right\|,
\end{aligned} \tag{3.30}$$

which gives a contradiction. Hence, we obtain that

$$\lim_{n\to\infty}\|x_n - u\| = 0, \tag{3.31}$$

and therefore $\{x_n\}$ is bounded. Moreover, $\{f(x_n)\}, \{z_n\}$, and $\{y_n\}$ are also bounded. Since $x_{n+1} = P_{C_{n+1}} f(x_n)$, we have

$$\left\langle f(x_n) - x_{n+1}, x_{n+1} - y\right\rangle \geq 0, \quad \forall y \in C_{n+1}. \tag{3.32}$$

Since $\Omega \subset C_{n+1}$, we get

$$\left\langle f(x_n) - x_{n+1}, x_{n+1} - y\right\rangle \geq 0, \quad \forall n \in \mathbb{N},\ y \in \Omega. \tag{3.33}$$

We have from $x_n \to u$ that

$$\left\langle f(u) - u, u - y\right\rangle \geq 0, \quad \forall y \in \Omega. \tag{3.34}$$

Step 3. There exists a subsequence $\{\|x_{n_i} - z_{n_i}\|\}$ of $\{\|x_n - z_n\|\}$ such that $\|x_{n_i} - z_{n_i}\| \to 0$ as $i \to \infty$.

We have from (3.13) and (3.31) that

$$\begin{aligned}
\|x_n - x_{n+1}\| &\leq \|x_n - u\| + \|u - u_{n+1}\| + \|u_{n+1} - x_{n+1}\| \\
&= \|x_n - u\| + \|u - u_{n+1}\| + \left\|P_{C_{n+1}} f(x_n) - P_{C_{n+1}} f(u)\right\| \\
&\leq \|x_n - u\| + \|u - u_{n+1}\| + \left\|f(x_n) - f(u)\right\| \longrightarrow 0.
\end{aligned} \tag{3.35}$$

From $x_{n+1} \in C_{n+1}$, we have that

$$\left\|y_n - x_{n+1}\right\| \le \|x_n - x_{n+1}\|, \tag{3.36}$$

and so $\|y_n - x_{n+1}\| \to 0$. We also have

$$\left\|y_n - x_n\right\| \le \left\|y_n - x_{n+1}\right\| + \|x_{n+1} - x_n\| \longrightarrow 0. \tag{3.37}$$

From $\liminf_{n\to\infty}\alpha_n < 1$, there exists a subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ and α_0 with $0 \le \alpha_0 < 1$ such that $\alpha_{n_i} \to \alpha_0$. Since $\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha)Tz_n\| = (1-\alpha_n)\|x_n - Tz_n\|$, we have

$$\|Tz_{n_i} - x_{n_i}\| \longrightarrow 0 \quad \text{as } i \longrightarrow \infty. \tag{3.38}$$

Using Lemma 2.2 (ii) and (3.6), we have

$$\begin{aligned}
\|z_n - z\|^2 &= \|T_{\lambda_n}(x_n - \lambda_n \Psi x_n) - T_{\lambda_n}(z - \lambda_n \Psi z)\|^2 \\
&\le \langle (x_n - \lambda_n \Psi x_n) - (z - \lambda_n \Psi z), z_n - z\rangle \\
&= -\langle (x_n - \lambda_n \Psi x_n) - (z - \lambda_n \Psi z), z - z_n\rangle \\
&= \frac{1}{2}\Big(\|(x_n - \lambda_n \Psi x_n) - (z - \lambda_n \Psi z)\|^2 + \|z_n - z\|^2 \\
&\qquad - \|(x_n - \lambda_n \Psi x_n) - (z - \lambda_n \Psi z) + (z - z_n)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_n - z\|^2 + \|z_n - z\|^2 - \|(x_n - z_n) - \lambda_n(\Psi x_n - \Psi z)\|^2\Big) \\
&= \frac{1}{2}\Big(\|x_n - z\|^2 + \|z_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, \Psi x_n - \Psi z\rangle - \lambda_n^2\|\Psi x_n - \Psi z\|^2\Big).
\end{aligned} \tag{3.39}$$

So, we have

$$\|z_n - z\|^2 \le \|x_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, \Psi x_n - \Psi z\rangle - \lambda_n^2\|\Psi x_n - \Psi z\|^2. \tag{3.40}$$

Let us consider

$$\begin{aligned}
\left\|y_n - z\right\|^2 &= \|\alpha_n(x_n - z) + (1-\alpha_n)(Tz_n - z)\|^2 \\
&\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|Tz_n - z\|^2 \\
&\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\
&= \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|T_{\lambda_n}(I - \lambda_n\Psi)x_n - T_{\lambda_n}(I - \lambda_n\Psi)z\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|(I-\lambda_n\Psi)x_n - (I-\lambda_n\Psi)z\|^2 \\
&= \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|(x_n - z) - \lambda_n(\Psi x_n - \Psi z)\|^2 \\
&= \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 + (1-\alpha_n)\lambda_n^2\|\Psi x_n - \Psi z\|^2 \\
&\quad - 2(1-\alpha_n)\lambda_n\langle x_n - z, \Psi x_n - \Psi z\rangle \\
&\leq \|x_n - z\|^2 + (1-\alpha_n)\lambda_n^2\|\Psi x_n - \Psi z\|^2 - 2(1-\alpha_n)\lambda_n\delta\|\Psi x_n - \Psi z\|^2 \\
&= \|x_n - z\|^2 + (1-\alpha_n)(\lambda_n - 2\delta)\lambda_n\|\Psi x_n - \Psi z\|^2 \\
&\leq \|x_n - z\|^2 + (1-\alpha_n)(b - 2\delta)b\|\Psi x_n - \Psi z\|^2.
\end{aligned} \tag{3.41}$$

In particular, we have

$$\begin{aligned}
(1-\alpha_{n_i})(2\delta - b)b\|\Psi x_{n_i} - \Psi z\|^2 &\leq \|x_{n_i} - z\|^2 - \left\|y_{n_i} - z\right\|^2 \\
&\leq \left\|x_{n_i} - y_{n_i}\right\|^2 + 2\left\|x_{n_i} - y_{n_i}\right\|\left\|y_{n_i} - z\right\|.
\end{aligned} \tag{3.42}$$

Since $\alpha_{n_i} \to \alpha_0$ with $\alpha_0 < 1$ and $\|x_{n_i} - y_{n_i}\| \to 0$, we obtain that

$$\|\Psi x_{n_i} - \Psi z\| \longrightarrow 0. \tag{3.43}$$

Using (3.40), we have

$$\begin{aligned}
\left\|y_n - z\right\|^2 &\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\
&\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n) \\
&\quad \times \left(\|x_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, \Psi x_n - \Psi z\rangle - \lambda_n^2\|\Psi x_n - \Psi z\|^2\right) \\
&\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\left(\|x_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|\Psi x_n - \Psi z\|\right) \\
&\leq \|x_n - z\|^2 - (1-\alpha_n)\|x_n - z_n\|^2 + 2(1-\alpha_n)\lambda_n(\|x_n\| + \|z_n\|)\|\Psi x_n - \Psi z\| \\
&\leq \|x_n - z\|^2 - (1-\alpha_n)\|x_n - z_n\|^2 + 2(1-\alpha_n)bM\|\Psi x_n - \Psi z\|,
\end{aligned} \tag{3.44}$$

where $M := \sup\{\|x_n\| + \|y_n\| : n \in \mathbb{N}\}$.

So, we have

$$\begin{aligned}
(1-\alpha_{n_i})\|x_{n_i} - z_{n_i}\|^2 &\leq \|x_{n_i} - z\|^2 - \left\|y_{n_i} - z\right\|^2 + 2(1-\alpha_{n_i})bM\|\Psi x_{n_i} - \Psi z\| \\
&\leq \left\|x_{n_i} - y_{n_i}\right\|^2 + 2\left\|x_{n_i} - y_{n_i}\right\|\left\|y_{n_i} - z\right\| + 2(1-\alpha_{n_i})bM\|\Psi x_{n_i} - \Psi z\|.
\end{aligned} \tag{3.45}$$

We have from $\alpha_{n_i} \to \alpha_0$, (3.37), and (3.43) that

$$\|x_{n_i} - z_{n_i}\| \longrightarrow 0. \tag{3.46}$$

Step 4. Finally, we prove that $u \in \Omega := F(T) \cap \mathrm{GEP}(F, \Psi)$.

Since $y_n = \alpha_n x_n + (1 - \alpha_n) T z_n$, we have $y_n - T z_n = \alpha_n (x_n - T z_n)$. So, from (3.38) we have

$$\left\|y_{n_i} - T z_{n_i}\right\| = \alpha_{n_i} \|x_{n_i} - T z_{n_i}\| \longrightarrow 0. \tag{3.47}$$

Since $\|z_{n_i} - T z_{n_i}\| \le \|z_{n_i} - x_{n_i}\| + \|x_{n_i} - y_{n_i}\| + \|y_{n_i} - T z_{n_i}\|$, from (3.37), (3.46), and (3.47) we have

$$\|z_{n_i} - T z_{n_i}\| \longrightarrow 0. \tag{3.48}$$

Since $x_{n_i} \to u$, we have $z_{n_i} \to u$. So, from (3.48) and the demiclosed property of T, we have

$$u \in F(T). \tag{3.49}$$

We next show that $u \in \mathrm{GEP}(F, \Psi)$. Since $z_n = T_{\lambda_n}(x_n - \lambda_n \Psi x_n)$, for any $y \in C$ we have

$$F(z_n, y) + \langle \Psi x_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \ge 0. \tag{3.50}$$

From (A2), we have

$$-F(y, z_n) + \langle \Psi x_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \ge 0 \tag{3.51}$$

and so

$$\langle \Psi x_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \ge F(y, z_n). \tag{3.52}$$

Replacing n by n_i, we have

$$\langle \Psi x_{n_i}, y - z_{n_i} \rangle + \left\langle y - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \ge F(y, z_{n_i}). \tag{3.53}$$

Note that Ψ is $1/\delta$-Lipschitz continuous, and from (3.46), we have

$$\|\Psi z_{n_i} - \Psi x_{n_i}\| \longrightarrow 0. \tag{3.54}$$

For $t \in (0,1]$ and $y \in C$, let $z_t^* = ty + (1-t)u$. Since C is convex, we have $z_t^* \in C$. So, from (3.53) we have

$$\begin{aligned}
\langle z_t^* - z_{n_i}, \Psi z_t^* \rangle &\geq \langle z_t^* - z_{n_i}, \Psi z_t^* \rangle - \langle z_t^* - z_{n_i}, \Psi x_{n_i} \rangle - \left\langle z_t^* - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle + F(z_t^*, z_{n_i}) \\
&= \langle z_t^* - z_{n_i}, \Psi z_t^* - \Psi x_{n_i} \rangle - \left\langle z_t^* - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle + F(z_t^*, z_{n_i}) \\
&= \langle z_t^* - z_{n_i}, \Psi z_t^* - \Psi z_{n_i} \rangle + \langle z_t^* - z_{n_i}, \Psi z_{n_i} - \Psi x_{n_i} \rangle \\
&\quad - \left\langle z_t^* - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle + F(z_t^*, z_{n_i}).
\end{aligned} \tag{3.55}$$

From $\langle z_t^* - z_{n_i}, \Psi z_t^* - \Psi z_{n_i} \rangle \geq 0$, we have

$$\langle z_t^* - z_{n_i}, \Psi z_t^* \rangle \geq \langle z_t^* - z_{n_i}, \Psi z_{n_i} - \Psi x_{n_i} \rangle - \left\langle z_t^* - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle + F(z_t^*, z_{n_i}). \tag{3.56}$$

Thus,

$$\langle z_t^* - z_{n_i}, \Psi z_t^* \rangle - \| z_t^* - z_{n_i} \| \| \Psi z_{n_i} - \Psi x_{n_i} \| \geq - \| z_t^* - z_{n_i} \| \left\| \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\| + F(z_t^*, z_{n_i}). \tag{3.57}$$

From Step 3 and (3.54), we obtain

$$\langle z_t^* - u, \Psi z_t^* \rangle \geq F(z_t^*, u). \tag{3.58}$$

From (A1), (A4), and (3.58), we have

$$\begin{aligned}
0 = F(z_t^*, z_t^*) &= F\big(z_t^*, ty + (1-t)u\big) \\
&\leq tF\big(z_t^*, y\big) + (1-t)F(z_t^*, u) \\
&\leq tF\big(z_t^*, y\big) + (1-t)\langle z_t^* - u, \Psi z_t^* \rangle \\
&\leq tF\big(z_t^*, y\big) + (1-t)t\langle y - u, \Psi z_t^* \rangle,
\end{aligned} \tag{3.59}$$

and hence

$$0 \leq F\big(z_t^*, y\big) + (1-t)\big\langle y - u, \Psi z_t^* \big\rangle. \tag{3.60}$$

Letting $t \downarrow 0$ and from (A3), we have that for each $y \in C$,

$$
\begin{aligned}
0 &\leq \lim_{t\downarrow 0}\left(F\left(z_t^*, y\right) + (1-t)\left\langle y-u, \Psi z_t^*\right\rangle\right) \\
&= \lim_{t\downarrow 0}\left(F\left(ty+(1-t)u, y\right) + (1-t)\left\langle y-u, t\Psi y + (1-t)\Psi u\right\rangle\right) \\
&\leq F(u,y) + \langle y-u, \Psi u\rangle.
\end{aligned}
\tag{3.61}
$$

This implies that $u \in \mathrm{GEP}(F,\Psi)$. So, we have $u \in F(T) \cap \mathrm{GEP}(F,\Psi)$. We obtain from (3.34) that $u = z_0$ and hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. □

By Theorem 3.3, we can obtain some new and interesting strong convergence theorems. Now we give some examples as follows.

Setting $f(x_n) = x, \forall n \in \mathbb{N}$ in Theorem 3.3, we obtain the following result.

Corollary 3.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4) and let Ψ be a δ-inverse strongly monotone mapping from C into H. Let $T : C \to C$ be a quasi-nonexpansive mapping which is demiclosed on C. Assume that $\Omega \neq \emptyset$ and let $C_1 = C$ and $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$
\begin{gathered}
F\left(z_n, y\right) + \left\langle \Psi x_n, y - z_n\right\rangle + \frac{1}{\lambda_n}\left\langle z_n - x_n, y - z_n\right\rangle \geq 0, \quad \forall y \in C, \\
y_n = \alpha_n x_n + (1-\alpha_n) T z_n, \\
C_{n+1} = \left\{ z \in C_n : \left\| y_n - z\right\| \leq \left\| x_n - z\right\| \right\}, \\
x_{n+1} = P_{C_{n+1}} x, \quad \forall n \in \mathbb{N},
\end{gathered}
\tag{3.62}
$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [0,2\delta)$ are sequences such that

$$
\liminf_{n\to\infty} \alpha_n < 1, \qquad 0 < a \leq \lambda_n \leq b < 2\delta,
\tag{3.63}
$$

for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 = P_\Omega z_0$.

Setting $\Psi \equiv 0$ in Theorem 3.3, we obtain the following result.

Corollary 3.5. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $T : C \to C$ be a quasi-nonexpansive mapping which is demiclosed on C. Assume that $\mathrm{EP}(F) \cap F(T) \neq \emptyset$ and f is a Meir-Keeler contraction of C into itself. Let $C_1 = C$ and $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{gathered} F(z_n, y) + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} f(x_n), \quad \forall n \in \mathbb{N}, \end{gathered} \tag{3.64}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [0,\infty)$ are sequences such that

$$\liminf_{n\to\infty} \alpha_n < 1, \qquad 0 < a \leq \lambda_n, \tag{3.65}$$

for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(T) \cap \mathrm{EP}(F)$.

Setting $\Psi \equiv 0$ and $f(x_n) = x$ for all $n \in \mathbb{N}$ in Theorem 3.3, we obtain the following result.

Corollary 3.6. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $T : C \to C$ be an quasi-nonexpansive mapping which is demiclosed on C and assume that $\mathrm{EP}(F) \cap F(T) \neq \emptyset$. Let $C_1 = C$ and $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{gathered} F(z_n, y) + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{3.66}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [0,\infty)$ are sequences such that

$$\liminf_{n\to\infty} \alpha_n < 1, \qquad 0 < a \leq \lambda_n, \tag{3.67}$$

for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(T) \cap \mathrm{EP}(F)$.

Next, using the CQ hybrid method introduced by Nakajo and Takahashi [16], we prove a strong convergence theorem of a quasi-nonexpansive mapping for solving the generalized equilibrium problem.

Theorem 3.7. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C\times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4) and let Ψ be a δ-inverse strongly monotone mapping from C into H. Let $T : C \to C$ be an quasi-nonexpansive mapping which is demiclosed on C. Assume that $\Omega \neq \emptyset$ and f is a Meir-Keeler contraction of C into itself. Let $Q_1 = C$ and $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{gathered} F(z_n, y) + \langle \Psi x_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in Q_{n-1} : \langle f(x_{n-1}) - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n} f(x_n), \quad \forall n \in \mathbb{N}, \end{gathered} \tag{3.68}$$

where $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,2\delta)$ satisfy

$$0 \leq \alpha_n \leq b < 1, \qquad 0 < c \leq \lambda_n \leq d < 2\delta, \tag{3.69}$$

for some $b, c, d \in \mathbb{R}$. Then, $\{x_n\}$ converges strongly to $z_0 \in \Omega$, which satisfies $z_0 = P_\Omega f(z_0)$.

Proof. As in the proof of Theorem 3.3, we have that the mapping $P_\Omega f$ of C onto Ω is a Meir-Keeler contraction on C. By Theorem 2.6, there exists a unique fixed point $z_0 \in C$ such that $z_0 = P_\Omega f(z_0)$. Next, it is clear that C_n is closed and convex. Next, we will show that Q_n is closed and convex for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, let $\{z_k\}$ be a sequence in Q_n such that $z_k \to z$. For each $k \in \mathbb{N}$, we observe that

$$0 \leq \langle x_n - z_k, f(x_{n-1}) - x_n\rangle = \frac{1}{2}\left(\|f(x_{n-1}) - z_k\|^2 - \|x_n - z_k\|^2 - \|f(x_{n-1}) - x_n\|^2\right). \tag{3.70}$$

Taking $k \to \infty$, we get $\langle x_n - z, f(x_{n-1}) - x_n\rangle \geq 0$ and then $z \in Q_n$. Therefore Q_n is closed.

Next, we will show that Q_n is convex. For any $u, v \in Q_n$, and $\alpha \in [0,1]$, put $z = \alpha u + (1-\alpha)v$. We claim that $z \in Q_n$. Since $u \in Q_n$, we have $\langle \alpha x_n - \alpha u, f(x_{n-1}) - x_n\rangle \geq 0$. Similarly, since $v \in Q_n$, we have $\langle (1-\alpha)x_n - (1-\alpha)v, f(x_{n-1}) - x_n\rangle \geq 0$. Thus,

$$\begin{aligned} 0 &\leq \langle \alpha x_n - \alpha u + (1-\alpha)x_n - (1-\alpha)v, f(x_{n-1}) - x_n\rangle \\ &= \langle x_n - \alpha u - (1-\alpha)v, f(x_{n-1}) - x_n\rangle \\ &= \langle x_n - z, f(x_{n-1}) - x_n\rangle. \end{aligned} \tag{3.71}$$

It follows that $z \in Q_n$, and therefore we have that Q_n is convex. We obtain from both C_n and Q_n which are closed convex sets for every $n \in \mathbb{N}$ that $C_n \cap Q_n$ is closed and convex for every $n \in \mathbb{N}$.

Next, we will show that $C_n \cap Q_n$ is nonempty. Let $z \in F(T) \cap \mathrm{GEP}(F, \Psi)$. We will show that $z \in C_n$ for any $n \in \mathbb{N}$. We notice that $z_n = T_{\lambda_n}(x_n - \lambda_n \Psi x_n)$ for each $n \in \mathbb{N}$ and

$z = T_{\lambda_n}(z - \lambda_n \Psi z)$. From Ψ which is an inverse strongly monotone mapping Lemma 2.2 (ii), Lemma 3.1, we obtain

$$\|z_n - z\| \le \|x_n - z\|, \quad \text{for any } n \in \mathbb{N}. \tag{3.72}$$

Since T is quasi-nonexpansive with the fixed point z and from (3.72), we have

$$\left\|y_n - z\right\|^2 \le \|x_n - z\|^2. \tag{3.73}$$

So, we have $z \in C_n$. Therefore $F(T) \cap \mathrm{GEP}(F, \Psi) \subset C_n$, for all $n \in \mathbb{N}$.

Next, we will show that

$$F(T) \cap \mathrm{GEP}(F, \Psi) \subset C_n \cap Q_n, \quad \forall n \in \mathbb{N}. \tag{3.74}$$

It is obvious that $F(T) \cap \mathrm{GEP}(F, \Psi) \subset C = Q_1$. Hence

$$F(T) \cap \mathrm{GEP}(F, \Psi) \subset C_1 \cap Q_1. \tag{3.75}$$

For any $k \in \mathbb{N}$, suppose that

$$F(T) \cap \mathrm{GEP}(F, \Psi) \subset C_k \cap Q_k. \tag{3.76}$$

Since $x_{k+1} = P_{C_k \cap Q_k} f(x_k)$, we have

$$\left\langle f(x_k) - x_{k+1}, x_{k+1} - z \right\rangle \ge 0, \quad \forall z \in C_k \cap Q_k. \tag{3.77}$$

In particular, for any $z \in F(T) \cap \mathrm{GEP}(F, \Psi)$, we obtain that

$$\left\langle f(x_k) - x_{k+1}, x_{k+1} - z \right\rangle \ge 0. \tag{3.78}$$

This shows that $z \in Q_{k+1}$. Hence $F(T) \cap \mathrm{GEP}(F, \Psi) \subset Q_{k+1}$. Therefore, we conclude that

$$F(T) \cap \mathrm{GEP}(F, \Psi) \subset C_{k+1} \cap Q_{k+1}. \tag{3.79}$$

By principle of mathematical induction, we can conclude that

$$F(T) \cap \mathrm{GEP}(F, \Psi) \subset C_n \cap Q_n, \quad \forall n \in \mathbb{N}. \tag{3.80}$$

Hence $\{x_n\}$ is well defined. Since $P_{\bigcap_{n=1}^{\infty} Q_n} f$ is a Meir-Keeler contraction on C, there exists a unique element $u \in C$ such that $u = P_{\bigcap_{n=1}^{\infty} Q_n} f(u)$. For each n, let $u_n = P_{Q_n} f(u)$. Since $F(T) \cap \mathrm{GEP}(F, \Psi) \subset Q_{n+1} \subset Q_n$, we have from Theorem 2.5 that $u_n \to u$. Notice that $x_n = P_{Q_n} f(x_{n-1})$. Thus, as in the proof of Theorem 3.3, we get $x_n \to u$ and hence $\{x_n\}$ is bounded. Moreover,

$$\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0, \qquad \lim_{n \to \infty} \left\|y_n - x_n\right\| = 0. \tag{3.81}$$

As the proof of Theorem 3.3, we have that

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)(\lambda_n - 2\delta)\lambda_n\|\Psi x_n - \Psi z\|^2. \tag{3.82}$$

Thus,

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-a)(d-2\delta)d\|\Psi x_n - \Psi z\|^2 \tag{3.83}$$

and so

$$\begin{aligned}(1-a)(2\delta - d)d\|\Psi x_n - \Psi z\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\ &\leq \|x_n - y_n\|^2 + 2\|x_n - y_n\|\|y_n - z\|.\end{aligned} \tag{3.84}$$

Since $(1-a)(2\delta - d)d > 0$ and $\|x_n - y_n\| \to 0$, we obtain that

$$\|\Psi x_n - \Psi z\| \longrightarrow 0. \tag{3.85}$$

Using (3.40) in Theorem 3.3, we have

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 - (1-b)\|x_n - z_n\|^2 + 2(1-a)dM\|\Psi x_n - \Psi z\|, \tag{3.86}$$

where $M := \sup\{\|x_n\| + \|y_n\|\}$. So, we have

$$\begin{aligned}(1-b)\|x_n - z_n\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 + 2(1-a)dM\|\Psi x_n - \Psi z\| \\ &\leq \|x_n - y_n\|^2 + 2\|x_n - y_n\|\|y_n - z\| + 2(1-a)dM\|\Psi x_n - \Psi z\|.\end{aligned} \tag{3.87}$$

We have from $1 - b > 0$, $\|x_n - y_n\| \to 0$ and (3.85) that

$$\|x_n - z_n\| \longrightarrow 0, \tag{3.88}$$

which implies that

$$z_n \longrightarrow u. \tag{3.89}$$

Notice that

$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha)Tz_n\| = (1-\alpha_n)\|x_n - Tz_n\|, \tag{3.90}$$

and from $\limsup_{n\to\infty}\alpha_n \leq b < 1$, we have

$$\|Tz_n - x_n\| \longrightarrow 0. \tag{3.91}$$

Since $y_n = \alpha_n x_n + (1-\alpha_n)Tz_n$, we have $y_n - Tz_n = \alpha_n(x_n - Tz_n)$. So, from $0 < a \leq \alpha_n \leq b < 1$ and (3.91), we obtain

$$\|y_n - Tz_n\| \longrightarrow 0 \tag{3.92}$$

and hence

$$\|z_n - Tz_n\| \longrightarrow 0. \tag{3.93}$$

From (3.89), (3.93), and the demiclosed property of T, we have $u \in F(T)$. As in the proof of Theorem 3.3 we have that $u \in F(T) \cap \mathrm{GEP}(F,\Psi)$. Since $F(T) \cap \mathrm{GEP}(F,\Psi) \subset Q_{n+1}$, we get

$$\langle f(x_n) - x_{n+1}, x_{n+1} - y \rangle \geq 0, \tag{3.94}$$

for all $n \in \mathbb{N}$ and $y \in F(T) \cap \mathrm{GEP}(F,\Psi)$. We have from $x_n \to u$ that

$$\langle f(u) - u, u - y \rangle \geq 0, \tag{3.95}$$

for all $y \in F(T) \cap \mathrm{GEP}(F,\Psi)$, which implies that $u = P_\Omega f(u)$. It follows that $u = z_0$, since $z_0 \in F(T) \cap \mathrm{GEP}(F,\Psi)$ of $P_\Omega f$ is unique. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. □

Setting $f(x_n) = x$, for all $n \in \mathbb{N}$ in Theorem 3.7, we obtain the following result.

Corollary 3.8. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4) and let Ψ be a δ-inverse strongly monotone mapping from C into H and let $T : C \to C$ be a quasi-nonexpansive mapping which is demiclosed on C. Assume that $\Omega := \mathrm{GEP}(F,\Psi) \cap F(T) \neq \emptyset$. Let $Q_1 = C$ and $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{gathered} F(z_n, y) + \langle \Psi x_n, y - z_n \rangle + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in Q_{n-1} : \langle x - x_n, x_n - z \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{3.96}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,2\delta)$ satisfy

$$0 \leq \alpha_n \leq b < 1, \qquad 0 < c \leq \lambda_n \leq d < 2\delta, \tag{3.97}$$

for some $a, b, c, d \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 = P_\Omega f(z_0)$.

Setting $\Psi \equiv 0$ in Theorem 3.7, we obtain the following result.

Corollary 3.9. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4) and let $T : C \to C$ be an quasi-nonexpansive mapping which is demiclosed on C. Assume that $\mathrm{EP}(F) \cap F(T) \neq \emptyset$ and f is a Meir-Keeler contraction of C into itself. Let $Q_1 = C$ and $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{gathered}
F(z_n, y) + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = \alpha_n x_n + (1-\alpha_n) T z_n, \\
C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in Q_{n-1} : \langle f(x_{n-1}) - x_n, x_n - z\rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} f(x_n), \quad \forall n \in \mathbb{N},
\end{gathered} \tag{3.98}$$

where $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy

$$0 \leq \alpha_n \leq b < 1, \qquad 0 < c \leq \lambda_n, \tag{3.99}$$

for some $a, b, c \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(T) \cap \mathrm{EP}(F)$.

Setting $\Psi \equiv 0$ and $f(x_n) = x, \forall n \in \mathbb{N}$ in Theorem 3.7, we obtain the following result.

Corollary 3.10. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4) and let $T : C \to C$ be a quasi-nonexpansive mapping which is demiclosed on H. Assume that $\mathrm{EP}(F) \cap F(T) \neq \emptyset$. Let $Q_1 = C$ and $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{gathered}
F(z_n, y) + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = \alpha_n x_n + (1-\alpha_n) T z_n, \\
C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in Q_{n-1} : \langle x - x_n, x_n - z\rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N},
\end{gathered} \tag{3.100}$$

where $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy

$$0 < a \leq \alpha_n \leq b < 1, \qquad 0 < c \leq \lambda_n, \tag{3.101}$$

for some $a, b, c \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(T) \cap \mathrm{EP}(F)$.

4. Applications

In this section, we present some convergence theorems deduced from the results in the previous section. Recall that a mapping $T : C \to H$ is said to be nonspreading if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2 \tag{4.1}$$

for all $x, y \in C$. Further, a mapping $T : C \to H$ is said to be hybrid if

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \tag{4.2}$$

for all $x, y \in C$. These mappings are deduced from a firmly nonexpansive mapping in a Hilbert space.

A mapping $F : C \to H$ is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \le \langle x - y, Fx - Fy\rangle, \tag{4.3}$$

for all $x, y \in C$; see, for instance, Browder [25] and Goebel and Kirk [26]. We also know that a firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space.

Recently, Kocourek et al. [27] introduced a more broad class of nonlinear mappings called generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2, \tag{4.4}$$

for all $x, y \in C$. Very recently, they defined a more broad class of mappings than the class of generalized hybrid mappings in a Hilbert space. A mapping $S : C \to H$ is called super hybrid if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\|Sx - Sy\|^2 + (1 - \alpha + \gamma)\|x - Sy\|^2 \\ &\quad \le (\beta + (\beta - \alpha)\gamma)\|Sx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \\ &\qquad + (\alpha - \beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sy\|^2, \end{aligned} \tag{4.5}$$

for all $x, y \in C$. We call such a mapping an (α, β, γ)-super hybrid mapping. We notice that an $(\alpha, \beta, 0)$-super hybrid mapping is (α, β)-generalized hybrid. So, the class of super hybrid mappings contains the class of generalized hybrid mappings. A super hybrid mapping is not quasi-nonexpansive generally. For more details, see [20]. Before proving, we need the following lemmas.

Lemma 4.1 (see [20]). *Let C be a nonempty subset of a Hilbert space H and let α, β, and γ be real numbers with $\gamma \neq -1$. Let S and T be mappings of C into H such that $T = (1/(1+\gamma))S + (\gamma/(1+\gamma))I$. Then, S is (α, β, γ)-super hybrid if and only if T is (α, β)-generalized hybrid. In this case, $F(S) = F(T)$.*

Lemma 4.2 (see [20]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to H$ be a generalized hybrid mapping. Then T is demiclosed on C.*

Setting $T := (1/(1+\gamma))S + (\gamma/(1+\gamma))I$ in Theorem 3.3, where S is a super hybrid mapping and γ is a real number, we obtain the following result.

Theorem 4.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4) and let Ψ be a δ-inverse strongly monotone mapping from C into H. Let α, β and γ be real numbers with $\gamma \neq -1$ and let $S : C \to H$ be an (α, β, γ)-super hybrid mapping such that $\mathrm{GEP}(F, \Psi) \cap F(S) \neq \emptyset$ and let f be a Meir-Keeler contraction of C into itself. Let $C_1 = C$ and $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{gathered} F(z_n, y) + \langle \Psi x_n, y - z_n \rangle + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\left(\frac{1}{1+\gamma} S z_n + \frac{\gamma}{1+\gamma} z_n\right), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} f(x_n), \quad \forall n \in \mathbb{N}, \end{gathered} \tag{4.6}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0, 2\delta)$ are sequences such that

$$\liminf_{n\to\infty} \alpha_n < 1, \qquad 0 < a \leq \lambda_n \leq b < 2\delta, \tag{4.7}$$

for some $a, b \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap \mathrm{GEP}(F,\Psi)} f(z_0)$.

Proof. Put $T = (1/(1+\gamma))S + (\gamma/(1+\gamma))I$; we have from Lemma 4.1 that T is a generalized hybrid mapping and $F(T) = F(S)$. Since $F(T) \neq \emptyset$, we have that T is quasi-nonexpansive. Following the proof of Theorem 3.3 and applying Lemma 4.2, we have the following result. □

Setting $f(x_n) = x, \forall n \in \mathbb{N}$ in Theorem 4.3, we obtain the following result.

Corollary 4.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4) and let Ψ be a δ-inverse strongly monotone mapping from C into H. Let α, β, and γ be real numbers with $\gamma \neq -1$ and let $S : C \to H$ be an (α, β, γ)-super hybrid mapping such that $\mathrm{GEP}(F, \Psi) \cap F(S) \neq \emptyset$. Let $C_1 = C$ and $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{gathered} F(z_n, y) + \langle \Psi x_n, y - z_n \rangle + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\left(\frac{1}{1+\gamma} S z_n + \frac{\gamma}{1+\gamma} z_n\right), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{4.8}$$

where $P_{C_{n+1}}$ *is the metric projection of* H *onto* C_{n+1} *and* $\{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,2\delta)$ *are sequences such that*

$$\liminf_{n\to\infty} \alpha_n < 1, \qquad 0 < a \le \lambda_n \le b < 2\delta, \tag{4.9}$$

for some $a, b \in \mathbb{R}$. *Then* $\{x_n\}$ *converges strongly to* $z_0 \in F(S) \cap \mathrm{GEP}(F, \Psi)$.

Setting $\Psi \equiv 0$ in Theorem 4.3, we obtain the following result.

Corollary 4.5. *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying (A1), (A2), (A3), and (A4). Let* α, β, *and* γ *be real numbers with* $\gamma \ne -1$ *and let* $S : C \to H$ *be an* (α, β, γ)*-super hybrid mapping such that* $\mathrm{EP}(F) \cap F(S) \ne \emptyset$ *and let* f *be a Meir-Keeler contraction of* C *into itself. Let* $C_1 = C$ *and* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 = x \in C$ *and*

$$\begin{gathered} F(z_n, y) + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\left(\frac{1}{1+\gamma} S z_n + \frac{\gamma}{1+\gamma} z_n\right), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} f(x_n), \quad \forall n \in \mathbb{N}, \end{gathered} \tag{4.10}$$

where $P_{C_{n+1}}$ *is the metric projection of* H *onto* C_{n+1} *and* $\{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$ *are sequences such that*

$$\liminf_{n\to\infty} \alpha_n < 1, \qquad 0 < a \le \lambda_n, \tag{4.11}$$

for some $a \in \mathbb{R}$. *Then* $\{x_n\}$ *converges strongly to* $z_0 \in F(S) \cap \mathrm{EP}(F)$.

Setting $\Psi \equiv 0$ and $f(x_n) = x$, for all $n \in \mathbb{N}$ in Theorem 4.3, we obtain the following result.

Corollary 4.6 (see [20], Theorem 5.2). *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying (A1), (A2), (A3), and (A4). Let* α, β, *and* γ *be real numbers with* $\gamma \ne -1$ *and let* $S : C \to H$ *be an* (α, β, γ)*-super hybrid mapping such that* $\mathrm{EP}(F) \cap F(S) \ne \emptyset$. *Let* $C_1 = C$ *and let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 = x \in C$ *and*

$$\begin{gathered} F(z_n, y) + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\left(\frac{1}{1+\gamma} S z_n + \frac{\gamma}{1+\gamma} z_n\right), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{4.12}$$

where $P_{C_{n+1}}$ *is the metric projection of* H *onto* C_{n+1} *and* $\{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$ *are sequences such that*

$$\liminf_{n\to\infty} \alpha_n < 1, \qquad 0 \leq \lambda_n, \tag{4.13}$$

for some $a \in \mathbb{R}$*. Then* $\{x_n\}$ *converges strongly to* $z_0 \in F(S) \cap \mathrm{EP}(F)$*.*

Setting $T := (1/(1+\gamma))S + (\gamma/(1+\gamma))I$ in Theorem 3.7, where S is an super hybrid mapping and γ is a real number, we obtain the following result.

Theorem 4.7. *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H*. Let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying (A1), (A2), (A3), and (A4) and let* Ψ *be a* δ*-inverse strongly monotone mapping from* C *into* H*. Let* α, β*, and* γ *be real numbers with* $\gamma \neq -1$ *and let* $S : C \to H$ *be an* (α,β,γ)*-super hybrid mapping such that* $\mathrm{GEP}(F,\Psi) \cap F(S) \neq \emptyset$ *and let* f *be a Meir-Keeler contraction of* C *into itself. Let* $Q_1 = C$ *and* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 = x \in C$ *and*

$$\begin{gathered}
F(z_n, y) + \langle \Psi x_n, y - z_n \rangle + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n \rangle \geq 0, \quad \forall y \in C, \\
y_n = \alpha_n x_n + (1-\alpha_n)\left(\frac{1}{1+\gamma} S z_n + \frac{\gamma}{1+\gamma} z_n\right), \\
C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in Q_{n-1} : \langle f(x_{n-1}) - x_n, x_n - z \rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} f(x_n), \quad \forall n \in \mathbb{N},
\end{gathered} \tag{4.14}$$

where $P_{C_n \cap Q_n}$ *is the metric projection of* H *onto* $C_n \cap Q_n$ *and* $\{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,2\delta)$ *satisfy*

$$0 \leq \alpha_n \leq b < 1, \qquad 0 < c \leq \lambda_n \leq d < 2\delta, \tag{4.15}$$

for some $b, c, d \in \mathbb{R}$*. Then* $\{x_n\}$ *converges strongly to* $z_0 = P_{F(S)\cap \mathrm{GEP}(F,\Psi)} f(z_0)$*.*

Proof. Put $T = (1/(1+\gamma))S + (\gamma/(1+\gamma))I$; we have from Lemma 4.1 that T is a generalized hybrid mapping and $F(T) = F(S)$. Since $F(T) \neq \emptyset$, we have that T is quasi-nonexpansive. Following the proof of Theorem 3.7 and applying Lemma 4.2, we obtain the following result. □

Setting $f(x_n) = x, \forall n \in \mathbb{N}$ in Theorem 4.7, we obtains the following result.

Corollary 4.8. *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H*. Let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying (A1), (A2), (A3), and (A4) and let* Ψ *be a* δ*-inverse strongly monotone mapping from* C *into* H*. Let* α, β*, and* γ *be real numbers with* $\gamma \neq -1$ *and let* $S : C \to H$

be an (α, β, γ)-super hybrid mapping such that $\mathrm{GEP}(F, \Psi) \cap F(S) \neq \emptyset$ *and let f be a Meir-Keeler contraction of C into itself. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{gathered}
F(z_n, y) + \langle \Psi x_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle z_n - x_n, y - z_n \rangle \geq 0, \quad \forall y \in C, \\
y_n = \alpha_n x_n + (1 - \alpha_n) \left(\frac{1}{1+\gamma} S z_n + \frac{\gamma}{1+\gamma} z_n \right), \\
C_n = \{ z \in C : \| y_n - z \| \leq \| x_n - z \| \}, \\
Q_n = \{ z \in C : \langle x - x_n, x_n - z \rangle \geq 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N},
\end{gathered} \tag{4.16}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0, 2\delta)$ satisfy

$$0 \leq \alpha_n \leq b < 1, \qquad 0 < c \leq \lambda_n \leq d < 2\delta, \tag{4.17}$$

for some $b, c, d \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap \mathrm{GEP}(F, \Psi)$.

Setting $\Psi \equiv 0$ in Theorem 4.7, we obtain the following result.

Corollary 4.9. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let α, β, and γ be real numbers with $\gamma \neq -1$ and let $S : C \to H$ be an (α, β, γ)-super hybrid mapping such that $\mathrm{EP}(F) \cap F(S) \neq \emptyset$ and let f be a Meir-Keeler contraction of C into itself. Let $Q_1 = C$ and $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{gathered}
F(z_n, y) + \frac{1}{\lambda_n} \langle z_n - x_n, y - z_n \rangle \geq 0, \quad \forall y \in C, \\
y_n = \alpha_n x_n + (1 - \alpha_n) \left(\frac{1}{1+\gamma} S z_n + \frac{\gamma}{1+\gamma} z_n \right), \\
C_n = \{ z \in C : \| y_n - z \| \leq \| x_n - z \| \}, \\
Q_n = \{ z \in Q_{n-1} : \langle f(x_{n-1}) - x_n, x_n - z \rangle \geq 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n} f(x_n), \quad \forall n \in \mathbb{N},
\end{gathered} \tag{4.18}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy

$$0 \leq \alpha_n \leq b < 1, \qquad 0 < c \leq \lambda_n, \tag{4.19}$$

for some $b, c \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap \mathrm{EP}(F)$.

Setting $\Psi \equiv 0$ and $f(x_n) = x, \forall n \in \mathbb{N}$ in Theorem 4.7, we obtain the following result.

Corollary 4.10 (see [20], Theorem 5.1). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let α, β, and γ be real numbers with $\gamma \neq -1$ and let $S : C \to H$ be an (α, β, γ)-super hybrid mapping such that $\mathrm{EP}(F) \cap F(S) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{gathered} F(z_n, y) + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\left(\frac{1}{1+\gamma}Sz_n + \frac{\gamma}{1+\gamma}z_n\right), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - z_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{4.20}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy

$$0 \leq \alpha_n \leq b < 1, \qquad 0 < c \leq \lambda_n, \tag{4.21}$$

for some $b, c \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap \mathrm{EP}(F)$.

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