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Research Article

Relative and Absolute Perturbation Bounds for Weighted Polar Decomposition

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Some new perturbation bounds for both weighted unitary polar factors and generalized nonnegative polar factors of the weighted polar decompositions are presented without the restriction that A and its perturbed matrix $\tilde{A}$ have the same rank. These bounds improve the corresponding recent results.

1. Introduction

Let $C^{m\times n}$, $C_r^{m\times n}$, $C_{\geq}^{m}$, $C_{>}^{m}$, and I_n denote the set of $m\times n$ complex matrices, subset of $C^{m\times n}$ consisting of matrices with rank r, set of the Hermitian nonnegative definite matrices of order m, subset of $C_{\geq}^{m}$ consisting of positive-definite matrices and $n\times n$ unit matrix, respectively. Without specification, we always assume that $m > n >\max\{r,s\}$ and the given weight matrices $M \in C_{>}^{m}, N \in C_{>}^{n}$. For $A \in C^{m\times n}$, we denote by $R(A), r(A), A^{*}, A_{MN}^{\#} = N^{-1}A^{*}M, A_{MN}^{\dagger}, \|A\|$ and $\|A\|_F$ the column space, rank, conjugate transpose, weighted conjugate transpose (or adjoint), weighted Moore-Penrose inverse, unitarily invariant norm, and Frobenius norm of A, respectively. The definitions of $A_{MN}^{\#}$ and $A_{MN}^{\dagger}$ can be found in details in [1, 2]. The weighted polar decomposition (MN-WPD) of $A\in C^{m\times n}$ is given by

$$A = QH, \tag{1.1}$$

where Q is an (M,N) weighted partial isometric matrix [3, 4] and H satisfies $NH \in C_{\geq}^{n}$. In this case, Q and H are called the (M,N) weighted unitary polar factor and generalized nonnegative polar factor, respectively, of this decomposition.

Yang and Li [5] proved that the MN-WPD is unique under the condition

$$R\left(Q^{\#}_{MN}\right) = R(H). \tag{1.2}$$

In this paper, we always assume that the MN-WPD satisfies condition (1.2).

If $M = I_m$ and $N = I_n$, then the MN-WPD is reduced to the generalized polar decomposition and Q and H are reduced to the subunitary polar factor and nonnegative polar factor, respectively. Further, if $r(A) = n$, then the MN-WPD is just the polar decomposition and Q and H are just the unitary polar factor and positive polar factor.

The problem on estimating the perturbation bounds for both polar decomposition and generalized polar decomposition under the assumption that the matrix and its perturbed matrix have the same rank [6–15] attracted most attention, and only some attention was given without the restriction [16, 17]. However, the arbitrary perturbation case seems important in both theoretical and practical problems. Now we list some published bounds for (generalized) polar decomposition without the restriction that A and $\widetilde{A}$ have the same rank.

Let $A \in C_r^{m\times n}, \widetilde{A} = A + E \in C_s^{m\times n}$ have the (generalized) polar decompositions $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$. For the perturbation bound of the (subunitary) unitary polar factors, the following two results can be found in [16]

$$\left\|\widetilde{Q} - Q\right\|_F \leq \frac{1}{\min\{\sigma_r, \widetilde{\sigma}_s\}}\|E\|_F, \tag{1.3}$$

$$\left\|\widetilde{Q} - Q\right\|_F \leq \frac{\sqrt{2}}{2}\sqrt{\|A^{\dagger}E\|_F^2 + \|EA^{\dagger}\|_F^2 + \left\|\widetilde{A}^{\dagger}E\right\|_F^2 + \left\|E\widetilde{A}^{\dagger}\right\|_F^2}. \tag{1.4}$$

For the nonnegative polar factors, the perturbation bounds obtained by Chen [17] are

$$\left\|\widetilde{H} - H\right\| \leq \left(\frac{\sigma_1 + \widetilde{\sigma}_1}{\sigma_r + \widetilde{\sigma}_s} + 2\right)\|E\|, \tag{1.5}$$

$$\left\|\widetilde{H} - H\right\| \leq \frac{\sigma_1\widetilde{\sigma}_1}{\sigma_1 + \widetilde{\sigma}_1}\left(\left\|EA^{\dagger}\right\| + \left\|E\widetilde{A}^{\dagger}\right\|\right) + \sigma_1\left\|A^{\dagger}E\right\| + \widetilde{\sigma}_1\left\|\widetilde{A}^{\dagger}E\right\|. \tag{1.6}$$

It is known that different elements of a vector are usually needed to be given some different weights in practice (e.g., the residual of the linear system), and the problems with weights, such as weighted generalized inverses problem and weighted least square problem, draw more and more attention, see, for example, [1, 2, 18, 19]. As a generalization of the (generalized) polar decomposition, MN-WPD may be useful for these problems. Therefore, it is of interest to study MN-WPD and its related properties.

Our goal of this paper is mainly to generalize the perturbation bounds in (1.3)–(1.6) to those for the weighted polar factors of the MN-WPDs in the corresponding weighted norms. The rest of this paper is organized as follows.

In Section 2, we list notation and some lemmas which are useful in the sequel. In Section 3, we present an absolute perturbation bound and a relative perturbation bound for the weighted unitary polar factors, respectively, and some perturbation bounds for the generalized nonnegative polar factors are also given in Section 4.

2. Notation and Some Lemmas

Firstly, we introduce the definitions of the weighted norms.

Definition 2.1. Let $A \in C^{m\times n}$. The norms $\|A\|_{(MN)} = \|M^{1/2}AN^{-1/2}\|$ and $\|A\|_{F(MN)} = \|M^{1/2}AN^{-1/2}\|_F$ are called the weighted unitarily invariant norm and weighted Frobenius norm of A, respectively. The definitions of $\|A\|_{(MN)}$ and $\|A\|_{F(MN)}$ can be also found in [20, 21].

Let $A \in C_r^{m\times n}$ and $\widetilde{A} \in C_s^{m\times n}$ have their weighted singular value decompositions (MN-SVDs):

$$A = U\Sigma V^* = (U_1, U_2)\begin{pmatrix}\Sigma_1 & 0\\ 0 & 0\end{pmatrix}(V_1, V_2)^* = U_1\Sigma_1 V_1^*, \tag{2.1}$$

$$\widetilde{A} = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^* = \left(\widetilde{U}_1, \widetilde{U}_2\right)\begin{pmatrix}\widetilde{\Sigma}_1 & 0\\ 0 & 0\end{pmatrix}\left(\widetilde{V}_1, \widetilde{V}_2\right)^* = \widetilde{U}_1\widetilde{\Sigma}_1\widetilde{V}_1^*. \tag{2.2}$$

Then the MN-WPDs of $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ can be obtained by

$$\begin{aligned} Q &= U_1V_1^*, \qquad H = N^{-1}V_1\Sigma_1V_1^*,\\ \widetilde{Q} &= \widetilde{U}_1\widetilde{V}_1^*, \qquad \widetilde{H} = N^{-1}\widetilde{V}_1\widetilde{\Sigma}_1\widetilde{V}_1^*, \end{aligned} \tag{2.3}$$

where $U = (U_1, U_2)$, $\widetilde{U} = (\widetilde{U}_1, \widetilde{U}_2) \in C^{m\times m}$ and $V = (V_1, V_2)$, $\widetilde{V} = (\widetilde{V}_1, \widetilde{V}_2) \in C^{n\times n}$ satisfy $U^*MU = \widetilde{U}^*M\widetilde{U} = I_m$ and $V^*N^{-1}V = \widetilde{V}^*N^{-1}\widetilde{V} = I_n$, and $U_1 \in C_r^{m\times r}$, $\widetilde{U}_1 \in C_s^{m\times s}$, $V_1 \in C_r^{n\times r}$, $\widetilde{V}_1 \in C_s^{n\times s}$, $\Sigma_1 = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$ and $\widetilde{\Sigma}_1 = \mathrm{diag}(\widetilde{\sigma}_1, \widetilde{\sigma}_2, \ldots, \widetilde{\sigma}_s)$. Here $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0$ and $\widetilde{\sigma}_1 \ge \widetilde{\sigma}_2 \ge \cdots \ge \widetilde{\sigma}_s > 0$ are the nonzero (M, N) weighted singular values of A and $\widetilde{A}$, respectively.

The following three lemmas can be found from [22], [23] and [16], respectively.

Lemma 2.2. *Let B_1 and B_2 be two Hermitian matrices and let P be a complex matrix. Suppose that there are two disjoint intervals separated by a gap of width at least η, where one interval contains the spectrum of B_1 and the other contains that of B_2. If $\eta > 0$, then there exists a unique solution X to the matrix equation $B_1X - XB_2 = P$ and, moreover,*

$$\|X\| \le \frac{1}{\eta}\|P\|. \tag{2.4}$$

Lemma 2.3. *Let $\Omega \in C^{s\times s}$ and $\Gamma \in C^{t\times t}$ be two Hermitian matrices, and let $E, F \in C^{s\times t}$. If $\lambda(\Omega) \cap \lambda(\Gamma) = \varnothing$, then $\Omega X - X\Gamma = \Omega E + F\Gamma$ has a unique solution $X \in C^{s\times t}$, and, moreover,*

$$\|X\|_F \le \frac{1}{\eta}\sqrt{\|E\|_F^2 + \|F\|_F^2}, \tag{2.5}$$

where $\eta = \min_{\lambda\in\lambda(\Omega), \widetilde{\lambda}\in\lambda(\Gamma)}\left(|\lambda - \widetilde{\lambda}| / \sqrt{|\widetilde{\lambda}|^2 + |\lambda|^2}\right)$.

Lemma 2.4. *Let $S = (S_1, S_2) \in C^{m\times m}$ and $T = (T_1, T_2) \in C^{n\times n}$ be both unitary matrices, where $S_1 \in C^{m\times r}, T_1 \in C^{n\times s}$. Then for any matrix $B \in C^{m\times n}$, one has*

$$\|B\|_F^2 = \left\|S_1^* B T_1\right\|_F^2 + \left\|S_1^* B T_2\right\|_F^2 + \left\|S_2^* B T_2\right\|_F^2 + \left\|S_2^* B T_2\right\|_F^2. \tag{2.6}$$

3. Perturbation Bounds for the Weighted Unitary Polar Factors

In this section, we present an absolute perturbation bound and a relative perturbation bound for the weighted unitary polar factors.

Theorem 3.1. *Let $A \in C_r^{m\times n}$ and $\widetilde{A} = A + E \in C_s^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$\left\|\widetilde{Q} - Q\right\|_{F(MN)} \le \frac{1}{\min\{\sigma_r, \widetilde{\sigma}_s\}} \|E\|_{F(MN)}. \tag{3.1}$$

Proof. By (2.1), and (2.2) the perturbation E can be written as

$$E = \widetilde{A} - A = \widetilde{U}_1 \widetilde{\Sigma}_1 \widetilde{V}_1^* - U_1 \Sigma_1 V_1^*, \tag{3.2}$$

which together with the facts that $U_1^* M U_1 = V_1^* N^{-1} V_1 = I_r$ and $\widetilde{U}_1^* M \widetilde{U}_1 = \widetilde{V}_1^* N^{-1} \widetilde{V}_1 = I_s$ gives

$$U_1^* M E N^{-1} \widetilde{V}_1 = U_1^* M \widetilde{U}_1 \widetilde{\Sigma}_1 - \Sigma_1 V_1^* N^{-1} \widetilde{V}_1, \tag{3.3}$$

$$\widetilde{U}_1^* M E N^{-1} V_1 = \widetilde{\Sigma}_1 \widetilde{V}_1^* N^{-1} V_1 - \widetilde{U}_1^* M U_1 \Sigma_1, \tag{3.4}$$

$$\widetilde{U}_2^* M E N^{-1} V_1 = -\widetilde{U}_2^* M U_1 \Sigma_1, \qquad U_2^* M E N^{-1} \widetilde{V}_1 = U_2^* M \widetilde{U}_1 \widetilde{\Sigma}_1, \tag{3.5}$$

$$\widetilde{U}_1^* M E N^{-1} V_2 = \widetilde{\Sigma}_1 \widetilde{V}_1^* N^{-1} V_2, \qquad U_1^* M E N^{-1} \widetilde{V}_2 = -\Sigma_1 V_1^* N^{-1} \widetilde{V}_2. \tag{3.6}$$

Taking the conjugate transpose on both sides of (3.4) and subtracting it from (3.3) leads to

$$\Sigma_1 \left(U_1^* M \widetilde{U}_1 - V_1^* N^{-1} \widetilde{V}_1\right) + \left(U_1^* M \widetilde{U}_1 - V_1^* N^{-1} \widetilde{V}_1\right) \widetilde{\Sigma}_1 = U_1^* M E N^{-1} \widetilde{V}_1 - V_1^* N^{-1} E^* M \widetilde{U}_1. \tag{3.7}$$

Applying Lemma 2.2 to (3.7) for the Frobenius norm leads to

$$\left\|U_1^* M \widetilde{U}_1 - V_1^* N^{-1} \widetilde{V}_1\right\|_F \le \frac{1}{\sigma_r + \widetilde{\sigma}_s} \left(\left\|U_1^* M E N^{-1} \widetilde{V}_1\right\|_F + \left\|V_1^* N^{-1} E^* M \widetilde{U}_1\right\|_F\right). \tag{3.8}$$

Since

$$U^*M\left(\widetilde{Q}-Q\right)N^{-1}\widetilde{V}=\begin{pmatrix}U_1^*M\widetilde{U}_1-V_1^*N^{-1}\widetilde{V}_1 & -V_1^*N^{-1}\widetilde{V}_2\\ U_2^*M\widetilde{U}_1 & 0\end{pmatrix},$$
$$\widetilde{U}^*M\left(\widetilde{Q}-Q\right)N^{-1}V=\begin{pmatrix}\widetilde{V}_1^*N^{-1}V_1-\widetilde{U}_1^*MU_1 & \widetilde{V}_1^*N^{-1}V_2\\ -\widetilde{U}_2^*MU_1 & 0\end{pmatrix}, \tag{3.9}$$

it follows from Definition 2.1 and the fact that $U^*M^{1/2}$, $\widetilde{U}^*M^{1/2}$, $N^{-1/2}V$, and $N^{-1/2}\widetilde{V}$ are all unitary matrices that

$$\begin{aligned}\left\|\widetilde{Q}-Q\right\|_{F(MN)}^2&=\left\|U^*M\left(\widetilde{Q}-Q\right)N^{-1}\widetilde{V}\right\|_F^2\\&=\left\|U_1^*M\widetilde{U}_1-V_1^*N^{-1}\widetilde{V}_1\right\|_F^2+\left\|V_1^*N^{-1}\widetilde{V}_2\right\|_F^2+\left\|U_2^*M\widetilde{U}_1\right\|_F^2,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\left\|\widetilde{Q}-Q\right\|_{F(MN)}^2&=\left\|\widetilde{U}^*M\left(\widetilde{Q}-Q\right)N^{-1}V\right\|_F^2\\&=\left\|\widetilde{V}_1^*N^{-1}V_1-\widetilde{U}_1^*MU_1\right\|_F^2+\left\|\widetilde{V}_1^*N^{-1}V_2\right\|_F^2+\left\|\widetilde{U}_2^*MU_1\right\|_F^2.\end{aligned} \tag{3.11}$$

Adding (3.10) to (3.11) gives

$$\begin{aligned}2\left\|\widetilde{Q}-Q\right\|_{F(MN)}^2&=\left\|U_1^*M\widetilde{U}_1-V_1^*N^{-1}\widetilde{V}_1\right\|_F^2+\left\|V_1^*N^{-1}\widetilde{V}_2\right\|_F^2+\left\|U_2^*M\widetilde{U}_1\right\|_F^2\\&\quad+\left\|\widetilde{V}_1^*N^{-1}V_1-\widetilde{U}_1^*MU_1\right\|_F^2+\left\|\widetilde{V}_1^*N^{-1}V_2\right\|_F^2+\left\|\widetilde{U}_2^*MU_1\right\|_F^2\\&=2\left\|U_1^*M\widetilde{U}_1-V_1^*N^{-1}\widetilde{V}_1\right\|_F^2+\left\|V_1^*N^{-1}\widetilde{V}_2\right\|_F^2+\left\|\widetilde{V}_1^*N^{-1}V_2\right\|_F^2\\&\quad+\left\|U_2^*M\widetilde{U}_1\right\|_F^2+\left\|\widetilde{U}_2^*MU_1\right\|_F^2.\end{aligned} \tag{3.12}$$

Combing (3.5), (3.6), (3.8), (3.12), Lemma 2.4, and the fact that $\|U_1^*M\widetilde{U}_1-V_1^*N^{-1}\widetilde{V}_1\|_F^2=\|\widetilde{V}_1^*N^{-1}V_1-\widetilde{U}_1^*MU_1\|_F^2$ gets

$$\begin{aligned}2\left\|\widetilde{Q}-Q\right\|_{F(MN)}^2&\le 2\left(\frac{1}{\sigma_r+\widetilde{\sigma}_s}\right)^2\left(\left\|U_1^*MEN^{-1}\widetilde{V}_1\right\|_F+\left\|V_1^*N^{-1}E^*M\widetilde{U}_1\right\|_F\right)^2\\&\quad+\frac{1}{\sigma_r^2}\left\|U_1^*MEN^{-1}\widetilde{V}_2\right\|_F^2+\frac{1}{\widetilde{\sigma}_s^2}\left\|U_2^*MEN^{-1}\widetilde{V}_1\right\|_F^2\\&\quad+\frac{1}{\widetilde{\sigma}_s^2}\left\|\widetilde{U}_1^*MEN^{-1}V_2\right\|_F^2+\frac{1}{\sigma_r^2}\left\|\widetilde{U}_2^*MEN^{-1}V_1\right\|_F^2\\&\le\left(\frac{2}{\sigma_r+\widetilde{\sigma}_s}\right)^2\left(\left\|U_1^*MEN^{-1}\widetilde{V}_1\right\|_F^2+\left\|V_1^*N^{-1}E^*M\widetilde{U}_1\right\|_F^2\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\min\{\sigma_r^2,\widetilde{\sigma}_s^2\}}\left(\left\|U_1^*MEN^{-1}\widetilde{V}_2\right\|_F^2+\left\|U_2^*MEN^{-1}\widetilde{V}_1\right\|_F^2\right.\\
&\qquad\left.+\left\|\widetilde{U}_1^*MEN^{-1}V_2\right\|_F^2+\left\|\widetilde{U}_2^*MEN^{-1}V_1\right\|_F^2\right)\\
&\le\frac{1}{\min\{\sigma_r^2,\widetilde{\sigma}_s^2\}}\left(\left(\left\|U_1^*MEN^{-1}\widetilde{V}_1\right\|_F^2+\left\|U_1^*MEN^{-1}\widetilde{V}_2\right\|_F^2\right.\right.\\
&\qquad\left.+\left\|U_2^*MEN^{-1}\widetilde{V}_1\right\|_F^2\right)+\left(\left\|\widetilde{U}_1^*MEN^{-1}V_1\right\|_F^2\right.\\
&\qquad+\left\|\widetilde{U}_1^*MEN^{-1}V_2\right\|_F^2\\
&\qquad\left.\left.+\left\|\widetilde{U}_2^*MEN^{-1}V_1\right\|_F^2\right)\right)\\
&\le\frac{2}{\min\{\sigma_r^2,\widetilde{\sigma}_s^2\}}\left\|M^{1/2}EN^{-1/2}\right\|_F^2=\frac{2}{\min\{\sigma_r^2,\widetilde{\sigma}_s^2\}}\|E\|_{F(MN)}^2,
\end{aligned}
\tag{3.13}
$$

which proves the theorem. □

Remark 3.2. If $M = I_m$ and $N = I_n$ in Theorem 3.1, the bound (3.1) is reduced to bound (1.3).

Theorem 3.3. *Let $A \in C_r^{m\times n}$ and $\widetilde{A} = A + E \in C_s^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$
\left\|\widetilde{Q}-Q\right\|_{F(MN)}\le\frac{\sqrt{2}}{2}\sqrt{\left\|\widetilde{A}_{MN}^{\dagger}E\right\|_{F(NN)}^2+\left\|A_{MN}^{\dagger}E\right\|_{F(NN)}^2+\left\|EA_{MN}^{\dagger}\right\|_{F(MM)}^2+\left\|E\widetilde{A}_{MN}^{\dagger}\right\|_{F(MM)}^2}.
\tag{3.14}
$$

Proof. From the MN-SVDs of A and $\widetilde{A}$ in (2.1) and (2.2) and the facts that $U^*MU = \widetilde{U}^*M\widetilde{U} = I_m$ and $V^*N^{-1}V = \widetilde{V}^*N^{-1}\widetilde{V} = I_n$, the weighted Moore-Penrose inverses of A and $\widetilde{A}$ can be written as

$$
A_{MN}^{\dagger}=N^{-1}V_1\Sigma_1^{-1}U_1^*M,\qquad \widetilde{A}_{MN}^{\dagger}=N^{-1}\widetilde{V}_1\widetilde{\Sigma}_1^{-1}\widetilde{U}_1^*M.
\tag{3.15}
$$

Premultiplying the equation $\widetilde{A} - A = E$ by $A_{MN}^{\dagger}$ leads to

$$
A_{MN}^{\dagger}\widetilde{A}-A_{MN}^{\dagger}A=A_{MN}^{\dagger}E,
\tag{3.16}
$$

that is,

$$
N^{-1}V_1\Sigma_1^{-1}U_1^*M\widetilde{U}_1\widetilde{\Sigma}_1\widetilde{V}_1^*-N^{-1}V_1V_1^*=A_{MN}^{\dagger}E.
\tag{3.17}
$$

By (3.17), we can obtain

$$U_1^* M\widetilde{U}_1\widetilde{\Sigma}_1 - \Sigma_1 V_1^* N^{-1}\widetilde{V}_1 = \Sigma_1 V_1^* A_{MN}^{\dagger} E N^{-1}\widetilde{V}_1, \qquad -V_1^* N^{-1}\widetilde{V}_2 = V_1^* A_{MN}^{\dagger} E N^{-1}\widetilde{V}_2. \tag{3.18}$$

Similarly, by $\widetilde{A}_{MN}^{\dagger}\widetilde{A} - \widetilde{A}_{MN}^{\dagger}A = \widetilde{A}_{MN}^{\dagger}E$, $\widetilde{A}A_{MN}^{\dagger} - AA_{MN}^{\dagger} = EA_{MN}^{\dagger}$ and $\widetilde{A}\widetilde{A}_{MN}^{\dagger} - A\widetilde{A}_{MN}^{\dagger} = E\widetilde{A}_{MN}^{\dagger}$, we get

$$\widetilde{\Sigma}_1\widetilde{V}_1^* N^{-1}V_1 - \widetilde{U}_1^* M U_1\Sigma_1 = \widetilde{\Sigma}_1\widetilde{V}_1^*\widetilde{A}_{MN}^{\dagger} E N^{-1} V_1, \qquad \widetilde{V}_1^* N^{-1} V_2 = \widetilde{V}_1^*\widetilde{A}_{MN}^{\dagger} E N^{-1} V_2, \tag{3.19}$$

$$\widetilde{\Sigma}_1\widetilde{V}_1^* N^{-1}V_1 - \widetilde{U}_1^* M U_1\Sigma_1 = \widetilde{U}_1^* M E A_{MN}^{\dagger} U_1\Sigma_1, \qquad -\widetilde{U}_2^* M U_1 = \widetilde{U}_2^* M E A_{MN}^{\dagger} U_1, \tag{3.20}$$

$$U_1^* M\widetilde{U}_1\widetilde{\Sigma}_1 - \Sigma_1 V_1^* N^{-1}\widetilde{V}_1 = U_1^* M E\widetilde{A}_{MN}^{\dagger}\widetilde{U}_1\widetilde{\Sigma}_1, \qquad U_2^* M\widetilde{U}_1 = U_2^* M E\widetilde{A}_{MN}^{\dagger}\widetilde{U}_1, \tag{3.21}$$

respectively. By the first equations in (3.18)–(3.21), we derive

$$\begin{aligned} &\left(U_1^* M\widetilde{U}_1 - V_1^* N^{-1}\widetilde{V}_1\right)\widetilde{\Sigma}_1 + \Sigma_1\left(U_1^* M\widetilde{U}_1 - V_1^* N^{-1}\widetilde{V}_1\right) \\ &\quad = \Sigma_1 V_1^* A_{MN}^{\dagger} E N^{-1}\widetilde{V}_1 - \left(\widetilde{V}_1^*\widetilde{A}_{MN}^{\dagger} E N^{-1} V_1\right)^*\widetilde{\Sigma}_1, \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\left(U_1^* M\widetilde{U}_1 - V_1^* N^{-1}\widetilde{V}_1\right)\widetilde{\Sigma}_1 + \Sigma_1\left(U_1^* M\widetilde{U}_1 - V_1^* N^{-1}\widetilde{V}_1\right) \\ &\quad = U_1^* M E\widetilde{A}_{MN}^{\dagger}\widetilde{U}_1\widetilde{\Sigma}_1 - \Sigma_1\left(\widetilde{U}_1^* M E A_{MN}^{\dagger} U_1\right)^*. \end{aligned} \tag{3.23}$$

Applying Lemma 2.3 to (3.22) and (3.23), respectively, and noting that

$$\eta = \min_{1\le i\le s, 1\le j\le r}\frac{\widetilde{\sigma}_i + \sigma_j}{\sqrt{\widetilde{\sigma}_i^2 + \sigma_j^2}} \ge 1, \tag{3.24}$$

we find that

$$\left\|U_1^* M\widetilde{U}_1 - V_1^* N^{-1}\widetilde{V}_1\right\|_F^2 \le \left\|V_1^* A_{MN}^{\dagger} E N^{-1}\widetilde{V}_1\right\|_F^2 + \left\|\widetilde{V}_1^*\widetilde{A}_{MN}^{\dagger} E N^{-1} V_1\right\|_F^2, \tag{3.25}$$

$$\left\|U_1^* M\widetilde{U}_1 - V_1^* N^{-1}\widetilde{V}_1\right\|_F^2 \le \left\|U_1^* M E\widetilde{A}_{MN}^{\dagger}\widetilde{U}_1\right\|_F^2 + \left\|\widetilde{U}_1^* M E A_{MN}^{\dagger} U_1\right\|_F^2. \tag{3.26}$$

From (3.12), the second equations in (3.18)–(3.21), (3.25), (3.26), and Lemma 2.4, we deduce that

$$
\begin{aligned}
2\left\|\widetilde{Q}-Q\right\|_{F(MN)}^2 &\le \left\|V_1^* A_{MN}^{\dagger} E N^{-1}\widetilde{V}_1\right\|_F^2 + \left\|\widetilde{V}_1^* \widetilde{A}_{MN}^{\dagger} E N^{-1} V_1\right\|_F^2 \\
&\quad + \left\|U_1^* M E \widetilde{A}_{MN}^{\dagger}\widetilde{U}_1\right\|_F^2 + \left\|\widetilde{U}_1^* M E A_{MN}^{\dagger} U_1\right\|_F^2 \\
&\quad + \left\|V_1^* A_{MN}^{\dagger} E N^{-1}\widetilde{V}_2\right\|_F^2 + \left\|U_2^* M E \widetilde{A}_{MN}^{\dagger}\widetilde{U}_1\right\|_F^2 \\
&\quad + \left\|\widetilde{V}_1^* \widetilde{A}_{MN}^{\dagger} E N^{-1} V_2\right\|_F^2 + \left\|\widetilde{U}_2^* M E A_{MN}^{\dagger} U_1\right\|_F^2 \\
&= \left(\left\|V_1^* A_{MN}^{\dagger} E N^{-1}\widetilde{V}_1\right\|_F^2 + \left\|V_1^* A_{MN}^{\dagger} E N^{-1}\widetilde{V}_2\right\|_F^2\right) \\
&\quad + \left(\left\|\widetilde{V}_1^* \widetilde{A}_{MN}^{\dagger} E N^{-1} V_1\right\|_F^2 + \left\|\widetilde{V}_1^* \widetilde{A}_{MN}^{\dagger} E N^{-1} V_2\right\|_F^2\right) \\
&\quad + \left(\left\|U_1^* M E \widetilde{A}_{MN}^{\dagger}\widetilde{U}_1\right\|_F^2 + \left\|U_2^* M E \widetilde{A}_{MN}^{\dagger}\widetilde{U}_1\right\|_F^2\right) \\
&\quad + \left(\left\|\widetilde{U}_1^* M E A_{MN}^{\dagger} U_1\right\|_F^2 + \left\|\widetilde{U}_2^* M E A_{MN}^{\dagger} U_1\right\|_F^2\right) \\
&\le \left\|N^{1/2} A_{MN}^{\dagger} E N^{-1/2}\right\|_F^2 + \left\|N^{1/2} \widetilde{A}_{MN}^{\dagger} E N^{-1/2}\right\|_F^2 \\
&\quad + \left\|M^{1/2} E \widetilde{A}_{MN}^{\dagger} M^{-1/2}\right\|_F^2 + \left\|M^{1/2} E A_{MN}^{\dagger} M^{-1/2}\right\|_F^2 \\
&= \left\|A_{MN}^{\dagger} E\right\|_{F(NN)}^2 + \left\|\widetilde{A}_{MN}^{\dagger} E\right\|_{F(NN)}^2 + \left\|E\widetilde{A}_{MN}^{\dagger}\right\|_{F(MM)}^2 \\
&\quad + \left\|E A_{MN}^{\dagger}\right\|_{F(MM)}^2,
\end{aligned}
\tag{3.27}
$$

which proves the theorem. □

Remark 3.4. If $M = I_m$ and $N = I_n$ in Theorem 3.3, the bound (3.14) is reduced to bound (1.4).

4. Perturbation Bounds for the Generalized Nonnegative Polar Factors

In this section, two absolute perturbation bounds and a relative perturbation bound for the generalized nonnegative polar factors are given.

Theorem 4.1. *Let $A \in C_r^{m\times n}$ and $\widetilde{A} = A + E \in C_s^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$
\left\|\widetilde{H}-H\right\|_{(NN)} \le \left(\frac{\sigma_1+\widetilde{\sigma}_1}{\sigma_r+\widetilde{\sigma}_s}+2\right)\|E\|_{(MN)}. \tag{4.1}
$$

Proof. By (2.1), (2.2), and (2.3), we have

$$
NH^2 = A^* M A, \qquad N\widetilde{H}^2 = \widetilde{A}^* M \widetilde{A}, \tag{4.2}
$$

which give

$$N\widetilde{H}\left(\widetilde{H}-H\right)+N\left(\widetilde{H}-H\right)H=\widetilde{A}^{*}M\left(\widetilde{A}-A\right)+\left(\widetilde{A}-A\right)^{*}MA. \tag{4.3}$$

Let $\Delta H=\widetilde{H}-\mathrm{H}$, we rewrite (4.3)

$$N\widetilde{H}\Delta H+N\Delta HH=\widetilde{A}^{*}ME+E^{*}MA, \tag{4.4}$$

that is,

$$\widetilde{V}_1\widetilde{\Sigma}_1\widetilde{V}_1^{*}\Delta H+N\Delta HN^{-1}V_1\Sigma_1V_1^{*}=\widetilde{V}_1\widetilde{\Sigma}_1\widetilde{U}_1^{*}ME+E^{*}MU_1\Sigma_1V_1^{*}. \tag{4.5}$$

Premultiplying and postmultiplying both sides of (4.5) by, respectively, $\widetilde{V}_1^{*}N^{-1}$ and $N^{-1}V_1$ give

$$\widetilde{\Sigma}_1\widetilde{V}_1^{*}\Delta HN^{-1}V_1+\widetilde{V}_1^{*}\Delta HN^{-1}V_1\Sigma_1=\widetilde{\Sigma}_1\widetilde{U}_1^{*}MEN^{-1}V_1+\widetilde{V}_1^{*}N^{-1}E^{*}MU_1\Sigma_1. \tag{4.6}$$

Similarly, we have

$$\widetilde{V}_1^{*}\Delta HN^{-1}V_2=\widetilde{U}_1^{*}MEN^{-1}V_2,\qquad \widetilde{V}_2^{*}\Delta HN^{-1}V_1=\widetilde{V}_2^{*}N^{-1}E^{*}MU_1. \tag{4.7}$$

Applying Lemma 2.2 to (4.6) gives

$$\begin{aligned}
\left\|\widetilde{V}_1^{*}\Delta HN^{-1}V_1\right\| &\le \frac{1}{\sigma_r+\widetilde{\sigma}_s}\left\|\widetilde{\Sigma}_1\widetilde{U}_1^{*}MEN^{-1}V_1+\widetilde{V}_1^{*}N^{-1}E^{*}MU_1\Sigma_1\right\| \\
&\le \frac{\widetilde{\sigma}_1}{\sigma_r+\widetilde{\sigma}_s}\left\|\widetilde{U}_1^{*}MEN^{-1}V_1\right\|+\frac{\sigma_1}{\sigma_r+\widetilde{\sigma}_s}\left\|\widetilde{V}_1^{*}N^{-1}E^{*}MU_1\right\| \\
&\le \frac{\widetilde{\sigma}_1}{\sigma_r+\widetilde{\sigma}_s}\|E\|_{(MN)}+\frac{\sigma_1}{\sigma_r+\widetilde{\sigma}_s}\|E\|_{(MN)} \\
&= \frac{\widetilde{\sigma}_1+\sigma_1}{\sigma_r+\widetilde{\sigma}_s}\|E\|_{(MN)}.
\end{aligned} \tag{4.8}$$

Notice that

$$\widetilde{V}^{*}\Delta HN^{-1}V=\begin{pmatrix}\widetilde{V}_1^{*}\Delta HN^{-1}V_1 & \widetilde{V}_1^{*}\Delta HN^{-1}V_2\\ \widetilde{V}_2^{*}\Delta HN^{-1}V_1 & 0\end{pmatrix}. \tag{4.9}$$

Combining (4.7)–(4.9) gives

$$\begin{aligned}
\|\Delta H\|_{(NN)} &= \left\|\widetilde{V}^{*}\Delta H N^{-1}V\right\| \\
&\leq \left\|\widetilde{V}_1^{*}\Delta H N^{-1}V_1\right\| + \left\|\widetilde{V}_1^{*}\Delta H N^{-1}V_2\right\| + \left\|\widetilde{V}_2^{*}\Delta H N^{-1}V_1\right\| \\
&\leq \frac{\widetilde{\sigma}_1+\sigma_1}{\sigma_r+\widetilde{\sigma}_s}\|E\|_{(MN)} + \left\|\widetilde{U}_1^{*}MEN^{-1}V_2\right\| + \left\|\widetilde{V}_2^{*}N^{-1}E^{*}MU_1\right\| \\
&\leq \frac{\widetilde{\sigma}_1+\sigma_1}{\sigma_r+\widetilde{\sigma}_s}\|E\|_{(MN)} + \|E\|_{(MN)} + \|E\|_{(MN)} \\
&= \left(\frac{\widetilde{\sigma}_1+\sigma_1}{\sigma_r+\widetilde{\sigma}_s}+2\right)\|E\|_{(MN)},
\end{aligned} \tag{4.10}$$

which proves the theorem. □

Remark 4.2. If $M = I_m$ and $N = I_n$ in Theorem 4.1, the bound (4.1) is reduced to bound (1.5).

If $r = n,\ s < n$ or $s = n,\ r < n$ or $r = s = n$, we can easily derive the following three corollaries.

Corollary 4.3. *Let $A \in C_n^{m\times n}$ and $\widetilde{A} = A + E \in C_s^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$\left\|\widetilde{H} - H\right\|_{(NN)} \leq \left(\frac{\sigma_1+\widetilde{\sigma}_1}{\sigma_n+\widetilde{\sigma}_s}+1\right)\|E\|_{(MN)}. \tag{4.11}$$

Corollary 4.4. *Let $A \in C_r^{m\times n}$ and $\widetilde{A} = A + E \in C_n^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$\left\|\widetilde{H} - H\right\|_{(NN)} \leq \left(\frac{\sigma_1+\widetilde{\sigma}_1}{\sigma_r+\widetilde{\sigma}_n}+1\right)\|E\|_{(MN)}. \tag{4.12}$$

Corollary 4.5. *Let $A, \widetilde{A} = A + E \in C_n^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$\left\|\widetilde{H} - H\right\|_{(NN)} \leq \frac{\sigma_1+\widetilde{\sigma}_1}{\sigma_n+\widetilde{\sigma}_n}\|E\|_{(MN)}. \tag{4.13}$$

If we take the weighted Frobenius norm as the specific weighted unitarily invariant norm in Theorem 4.1, an alternative absolute perturbation bound can be derived as follows.

Theorem 4.6. *Let $A \in C_r^{m\times n}$ and $\widetilde{A} = A + E \in C_s^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$\left\|\widetilde{H} - H\right\|_{F(NN)} \leq \left(2 + \sqrt{2}\max\left\{\frac{\sqrt{\sigma_1^2+\widetilde{\sigma}_s^2}}{\sigma_1+\widetilde{\sigma}_s}, \frac{\sqrt{\sigma_r^2+\widetilde{\sigma}_1^2}}{\sigma_r+\widetilde{\sigma}_1}\right\}\right)\|E\|_{F(MN)}. \tag{4.14}$$

Proof. Applying Lemma 2.3 to (4.6) gives

$$\begin{aligned}\left\|\widetilde{V}_1^* \Delta H N^{-1} V_1\right\|_F^2 &\le \max_{\substack{1\le i\le r\\ 1\le j\le s}} \frac{\sigma_i^2+\widetilde{\sigma}_j^2}{\left(\sigma_i+\widetilde{\sigma}_j\right)^2}\left(\left\|\widetilde{U}_1^* M E N^{-1} V_1\right\|_F^2+\left\|\widetilde{V}_1^* N^{-1} E^* M U_1\right\|_F^2\right)\\ &\le 2\max_{\substack{1\le i\le r\\ 1\le j\le s}} \frac{\sigma_i^2+\widetilde{\sigma}_j^2}{\left(\sigma_i+\widetilde{\sigma}_j\right)^2}\|E\|_{F(MN)}^2.\end{aligned} \tag{4.15}$$

From [16], we know

$$\max_{\substack{1\le i\le r\\ 1\le j\le s}} \frac{\widetilde{\sigma}_i^2+\widetilde{\sigma}_j^2}{\left(\sigma_i+\widetilde{\sigma}_j\right)^2} = \max\left\{\frac{\sigma_1^2+\widetilde{\sigma}_s^2}{\left(\sigma_1+\widetilde{\sigma}_s\right)^2}, \frac{\sigma_r^2+\widetilde{\sigma}_1^2}{\left(\sigma_r+\widetilde{\sigma}_1\right)^2}\right\}, \tag{4.16}$$

which together with (4.7), (4.9), and (4.15) gives

$$\begin{aligned}\|\Delta H\|_{F(NN)} &= \left\|\widetilde{V}^* \Delta H N^{-1} V\right\|_F\\ &\le \left\|\widetilde{V}_1^* \Delta H N^{-1} V_1\right\|_F+\left\|\widetilde{V}_1^* \Delta H N^{-1} V_2\right\|_F+\left\|\widetilde{V}_2^* \Delta H N^{-1} V_1\right\|_F\\ &\le \sqrt{2}\max\left\{\frac{\sqrt{\sigma_1^2+\widetilde{\sigma}_s^2}}{\sigma_1+\widetilde{\sigma}_s}, \frac{\sqrt{\sigma_r^2+\widetilde{\sigma}_1^2}}{\sigma_r+\widetilde{\sigma}_1}\right\}\|E\|_{F(MN)}\\ &\quad+\left\|\widetilde{U}_1^* M E N^{-1} V_2\right\|_F+\left\|\widetilde{V}_2^* N^{-1} E^* M U_1\right\|_F\\ &\le \sqrt{2}\max\left\{\frac{\sqrt{\sigma_1^2+\widetilde{\sigma}_s^2}}{\sigma_1+\widetilde{\sigma}_s}, \frac{\sqrt{\sigma_r^2+\widetilde{\sigma}_1^2}}{\sigma_r+\widetilde{\sigma}_1}\right\}\|E\|_{F(MN)}+\|E\|_{F(MN)}+\|E\|_{F(MN)}\\ &= \left(\sqrt{2}\max\left\{\frac{\sqrt{\sigma_1^2+\widetilde{\sigma}_s^2}}{\sigma_1+\widetilde{\sigma}_s}, \frac{\sqrt{\sigma_r^2+\widetilde{\sigma}_1^2}}{\sigma_r+\widetilde{\sigma}_1}\right\}+2\right)\|E\|_{F(MN)}.\end{aligned} \tag{4.17}$$

Hence, we complete the theorem. □

Similarly, we can obtain the following three corollaries.

Corollary 4.7. *Let $A \in C_n^{m\times n}$ and $\widetilde{A} = A + E \in C_s^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$\left\|\widetilde{H}-H\right\|_{F(NN)} \le \left(1+\sqrt{2}\max\left\{\frac{\sqrt{\sigma_1^2+\widetilde{\sigma}_s^2}}{\sigma_1+\widetilde{\sigma}_s}, \frac{\sqrt{\sigma_n^2+\widetilde{\sigma}_1^2}}{\sigma_n+\widetilde{\sigma}_1}\right\}\right)\|E\|_{F(MN)}. \tag{4.18}$$

Corollary 4.8. *Let $A \in C_r^{m\times n}$ and $\widetilde{A} = A + E \in C_n^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$\left\|\widetilde{H} - H\right\|_{F(NN)} \leq \left(1 + \sqrt{2}\max\left\{\frac{\sqrt{\sigma_1^2 + \widetilde{\sigma}_n^2}}{\sigma_1 + \widetilde{\sigma}_n}, \frac{\sqrt{\sigma_r^2 + \widetilde{\sigma}_1^2}}{\sigma_r + \widetilde{\sigma}_1}\right\}\right)\|E\|_{F(MN)}. \tag{4.19}$$

Corollary 4.9. *Let $A, \widetilde{A} = A + E \in C_n^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$\left\|\widetilde{H} - H\right\|_{F(NN)} \leq \sqrt{2}\max\left\{\frac{\sqrt{\sigma_1^2 + \widetilde{\sigma}_n^2}}{\sigma_1 + \widetilde{\sigma}_n}, \frac{\sqrt{\sigma_n^2 + \widetilde{\sigma}_1^2}}{\sigma_n + \widetilde{\sigma}_1}\right\}\|E\|_{F(MN)}. \tag{4.20}$$

The relative perturbation bound for the generalized nonnegative polar factors is given in the following theorem.

Theorem 4.10. *Let $A \in C_r^{m\times n}$ and $\widetilde{A} = A + E \in C_s^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$\begin{aligned}\left\|\widetilde{H} - H\right\|_{(NN)} \leq \frac{\sigma_1\widetilde{\sigma}_1}{\sigma_1 + \widetilde{\sigma}_1}\left(\left\|EA_{MN}^{\dagger}\right\|_{(MM)} + \left\|E\widetilde{A}_{MN}^{\dagger}\right\|_{(MM)}\right) \\ + \sigma_1\left\|A_{MN}^{\dagger}E\right\|_{(NN)} + \widetilde{\sigma}_1\left\|\widetilde{A}_{MN}^{\dagger}E\right\|_{(NN)}.\end{aligned} \tag{4.21}$$

Proof. From the proof of Theorem 3.3, we know that

$$A_{MN}^{\dagger} = N^{-1}V_1\Sigma_1^{-1}U_1^*M, \qquad \widetilde{A}_{MN}^{\dagger} = N^{-1}\widetilde{V}_1\widetilde{\Sigma}_1^{-1}\widetilde{U}_1^*M. \tag{4.22}$$

Premultiplying and postmultiplying both sides of (4.5) by, respectively, $(\widetilde{A}_{MN}^{\dagger})^*$ and $A_{MN}^{\dagger}$ give

$$\begin{aligned}&M\widetilde{U}_1\widetilde{V}_1^*\Delta HN^{-1}V_1\Sigma_1^{-1}U_1^*M + M\widetilde{U}_1\widetilde{\Sigma}_1^{-1}\widetilde{V}_1^*\Delta HN^{-1}V_1U_1^*M \\ &\quad = M\widetilde{U}_1\widetilde{U}_1^*MEN^{-1}V_1\Sigma_1^{-1}U_1^*M + M\widetilde{U}_1\widetilde{\Sigma}_1^{-1}\widetilde{V}_1^*N^{-1}E^*MU_1U_1^*M \\ &\quad = M\widetilde{U}_1\widetilde{U}_1^*MEA_{MN}^{\dagger} + \left(E\widetilde{A}_{MN}^{\dagger}\right)^*MU_1U_1^*M.\end{aligned} \tag{4.23}$$

Premultiplying and postmultiplying both sides of (4.23) by, respectively, $\widetilde{U}_1^*$ and U_1 give

$$\widetilde{V}_1^*\Delta HN^{-1}V_1\Sigma_1^{-1} + \widetilde{\Sigma}_1^{-1}\widetilde{V}_1^*\Delta HN^{-1}V_1 = \widetilde{U}_1^*MEA_{MN}^{\dagger}U_1 + \widetilde{U}_1^*\left(E\widetilde{A}_{MN}^{\dagger}\right)^*MU_1, \tag{4.24}$$

which together with Lemma 2.2 gives

$$\begin{aligned}\left\|\widetilde{V}_1^*\Delta HN^{-1}V_1\right\| &\leq \frac{\sigma_1\widetilde{\sigma}_1}{\sigma_1+\widetilde{\sigma}_1}\left(\left\|\widetilde{U}_1^*MEA_{MN}^{\dagger}U_1\right\| + \left\|\widetilde{U}_1^*\left(E\widetilde{A}_{MN}^{\dagger}\right)^*MU_1\right\|\right)\\ &\leq \frac{\sigma_1\widetilde{\sigma}_1}{\sigma_1+\widetilde{\sigma}_1}\left(\left\|EA_{MN}^{\dagger}\right\|_{(MM)} + \left\|E\widetilde{A}_{MN}^{\dagger}\right\|_{(MM)}\right).\end{aligned} \tag{4.25}$$

By (4.7) and the facts that $AA_{MN}^{\dagger} = U_1U_1^*M$ and $\widetilde{A}\widetilde{A}_{MN}^{\dagger} = \widetilde{U}_1\widetilde{U}_1^*M$, we have

$$\begin{aligned}\widetilde{V}_1^*\Delta HN^{-1}V_2 &= \widetilde{U}_1^*MEN^{-1}V_2 = \widetilde{U}_1^*M\widetilde{U}_1\widetilde{U}_1^*MEN^{-1}V_2\\ &= \widetilde{U}_1^*M\widetilde{A}\widetilde{A}_{MN}^{\dagger}EN^{-1}V_2 = \widetilde{\Sigma}_1\widetilde{V}_1^*\widetilde{A}_{MN}^{\dagger}EN^{-1}V_2,\\ \widetilde{V}_2^*\Delta HN^{-1}V_1 &= \widetilde{V}_2^*N^{-1}E^*MU_1 = \widetilde{V}_2^*N^{-1}E^*MU_1U_1^*MU_1\\ &= \widetilde{V}_2^*N^{-1}\left(A_{MN}^{\dagger}E\right)^*A^*MU_1 = \widetilde{V}_2^*N^{-1}\left(A_{MN}^{\dagger}E\right)^*V_1\Sigma_1.\end{aligned} \tag{4.26}$$

It follows from (4.9), (4.25) and (4.26) that

$$\begin{aligned}\|\Delta H\|_{(NN)} &= \left\|N^{1/2}\Delta HN^{-1/2}\right\| = \left\|\widetilde{V}^*\Delta HN^{-1}V\right\|\\ &\leq \left\|\widetilde{V}_1^*\Delta HN^{-1}V_1\right\| + \left\|\widetilde{V}_1^*\Delta HN^{-1}V_2\right\| + \left\|\widetilde{V}_2^*\Delta HN^{-1}V_1\right\|\\ &\leq \frac{\sigma_1\widetilde{\sigma}_1}{\sigma_1+\widetilde{\sigma}_1}\left(\left\|EA_{MN}^{\dagger}\right\|_{(MM)} + \left\|E\widetilde{A}_{MN}^{\dagger}\right\|_{(MM)}\right)\\ &\quad + \left\|\widetilde{\Sigma}_1\widetilde{V}_1^*\widetilde{A}_{MN}^{\dagger}EN^{-1}V_2\right\| + \left\|\widetilde{V}_2^*N^{-1}\left(A_{MN}^{\dagger}E\right)^*V_1\Sigma_1\right\|\\ &\leq \frac{\sigma_1\widetilde{\sigma}_1}{\sigma_1+\widetilde{\sigma}_1}\left(\left\|EA_{MN}^{\dagger}\right\|_{(MM)} + \left\|E\widetilde{A}_{MN}^{\dagger}\right\|_{(MM)}\right)\\ &\quad + \widetilde{\sigma}_1\left\|\widetilde{A}_{MN}^{\dagger}E\right\|_{(NN)} + \sigma_1\left\|A_{MN}^{\dagger}E\right\|_{(NN)},\end{aligned} \tag{4.27}$$

which proves the theorem. □

Remark 4.11. If $M = I_m$ and $N = I_n$ in Theorem 4.10, the bound (4.21) is reduced to bound (1.6).

The following three corollaries can be also easily obtained.

Corollary 4.12. *Let $A \in C_n^{m\times n}$ and $\widetilde{A} = A + E \in C_s^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$\left\|\widetilde{H} - H\right\|_{(NN)} \leq \frac{\sigma_1\widetilde{\sigma}_1}{\sigma_1+\widetilde{\sigma}_1}\left(\left\|EA_{MN}^{\dagger}\right\|_{(MM)} + \left\|E\widetilde{A}_{MN}^{\dagger}\right\|_{(MM)}\right) + \sigma_1\left\|A_{MN}^{\dagger}E\right\|_{(NN)}. \tag{4.28}$$

Corollary 4.13. *Let $A \in C_r^{m\times n}$ and $\widetilde{A} = A + E \in C_n^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$\left\|\widetilde{H} - H\right\|_{(NN)} \leq \frac{\sigma_1\widetilde{\sigma}_1}{\sigma_1+\widetilde{\sigma}_1}\left(\left\|EA_{MN}^{\dagger}\right\|_{(MM)} + \left\|E\widetilde{A}_{MN}^{\dagger}\right\|_{(MM)}\right) + \widetilde{\sigma}_1\left\|\widetilde{A}_{MN}^{\dagger}E\right\|_{(NN)}. \tag{4.29}$$

Corollary 4.14. *Let $A \in C_n^{m\times n}$ and $\widetilde{A} = A + E \in C_n^{m\times n}$, and let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be their MN-WPDs of A and $\widetilde{A}$, respectively. Then*

$$\left\|\widetilde{H} - H\right\|_{(NN)} \leq \frac{\sigma_1\widetilde{\sigma}_1}{\sigma_1 + \widetilde{\sigma}_1}\left(\left\|EA^{\dagger}_{MN}\right\|_{(MM)} + \left\|E\widetilde{A}^{\dagger}_{MN}\right\|_{(MM)}\right). \tag{4.30}$$

5. Conclusion

In this paper, we obtain the relative and absolute perturbation bounds for the weighted polar decomposition without the restriction that the original matrix and its perturbed matrix have the same rank. These bounds are the corresponding generalizations of those for the (generalized) polar decomposition.

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References

[1] G. R. Wang and Y. M. Wei, *Generalized Inverses: Theory and Computations*, Science, Beijing, China, 2004.

[2] C. R. Rao and S. K. Mitra, *Generalized Inverse of Matrices and Its Applications*, John Wiley & Sons, New York, NY, USA, 1971.

[3] H. Yang and H. Li, "Weighted UDV^*-decomposition and weighted spectral decomposition for rectangular matrices and their applications," *Applied Mathematics and Computation*, vol. 198, no. 1, pp. 150–162, 2008.

[4] H. Yang and H. Li, "Weighted polar decomposition and WGL partial ordering of rectangular complex matrices," *SIAM Journal on Matrix Analysis and Applications*, vol. 30, no. 2, pp. 898–924, 2008.

[5] H. Yang and H. Y. Li, "Weighted polar decomposition," *Journal of Mathematical Research and Exposition*, vol. 29, no. 5, pp. 787–798, 2009.

[6] J. G. Sun and C. H. Chen, "Generalized polar decomposition," *Mathematica Numerica Sinica*, vol. 11, no. 3, pp. 262–273, 1989 (Chinese).

[7] C. H. Chen and J. G. Sun, "Perturbation bounds for the polar factors," *Journal of Computational Mathematics*, vol. 7, no. 4, pp. 397–401, 1989.

[8] R. C. Li, "A perturbation bound for the generalized polar decomposition," *BIT*, vol. 33, no. 2, pp. 304–308, 1993.

[9] R.-C. Li, "Relative perturbation bounds for the unitary polar factor," *BIT. Numerical Mathematics*, vol. 37, no. 1, pp. 67–75, 1997.

[10] W. Li and W. Sun, "Perturbation bounds of unitary and subunitary polar factors," *SIAM Journal on Matrix Analysis and Applications*, vol. 23, no. 4, pp. 1183–1193, 2002.

[11] R. Mathias, "Perturbation bounds for the polar decomposition," *SIAM Journal on Matrix Analysis and Applications*, vol. 14, no. 2, pp. 588–597, 1993.

[12] X. S. Chen and W. Li, "Perturbation bounds for polar decomposition under unitarily invariant norms," *Mathematica Numerica Sinica*, vol. 27, no. 2, pp. 121–128, 2005.

[13] R. C. Li, "New perturbation bounds for the unitary polar factor," *SIAM Journal on Matrix Analysis and Applications*, vol. 16, no. 1, pp. 327–332, 1995.

[14] W. Li and W. Sun, "New perturbation bounds for unitary polar factors," *SIAM Journal on Matrix Analysis and Applications*, vol. 25, no. 2, pp. 362–372, 2003.

[15] W. Li and W. Sun, "Some remarks on the perturbation of polar decompositions for rectangular matrices," *Numerical Linear Algebra with Applications*, vol. 13, no. 4, pp. 327–338, 2006.

[16] X.-S. Chen and W. Li, "Variations for the Q- and H-factors in the polar decomposition," *Calcolo*, vol. 45, no. 2, pp. 99–109, 2008.

[17] X. S. Chen, “Absolute and relative perturbation bounds for the Hermitian positive semidefinite polar factor under unitarily invariant norm,” *Journal of South China Normal University*, no. 3, pp. 1–3, 2010 (Chinese).
[18] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 15, Springer, New York, NY, USA, 2nd edition, 2003.
[19] C. R. Rao and M. B. Rao, *Matrix Algebra and Its Applications to Statistics and Econometrics*, World Scientific, River Edge, NJ, USA, 1998.
[20] H. Li and H. Yang, “Relative perturbation bounds for weighted polar decomposition,” *Computers & Mathematics with Applications*, vol. 59, no. 2, pp. 853–860, 2010.
[21] H. Yang and H. Li, “Perturbation bounds for weighted polar decomposition in the weighted unitarily invariant norm,” *Numerical Linear Algebra with Applications*, vol. 15, no. 8, pp. 685–700, 2008.
[22] C. Davis and W. M. Kahan, “The rotation of eigenvectors by a perturbations III,” *SIAM Journal on Scientific and Statistical Computing*, no. 12, pp. 488–504, 1991.
[23] R.-C. Li, “Relative perturbation theory. II. Eigenspace and singular subspace variations,” *SIAM Journal on Matrix Analysis and Applications*, vol. 20, no. 2, pp. 471–492, 1998.