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Research Article

On the Interval Baker-Thompson Rule

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The main aim of this paper is to give an axiomatic characterization of the interval Baker-Thompson rule.

1. Introduction

Airport situations have been paid much attention in the literature [1, 2]. In this context we focus on the appealing rule introduced by the economists Baker [3] and Thompson [4]. This rule, called the Baker-Thompson rule, provides a fair and easy share for the costs of the landings.

On the other hand, uncertainty is a daily presence in real life. It affects our decision making and may have influence on cooperation. Recently, various economic and operations research situations under uncertainty are studied. We refer here to Alparslan Gök et al. [5] who present and identify the interval Baker-Thompson rule for solving the aircraft fee problem of an airport with one runway when there is uncertainty regarding the costs of the pieces of the runway; Moretti et al. [6] for cost allocation problems arising from connection situations where edge costs are closed intervals of real numbers; Alparslan Gök et al. [7] for sequencing situations with interval data; Branzei and Alparslan Gök [8] for bankruptcy situations under uncertainty.

In this paper we give an axiomatic characterization of the interval Baker-Thompson rule. Our intuition is from Fragnelli and Marina [9] who study the axiomatic characterization of the classical Baker-Thompson rule.

In the sequel we first recall the classical airport situations. We consider the aircraft fee problem of an airport with one runway and suppose that the planes which are to land are classified into m types. For each $1 \leq j \leq m$, denote the set of landings of planes of type j by N_j and its cardinality by n_j. Then $N = \cup_{j=1}^{m} N_j$ represents the set of all landings. Let c_j represent the cost of a runway adequate for planes of type j. We assume that the types are ordered such that $0 = c_0 < c_1 < \cdots < c_m$. We consider the runway divided into m consecutive pieces P_j, $1 \leq j \leq m$, where P_1 is adequate for landings of planes of type 1; P_1 and P_2 together for landings of planes of type 2, and so on. The cost of piece P_j, $1 \leq j \leq m$, is the marginal cost $c_j - c_{j-1}$. According to this the Baker-Thompson rule is given by $BT_i = \sum_{k=1}^{j} [\sum_{r=k}^{m} n_r]^{-1} (c_k - c_{k-1})$ whenever $i \in N_j$. That is, every landing of planes of type j contributes to the cost of the pieces P_k, $1 \leq k \leq j$, equally allocated among its users $\cup_{r=k}^{m} N_r$.

Next, we recall some properties regarding an allocation problem of a classical airport situation [9]. Formally, an allocation rule for an allocation problem is a map F associating each allocation problem $(N, (c_k)_{k=1,\ldots,m})$, a unique point $F(N, (c_k)_{k=1,\ldots,m}) \in \mathbb{R}^N$ with $\sum_{i \in N} F_i(N, (c_k)_{k=1,\ldots,m}) = c_m$.

An allocation rule F satisfies individual equal sharing (IES) property if for every situation $(N, (c_k)_{k=1,\ldots,m})$, $F_i(N, (c_k)_{k=1,\ldots,m}) \geq c_r / m$ for each $i \in N_r$ and $r = 1, \ldots, m$.

An allocation rule F satisfies collective usage right (CUR) property if for every situation $(N, (c_k)_{k=1,\ldots,m})$, $F_i(N, (c_k)_{k=1,\ldots,m}) \leq c_r (\sum_{l=1,\ldots,r} n_l)^{-1}$ for each $i \in N_r$ and $r = 1, \ldots, m$.

An allocation rule F satisfies consistency on last group (CLAST) property if for every situation $(N, (c_k)_{k=1,\ldots,m})$ and for each $h \in N_m$,

$$F_i(N, (c_k)_{k=1,\ldots,m}) = F_i\left(\widehat{N}, (\widehat{c}_k)_{k=1,\ldots,m}\right), \quad i \in N \setminus \{h\}, \tag{1.1}$$

where $\widehat{N_l} = N_l, l = 1, \ldots, m-1$, $\widehat{N_m} = N_m \setminus \{h\}$ and $\widehat{c}_l = c_l - F_h(N, (c_k)_{k=1,\ldots,m})$, $l = 1, \ldots, m$.

Fragnelli and Marina [9] show that the Baker-Thompson rule satisfies the properties above and do characterization by using them. Our aim is to extend these results to the interval setting.

In this paper we take into account the airport situations where cost of pieces of the runway are intervals. Consider the aircraft fee problem of an airport with one runway. Assume that the planes which are to land are classified into m types. For each $1 \leq j \leq m$, denote the set of landings of planes of type j by N_j and its cardinality by n_j. Then $N = \cup_{j=1}^{m} N_j$ represents the set of all landings. Consider the runway is divided into m consecutive pieces P_j, $1 \leq j \leq m$, where P_1 is sufficient for landings of planes of type 1; P_1 and P_2 together for landings of planes of type 2, and so on. Let the interval T_j with nonnegative finite bounds represent the interval cost of piece P_j, $1 \leq j \leq m$. For a given airport interval situation $(N, (T_k)_{k=1,\ldots,m})$ the Baker-Thompson allocation for each player $i \in N_j$ is given by:

$$\beta_i = \sum_{k=1}^{j} \left(\sum_{r=k}^{m} n_r \right)^{-1} T_k. \tag{1.2}$$

The interval cost allocation rule β presented above called the interval Baker-Thompson rule [5].

Here for the piece P_k of the runway the users are $\cup_{r=k}^{m} N_r$ meaning that there are $\sum_{r=k}^{m} n_r$ users. So, $(\sum_{r=k}^{m} n_r)^{-1} T_k$ is the equal cost share of each user of the piece P_k. This means that a player $i \in N_j$ contributes to the cost of the pieces $P_1, \ldots, P_j$.

Further, we denote by $I(\mathbb{R})$ the set of all closed and bounded intervals in $\mathbb{R}$, and by $I(\mathbb{R})^N$ the set of all n-dimensional vectors with elements in $I(\mathbb{R})$. Let $I, J \in I(\mathbb{R})$ with $I = [\underline{I}, \overline{I}]$, $J = [\underline{J}, \overline{J}]$. Then $I + J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$ [10].

2. On the Interval Baker-Thompson Rule

Theorem 2.1. *Let $(N, (T_k)_{k=1,\dots,m})$ be an airport interval situation. Then the interval Baker-Thompson rule β for each player $i \in N_j$ is $\beta_i = [\underline{\beta}_i, \overline{\beta}_i]$.*

Proof. By (1.2) we have for all $i \in N_j$

$$\beta_i = \sum_{k=1}^{j}\left(\sum_{r=k}^{m} n_r\right)^{-1} T_k = \sum_{k=1}^{j}\left(\sum_{r=k}^{m} n_r\right)^{-1}\left[\underline{T}_k, \overline{T}_k\right] = \left[\left(\sum_{r=k}^{m} n_r\right)^{-1}\underline{T}_k, \left(\sum_{r=k}^{m} n_r\right)^{-1}\overline{T}_k\right] \tag{2.1}$$

$$= \left[\underline{\beta}_i, \overline{\beta}_i\right].$$

□

Theorem 2.1 shows that one can calculate the lower bound of the interval Baker-Thompson rule by using the lower bounds of the interval costs and the upper bound of the interval Baker-Thompson rule by using the upper bounds of the interval costs. These calculations can be done easily by using the classical Baker-Thompson rule. Similar approach can be found for the calculation of the interval Shapley value introduced by Alparslan Gök et al. [11] (Please see Proposition 4.7 of Alparslan Gök et al. [11]).

Next we give the following example to illustrate the calculation of the interval Baker-Thompson rule by using Proposition 2.3.

Example 2.2. (i) Let $(N = \{1, 2, 3\}, (T_k)_{k=1,2,3})$ be an airport interval situation with the interval costs $T_1 = [30, 45]$, $T_2 = [20, 40]$, $T_3 = [100, 120]$. Then, $\underline{\beta} = (10, 20, 120)$ and $\overline{\beta} = (15, 35, 155)$ and by Theorem 2.1, $\beta = ([10, 15], [20, 35], [120, 155])$.

(ii) Let $(N = \{1, 2\}, (T_k)_{k=1,2})$ be an airport interval situation with the interval costs $T_1 = [4, 6]$, $T_2 = [1, 8]$. Then, $\underline{\beta} = (2, 3)$ and $\overline{\beta} = (3, 11)$, and by Theorem 2.1, $\beta = ([2, 3], [3, 11])$.

2.1. Properties of the Interval Baker-Thompson Rule

We define an interval allocation rule for a given airport interval situation $(N, (T_k)_{k=1,\dots,m})$ as a map $\digamma$ associating each allocation situation $(N, (T_k)_{k=1,\dots,m})$ to a unique rule $\digamma(N, (T_k)_{k=1,\dots,m}) = \digamma(N, ([\underline{T}_k, \overline{T}_k])_{k=1,\dots,m}) \in \mathbb{IR}^N$ with

$$\sum_{i\in N} \digamma_i(N, (T_k)_{k=1,\dots,m}) = \sum_{i=1}^{m}\left[\underline{T}_i, \overline{T}_i\right] = \sum_{i=1}^{m} T_i. \tag{2.2}$$

An interval allocation rule $\digamma$ satisfies interval individual equal sharing (IIES) property if for every interval situation $(N, (T_k)_{k=1,\dots,m})$, $(N, (\underline{T}_k)_{k=1,\dots,m})$ and $(N, (\overline{T}_k)_{k=1,\dots,m})$ satisfies IES for each $i \in N_r$ and $r = 1, \dots, m$.

An interval allocation rule $\mathcal{F}$ satisfies interval collective usage right (ICUR) property if for every interval situation $(N, (T_k)_{k=1,\dots,m})$, $(N, (\underline{T}_k)_{k=1,\dots,m})$, and $(N, (\overline{T}_k)_{k=1,\dots,m})$ satisfy CUR for each $i \in N_r$ and $r = 1, \dots, m$.

An interval allocation rule $\mathcal{F}$ satisfies interval individual consistency on last group (ICLAST) property if for every interval situation $(N, (T_k)_{k=1,\dots,m})$, $(N, (\underline{T}_k)_{k=1,\dots,m})$, and $(N, (\overline{T}_k)_{k=1,\dots,m})$ satisfy CLAST for each $i \in N_r$ and $r = 1, \dots, m$.

Next we give some properties of the interval Baker-Thompson rule with the following proposition.

Proposition 2.3. *The interval Baker-Thompson rule β satisfies $IIES$, $ICUR$, and $ICLAST$.*

Proof. The proof can be obtained by following the steps of Fragnelli and Marina [9] for $\underline{\beta}_i$ and $\overline{\beta}_i$ for each $i \in N_j$ and $j = 1, \dots, m$. Then, by using Theorem 2.1 we are done. □

2.2. An Axiomatic Characterization of the Interval Baker-Thompson Rule

We give an axiomatic characterization of the interval Baker-Thompson rule with the following theorem.

Theorem 2.4. *The interval Baker-Thompson rule β is the unique rule satisfying IIES, ICUR, and ICLAST.*

Proof. From Proposition 2.3 we know that β satisfies the three properties. We only need to show the uniqueness. For uniqueness, it is clear from Fragnelli and Marina [9] that $\underline{\beta}_i$ and $\overline{\beta}_i$ for each $i \in N_j$ and $j = 1, \dots, m$, are the unique allocations satisfying the three properties IES, CUR and CLAST. Finally, by Theorem 2.1 we conclude that $\beta_i = [\underline{\beta}_i, \overline{\beta}_i]$ for each $i \in N_j$, $j = 1, \dots, m$ is unique. Hence β is the unique interval allocation satisfying IIES, ICUR and ICLAST. □

3. Final Remarks

Interval uncertainty is the simplest and the most natural type of uncertainty which may influence cooperation because lower and upper bounds for future outcomes or costs of cooperation can always be estimated based on available economic data.

In this paper we consider airport situations where the costs of the pieces of the runway are given by intervals. In this context we give an axiomatic characterization of the interval Baker-Thompson rule which was introduced by Alparslan Gök et al. [5]. Further, we note that the interval Baker-Thompson rule is interesting at an ex-ante stage to inform users about what they can expect to pay for the construction of the runway. At an ex post stage when all costs are known with certainty, the classical Baker-Thompson rule can be applied to pick up effective costs $x_i \in \beta_i$ for each $i \in N$.

We refer reader to Branzei et al. [12] on several procedures specifying how a certain interval solution might be used to transform an interval allocation into a payoff vector when uncertainty regarding the value of the grand coalition is resolved. In the sequel the straightforward extension of the axiomatic characterization of the classical Baker-Thompson rule to the interval setting is advantageous.

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