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Research Article

Some New Difference Inequalities and an Application to Discrete-Time Control Systems

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Two new nonlinear difference inequalities are considered, where the inequalities consist of multiple iterated sums, and composite function of nonlinear function and unknown function may be involved in each layer. Under several practical assumptions, the inequalities are solved through rigorous analysis, and explicit bounds for the unknown functions are given clearly. Further, the derived results are applied to the stability problem of a class of linear control systems with nonlinear perturbations.

1. Introduction

Being an important tool in the study of existence, uniqueness, boundedness, stability, invariant manifolds, and other qualitative properties of solutions of differential equations and integral equations, various generalizations of Gronwall inequalities [1, 2] and their applications have attracted great interests of many mathematicians [3–5]. Some recent works can be found in [6–16] and references therein. Along with the development of the theory of integral inequalities and the theory of difference equations, more and more attentions are paid to discrete versions of Gronwall type inequalities [17–24]. For instance, Pachpatte [17] considered the following discrete inequality:

$$u(n) \le u_0 + \sum_{s=n_0}^{n-1} f(s)[u(s) + h(s)] + \sum_{s=n_0}^{n-1} f(s)\left(\sum_{\tau=n_0}^{s-1} g(\tau)u(\tau)\right), \quad \forall n \in N_0. \tag{1.1}$$

In 2006, Cheung and Ren [18] studied

$$u^p(m,n) \leq c + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t) u^q(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t) u^q(s,t) w(u(s,t)). \tag{1.2}$$

Later, Zheng et al. [24] discussed the following discrete inequality:

$$u(n) \leq a(n) + \sum_{i=1}^{k} \sum_{s=0}^{n-1} f_i(n,s) w_i(u(s)). \tag{1.3}$$

However, the above results are not applicable to inequalities that consist of multiple iterated sums, in particular those in which composite function of nonlinear function and unknown function is involved in each layer of iterated sums. Hence, it is desirable to consider more general difference inequalities of these extended types. They can be used in the study of certain classes of difference equations or applied in many practical engineering problems.

Motivated by the results given in [7, 8, 11, 16–19, 21], in this paper we discuss the following two types of inequalities:

$$\begin{aligned} u(n) \leq{} & a(n) + \sum_{s=n_0}^{n-1} f_1(n,s) w(u(s)) + \sum_{s=n_0}^{n-1} f_1(n,s) w(u(s)) \sum_{\tau=n_0}^{s-1} f_2(s,\tau) w(u(\tau)) \\ & + \sum_{s=n_0}^{n-1} f_1(n,s) w(u(s)) \sum_{\tau=n_0}^{s-1} f_2(s,\tau) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi) w(u(\xi)), \end{aligned} \tag{1.4}$$

$$\begin{aligned} u(n) \leq{} & a(n) + \sum_{s=n_0}^{n-1} f_1(n,s) w_1(u(s)) + \sum_{s=n_0}^{n-1} f_1(n,s) w_1(u(s)) \sum_{\tau=n_0}^{s-1} f_2(s,\tau) w_2(u(\tau)) \\ & + \sum_{s=n_0}^{n-1} f_1(n,s) w_1(u(s)) \sum_{\tau=n_0}^{s-1} f_2(s,\tau) w_2(u(\tau)) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi) w_3(u(\xi)), \end{aligned} \tag{1.5}$$

for all $n \in N_0$. All the assumptions on (1.4) and (1.5) are given in the next sections. The inequalities (1.5) consist of multiple iterated sums, and composite function of nonlinear functions and unknown function may be involved in each layer. Under several practical assumptions, the inequalities are solved through rigorous analysis, and explicit bounds for the unknown functions are given clearly. Further, the derived results are applied to the stability problem of a class of linear control systems with nonlinear perturbations.

2. Main Result

In this section, we proceed to solving the difference inequalities (1.4) and (1.5) and present explicit bounds on the embedded unknown functions. Throughout this paper, let $\mathbf{N}$ denote the set of all natural numbers, and $N_0 = [n_0, K) \cap \mathbf{N}$ where n_0 and K are two constants, satisfying $K > n_0$.

The following theorem summarizes the result on the inequality (1.4).

Theorem 2.1. *Let $u(n)$ and $a(n)$ be nonnegative functions defined on N_0 with $a(n)$ nondecreasing on N_0. Moreover, let $f_i(n,s)$, $i = 1,2,3$ be nonnegative functions for $n_0 \leq s \leq n \leq K$ and nondecreasing in n for fixed $s \in N_0$. Suppose that $w(u)$ is a nondecreasing function on $[0,\infty)$ with $w(u) > 0$ for $u > 0$. Then, the discrete inequality (1.4) gives*

$$u(n) \leq W_1^{-1}\left[W_2^{-1}(U_1(n))\right], \quad \forall n \in [n_0, M_1) \cap \mathbf{N}, \tag{2.1}$$

where

$$\begin{aligned} U_1(n) = W_2\Bigg(& W_1(a(n)) + \sum_{s=n_0}^{n-1} f_1(n,s)\Bigg) \\ & + \sum_{s=n_0}^{n-1} f_1(n,s)\Bigg(\sum_{\tau=n_0}^{s-1} f_2(s,\tau) + \sum_{\tau=n_0}^{s-1} f_2(s,\tau)\sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)\Bigg), \end{aligned} \tag{2.2}$$

$$W_2(u) = \int_1^u \frac{ds}{w\left(W_1^{-1}(s)\right)}, \quad u > 0, \tag{2.3}$$

$$W_1(u) = \int_1^u \frac{ds}{w(s)}, \quad u > 0, \tag{2.4}$$

W_1^{-1}, W_2^{-1} are the inverse functions of W_1, W_2, respectively, and M_1 is the largest natural number such that

$$U_1(M_1) \in \mathrm{Dom}\left(W_2^{-1}\right), \qquad W_2^{-1}(U_1(M_1)) \in \mathrm{Dom}\left(W_1^{-1}\right). \tag{2.5}$$

Proof. Fix $M \in N_{M_1} = [n_0, M_1) \cap \mathbf{N}$, where M is chosen arbitrarily and M_1 is defined by (2.5). For $n \in N_M = [n_0, M] \cap \mathbf{N}$, from (1.4), we have

$$\begin{aligned} u(n) \leq a(M) &+ \sum_{s=n_0}^{n-1} f_1(M,s)w(u(s)) + \sum_{s=n_0}^{n-1} f_1(M,s)w(u(s))\sum_{\tau=n_0}^{s-1} f_2(s,\tau)w(u(\tau)) \\ &+ \sum_{s=n_0}^{n-1} f_1(M,s)w(u(s))\sum_{\tau=n_0}^{s-1} f_2(s,\tau)\sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)w(u(\xi)). \end{aligned} \tag{2.6}$$

Denote the right-hand side of (2.6) by $z_1(n)$, which is a positive and nondecreasing function on N_M with $z_1(n_0) = a(M)$. Then, (2.6) is equivalent to

$$u(n) \leq z_1(n), \quad \forall n \in N_M. \tag{2.7}$$

From (2.6) and (2.7), we observe that

$$\begin{aligned}\Delta z_1(n) &:= z_1(n+1) - z_1(n) \\ &\le f_1(M,n)w(z_1(n)) + f_1(M,n)w(z_1(n))\sum_{\tau=n_0}^{n-1} f_2(n,\tau)w(z_1(\tau)) \\ &\quad + f_1(M,n)w(z_1(n))\sum_{\tau=n_0}^{n-1} f_2(n,\tau)\sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)w(z_1(\xi)) \\ &= f_1(M,n)w(z_1(n))\Bigg[1 + \sum_{\tau=n_0}^{n-1} f_2(n,\tau)w(z_1(\tau)) \\ &\qquad + \sum_{\tau=n_0}^{n-1} f_2(n,\tau)\sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)w(z_1(\xi))\Bigg], \quad \forall n \in N_M.\end{aligned} \tag{2.8}$$

Furthermore, it follows from (2.8) that

$$\begin{aligned}\frac{\Delta z_1(n)}{w(z_1(n))} &\le f_1(M,n)\Bigg[1 + \sum_{\tau=n_0}^{n-1} f_2(n,\tau)w(z_1(\tau)) \\ &\qquad + \sum_{\tau=n_0}^{n-1} f_2(n,\tau)\sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)w(z_1(\xi))\Bigg], \quad \forall n \in N_M.\end{aligned} \tag{2.9}$$

On the other hand, by the mean-value theorem for integrals, for arbitrarily given integers $n, n+1 \in N_M$, there exists η in the open interval $(z_1(n), z_1(n+1))$ such that

$$\begin{aligned}W_1(z_1(n+1)) - W_1(z_1(n)) &= \int_{z_1(n)}^{z_1(n+1)} \frac{ds}{w(z_1(s))} = \frac{\Delta z_1(n)}{w(z_1(\eta))} \le \frac{\Delta z_1(n)}{w(z_1(n))} \\ &\le f_1(M,n)\Bigg[1 + \sum_{\tau=n_0}^{n-1} f_2(n,\tau)w(z_1(\tau)) \\ &\qquad + \sum_{\tau=n_0}^{n-1} f_2(n,\tau)\sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)w(z_1(\xi))\Bigg], \quad \forall n \in N_M,\end{aligned} \tag{2.10}$$

where W_1 is defined by (2.4). By setting $n = s$ in (2.10) and substituting $s = n_0, n_0 + 1, n_0 + 2, \ldots, n - 1$ successively, we obtain

$$W_1(z_1(n)) \leq W_1(z_1(n_0)) + \sum_{s=n_0}^{M-1} f_1(M,s) + \sum_{s=n_0}^{n-1} f_1(M,s) \times \left[\sum_{\tau=n_0}^{s-1} f_2(s,\tau) w(z_1(\tau)) + \sum_{\tau=n_0}^{s-1} f_2(s,\tau) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi) w(z_1(\xi))\right], \quad \forall n \in N_M. \tag{2.11}$$

Let $v_1(n)$ denote the right-hand side of (2.11), which is a positive and nondecreasing function on N_M with $v_1(n_0) = W_1(z_1(n_0)) + \sum_{s=n_0}^{M-1} f_1(M,s)$. Then, (2.11) is equivalent to

$$z_1(n) \leq W_1^{-1}(v_1(n)), \quad \forall n \in N_M. \tag{2.12}$$

By the definition of v_1, we obtain

$$\begin{aligned}\Delta v_1(n) &:= v_1(n+1) - v_1(n) \\ &= f_1(M,n) \left[\sum_{\tau=n_0}^{n-1} f_2(n,\tau) w(z_1(\tau)) + \sum_{\tau=n_0}^{n-1} f_2(n,\tau) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi) w(z_1(\xi))\right], \quad \forall n \in N_M.\end{aligned} \tag{2.13}$$

Considering (2.12), (2.13) and the monotonicity properties of w, W_1^{-1}, and z_1, we get

$$\frac{\Delta v_1(n)}{w(W_1^{-1}(v_1(n)))} \leq f_1(M,n) \left[\sum_{\tau=n_0}^{n-1} f_2(n,\tau) + \sum_{\tau=n_0}^{n-1} f_2(n,\tau) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)\right], \tag{2.14}$$

for all $n \in N_M$. Once again, performing the same procedure as in (2.10) and (2.11), (2.14) gives

$$W_2(v_1(n)) \leq W_2(v_1(n_0)) + \sum_{s=n_0}^{n-1} f_1(M,s) \left[\sum_{\tau=n_0}^{s-1} f_2(s,\tau) + \sum_{\tau=n_0}^{s-1} f_2(s,\tau) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)\right], \tag{2.15}$$

for all $n \in N_M$, where W_2 is defined in (2.3). In the sequel, (2.7), (2.12), and (2.15) render to

$$\begin{aligned} u(n) &\le z_1(n) \le W_1^{-1}(v_1(n)) \\ &= W_1^{-1}\Bigg[W_2^{-1}\Bigg(W_2\Bigg(W_1(a(M)) + \sum_{s=n_0}^{M-1} f_1(M,s)\Bigg) + \sum_{s=n_0}^{n-1} f_1(M,s) \\ &\qquad \times\Bigg(\sum_{\tau=n_0}^{s-1} f_2(s,\tau) + \sum_{\tau=n_0}^{s-1} f_2(s,\tau)\sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)\Bigg)\Bigg)\Bigg], \quad \forall n \in N_M. \end{aligned} \tag{2.16}$$

Let $n = M$ in (2.16), then, we have

$$\begin{aligned} u(n) \le W_1^{-1}\Bigg[W_2^{-1}\Bigg(W_2\Bigg(W_1(a(M)) + \sum_{s=n_0}^{M-1} f_1(M,s)\Bigg) + \sum_{s=n_0}^{M-1} f_1(M,s) \\ \times\Bigg(\sum_{\tau=n_0}^{s-1} f_2(s,\tau) + \sum_{\tau=n_0}^{s-1} f_2(s,\tau)\sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)\Bigg)\Bigg)\Bigg]. \end{aligned} \tag{2.17}$$

Noticing that M is chosen arbitrarily, (2.1) is directly induced by (2.17). The proof of Theorem 2.1 is complete. □

Now, we are in the position of solving the inequality (1.5).

Theorem 2.2. *Let the functions $u(n)$, $a(n)$, $f_i(n,s)$, $i = 1,2,3$, and $\varphi(u)$ be the same as in Theorem 2.1. Suppose that $w_i(u)$, $i = 1,2,3$ are nondecreasing functions on $[0,\infty)$ with $w_i(u) > 0$ for $u > 0$. If $u(n)$ satisfies the discrete inequality* (1.5), *then*

$$u(n) \le \Phi_1^{-1}\Big[\Phi_2^{-1}\Big(\Phi_3^{-1}(U_2(n))\Big)\Big], \quad \forall n \in N_{M_3} = [n_0, M_3) \cap \mathbf{N}, \tag{2.18}$$

where

$$\begin{aligned} U_2(n) = \Phi_3\Bigg(\Phi_2\Bigg(\Phi_1(a(n)) + \sum_{s=n_0}^{n-1} f_1(n,s)\Bigg) + \sum_{s=n_0}^{n-1} f_1(n,s)\sum_{\tau=n_0}^{s-1} f_2(s,\tau)\Bigg) \\ + \sum_{s=n_0}^{n-1} f_1(n,s)\sum_{\tau=n_0}^{s-1} f_2(s,\tau)\sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi), \end{aligned} \tag{2.19}$$

$$\Phi_1(u) = \int_1^u \frac{ds}{w_1(s)}, \quad u > 0, \tag{2.20}$$

$$\Phi_2(u) = \int_1^u \frac{ds}{w_2(\Phi_1^{-1}(s))}, \quad u > 0, \tag{2.21}$$

$$\Phi_3(u) = \int_1^u \frac{ds}{w_3(\Phi_1^{-1}(\Phi_2^{-1}(s)))}, \quad u > 0, \tag{2.22}$$

$\Phi_i^{-1},\ i=1,2,3$ are the inverse functions of $\Phi_i,\ i=1,2,3$, respectively, and M_2 is the largest natural number such that

$$U_2(M_2)\in \mathrm{Dom}\left(\Phi_3^{-1}\right),\qquad \Phi_3^{-1}(U_2(M_2))\in \mathrm{Dom}\left(\Phi_2^{-1}\right),$$
$$\Phi_2^{-1}\left(\Phi_3^{-1}(U_2(M_2))\right)\in \mathrm{Dom}\left(\Phi_1^{-1}\right). \tag{2.23}$$

Proof. Fix $M\in N_{M_2}=[n_0,M_2)\cap \mathbf{N}$, where M is chosen arbitrarily and M_2 is given in (2.23). For $n\in N_M$, from (1.5), we have

$$u(n)\le a(M)+\sum_{s=n_0}^{n-1}f_1(M,s)w_1(u(s))+\sum_{s=n_0}^{n-1}f_1(M,s)w_1(u(s))\sum_{\tau=n_0}^{s-1}f_2(s,\tau)w_2(u(\tau))$$
$$+\sum_{s=n_0}^{n-1}f_1(M,s)w_1(u(s))\sum_{\tau=n_0}^{s-1}f_2(s,\tau)w_2(u(s))\sum_{\xi=n_0}^{\tau-1}f_3(\tau,\xi)w_3(u(\xi)). \tag{2.24}$$

Let $z_2(n)$ represent the right-hand side of (2.24), which is a positive and nondecreasing function on N_{M_2} with $z_2(n_0)=a(M)$. Then, (2.24) is equivalent to

$$u(n)\le z_2(n),\quad \forall n\in N_M. \tag{2.25}$$

Using (2.24) and (2.25), $\Delta z_2(n):=z_2(n+1)-z_2(n)$ can be estimated as follows:

$$\Delta z_2(n)\le f_1(\mathrm{M},n)w_1(z_2(n))+f_1(M,n)w_1(z_2(n))\sum_{\tau=n_0}^{n-1}f_2(n,\tau)w_2(z_2(\tau))$$
$$+f_1(M,n)w_1(z_2(n))\sum_{\tau=n_0}^{n-1}f_2(n,\tau)w_2(z_2(n))\sum_{\xi=n_0}^{\tau-1}f_3(\tau,\xi)w_3(z_2(\xi))$$
$$=f_1(M,n)w_1(z_2(n))\Bigg[1+\sum_{\tau=n_0}^{n-1}f_2(n,\tau)w_2(z_2(\tau))$$
$$+\sum_{\tau=n_0}^{n-1}f_2(n,\tau)w_2(z_2(\tau))\sum_{\xi=n_0}^{\tau-1}f_3(\tau,\xi)w_3(z_2(\xi))\Bigg],\quad \forall n\in N_M, \tag{2.26}$$

Implying

$$\frac{\Delta z_2(n)}{w_1(z_2(n))}\le f_1(M,n)\Bigg[1+\sum_{\tau=n_0}^{n-1}f_2(n,\tau)w_2(z_2(\tau))$$
$$+\sum_{\tau=n_0}^{n-1}f_2(n,\tau)w_2(z_2(\tau))\sum_{\xi=n_0}^{\tau-1}f_3(\tau,\xi)w_3(z_2(\xi))\Bigg], \tag{2.27}$$

for all $n \in N_M$. Performing the same derivation as in (2.10) and (2.11), we obtain from (2.27) that

$$\begin{aligned} \Phi_1(z_2(n)) \le \Phi_1(z_2(n_0)) &+ \sum_{s=n_0}^{M-1} f_1(M,s) + \sum_{s=n_0}^{n-1} f_1(M,s) \\ &\times \left[\sum_{\tau=n_0}^{s-1} f_2(s,\tau) w_2(z_2(\tau)) \right. \\ &\left. + \sum_{\tau=n_0}^{s-1} f_2(s,\tau) w_2(z_2(\tau)) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi) w_3(z_2(\xi)) \right], \quad \forall n \in N_M, \end{aligned} \tag{2.28}$$

where Φ_1 is defined in (2.20). Denote by $v_2(n)$ the right-hand side of (2.28), which is a positive and nondecreasing function on N_{M_2} with $v_2(n_0) = \Phi_1(z_2(n_0)) + \sum_{s=n_0}^{M-1} f_1(M,s) = \Phi_1(a(M)) + \sum_{s=n_0}^{M-1} f_1(M,s)$. Then, (2.28) is equivalent to

$$z_2(n) \le \Phi_1^{-1}(v_2(n)), \quad \forall n \in N_M. \tag{2.29}$$

By the definition of v_2, we obtain

$$\begin{aligned} \Delta v_2(n) &:= v_2(n+1) - v_2(n) \\ &= f_1(M,n) \left[\sum_{\tau=n_0}^{n-1} f_2(n,\tau) w_2(z_2(\tau)) \right. \\ &\left. \quad + \sum_{\tau=n_0}^{n-1} f_2(n,\tau) w_2(z_2(\tau)) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi) w_3(z_2(\xi)) \right], \quad \forall n \in N_M. \end{aligned} \tag{2.30}$$

From (2.29), (2.30) and the monotonicity of w_2, Φ_1^{-1}, and z_2, we get

$$\frac{\Delta v_2(n)}{w_2(\Phi_1^{-1}(v_2(n)))} \le f_1(M,n) \left[\sum_{\tau=n_0}^{n-1} f_2(n,\tau) + \sum_{\tau=n_0}^{n-1} f_2(n,\tau) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi) w_3\Big(\Phi_1^{-1}(v_2(\xi))\Big) \right], \tag{2.31}$$

for all $n \in N_M$. Similarly to (2.28), it follows from (2.31) that

$$\begin{aligned} \Phi_2(v_2(n)) \le \Phi_2(v_2(n_0)) &+ \sum_{s=n_0}^{M-1} f_1(M,s) \sum_{\tau=n_0}^{s-1} f_2(s,\tau) \\ &+ \sum_{s=n_0}^{n-1} f_1(M,s) \sum_{\tau=n_0}^{s-1} f_2(s,\tau) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi) w_3\Big(\Phi_1^{-1}(v_2(n))\Big), \end{aligned} \tag{2.32}$$

for all $n \in N_M$, where Φ_2 is defined in (2.21). Let $v_3(n)$ denote the right-hand side of (2.32), which is a positive and nondecreasing function on N_{M_2} with

$$\begin{aligned} v_3(n_0) &= \Phi_2(v_2(n_0)) + \sum_{s=n_0}^{M-1} f_1(M,s) \sum_{\tau=n_0}^{s-1} f_2(s,\tau) \\ &= \Phi_2\left(\Phi_1(a(M)) + \sum_{s=n_0}^{M-1} f_1(M,s)\right) + \sum_{s=n_0}^{M-1} f_1(M,s) \sum_{\tau=n_0}^{s-1} f_2(s,\tau). \end{aligned} \tag{2.33}$$

Then, (2.32) is equivalent to

$$v_2(n) \le \Phi_2^{-1}(v_3(n)), \quad \forall n \in N_M. \tag{2.34}$$

By the definition of v_3,

$$\begin{aligned} \Delta v_3(n) &:= v_3(n+1) - v_3(n) \\ &= f_1(M,n) \sum_{\tau=n_0}^{n-1} f_2(n,\tau) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi) w_3\left(\Phi_1^{-1}(v_2(\xi))\right), \quad \forall n \in N_M. \end{aligned} \tag{2.35}$$

In consequence, (2.34), (2.35) and the monotonicity properties of w_3, Φ_1^{-1}, and v_2 lead to

$$\frac{\Delta v_3(n)}{w_3(\Phi_1^{-1}(\Phi_2^{-1}(v_3(n))))} \le f_1(M,n) \sum_{\tau=n_0}^{n-1} f_2(n,\tau) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi), \quad \forall n \in N_M. \tag{2.36}$$

Similarly to (2.28) and (2.32), we obtain from (2.36) that

$$\Phi_3(v_3(n)) \le \Phi_3(v_3(n_0)) + \sum_{s=n_0}^{n-1} f_1(M,s) \sum_{\tau=n_0}^{s-1} f_2(s,\tau) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi), \quad \forall n \in N_M, \tag{2.37}$$

where Φ_3 is defined in (2.22).

Summarizing the results in (2.25), (2.29), (2.34), and (2.37), we can conclude that

$$\begin{aligned} u(n) &\le z_2(n) \le \Phi_1^{-1}[v_2(n)] \le \Phi_1^{-1}\left[\Phi_2^{-1}(v_3(n))\right] \\ &\le \Phi_1^{-1}\left[\Phi_2^{-1}\left(\Phi_3^{-1}\left(\Phi_3(v_3(n_0)) + \sum_{s=n_0}^{n-1} f_1(M,s) \sum_{\tau=n_0}^{s-1} f_2(s,\tau) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)\right)\right)\right] \end{aligned}$$

$$= \Phi_1^{-1}\left[\Phi_2^{-1}\left(\Phi_3^{-1}\left(\Phi_3\left(\Phi_2\left(\Phi_1(a(M)) + \sum_{s=n_0}^{M-1} f_1(M,s)\right) + \sum_{s=n_0}^{M-1} f_1(M,s)\sum_{\tau=n_0}^{s-1} f_2(s,\tau)\right) + \sum_{s=n_0}^{n-1} f_1(M,s)\sum_{\tau=n_0}^{s-1} f_2(s,\tau)\sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)\right)\right)\right], \tag{2.38}$$

for all $n \in N_M$. As $n = M$, (2.38) yields

$$u(M) \leq \Phi_1^{-1}\left[\Phi_2^{-1}\left(\Phi_3^{-1}\left(\Phi_3\left(\Phi_2\left(\Phi_1(a(M)) + \sum_{s=n_0}^{M-1} f_1(M,s)\right) + \sum_{s=n_0}^{M-1} f_1(M,s)\sum_{\tau=n_0}^{s-1} f_2(s,\tau)\right) + \sum_{s=n_0}^{M-1} f_1(M,s)\sum_{\tau=n_0}^{s-1} f_2(s,\tau)\sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi)\right)\right)\right]. \tag{2.39}$$

Since M is chosen arbitrarily in (2.39), the inequality (2.18) is derived. This completes the proof of Theorem 2.2. □

3. Applications

In this section, the result of Theorem 2.2 is applied to explore the asymptotic stability behavior of a class of discrete-time control systems [17]

$$x(n+1) = A(n)x(n) + f(n, x(n), \sigma(n)), \qquad x(n_0) = x_0, \tag{3.1}$$

where

$$\sigma(n) = \theta(n) + \sum_{s=n_0}^{n-1} k(n, s, x(s)). \tag{3.2}$$

Control system (3.1) can be regarded as the perturbation counterpart of the following closed-loop system:

$$y(n+1) = A(n)y(n), \qquad y(n_0) = x_0. \tag{3.3}$$

The functions $x,\ y,\ \theta,\ \sigma$ are defined on $N \to \mathbf{R}^r$, the r-dimensional vector space, $A(n)$ is an $r \times r$ matrix with $\det A(n) \neq 0$, and the functions f and k are defined on $N \times \mathbf{R}^r \times \mathbf{R}^r$ and $N \times N \times \mathbf{R}^r$, respectively. Moreover, f and k are supposed to meet the following constraints:

$$\left|f(n,x(n),\sigma(n))\right| \leq g_1(n)e^{-\alpha n}w_1(|x(n)|e^{\alpha n})(1+|\sigma(n)|), \tag{3.4}$$

$$|k(n,s,x(s))| \leq g_2(n,s)w_2(|x(n)|e^{\alpha n})\left(1+\sum_{\tau=n_0}^{s-1} g_3(s,\tau)w_3(|x(\tau)|e^{\alpha\tau})\right), \tag{3.5}$$

where $\alpha > 0$ is a constant, $g_i,\ i = 1,2,3$ are nonnegative real-valued functions defined on N_0 and $N_0 \times N_0$, respectively, $g_2(n,s)$ and $g_3(n,s)$ are nondecreasing in n for fixed $s \in N_0$, and $w_i(u),\ i = 1,2,3$ are positive and continuous functions defined on $[0,\infty)$. The symbol $|\cdot|$ denotes norm on $\mathbf{R}^r$ as well as a corresponding consistent matrix norm.

Corollary 3.1. *Consider the discrete-time control systems* (3.1) *and* (3.2), *where the perturbation-related functions f and k satisfy the conditions* (3.4) *and* (3.5). *Assume that the fundamental solution matrix $Y(n)$ of the linear system* (3.3) *satisfies*

$$\left|Y(n)Y^{-1}(s)\right| \leq C\exp(-\alpha(n-s)), \quad 0 \leq s \leq n \leq \infty, \tag{3.6}$$

where $C > 0$ is a constant. Then, any solutions of the control systems (3.1) *and* (3.2), *denoted by $x_\sigma(n,n_0,x_0)$, can be estimated by*

$$|x_\sigma(n,n_0,x_0)| \leq \exp(-\alpha n)\left\{\Phi_4^{-1}\left[\Phi_5^{-1}\left(\Phi_6^{-1}(U_4(n))\right)\right]\right\}, \quad \forall n \in N_{M_4} = [n_0, M_4) \cap \mathbf{N}, \tag{3.7}$$

where

$$\begin{aligned} U_4(n) = {} & \Phi_6\Bigg(\Phi_5\Bigg(\Phi_4(|x_0|C\exp(\alpha n_0)) + \sum_{s=n_0}^{n-1} Ce^{\alpha}g_1(s)(1+|\theta(s)|)\Bigg) \\ & + \sum_{s=n_0}^{n-1} Ce^{\alpha}g_1(s)(1+|\theta(s)|)\sum_{\tau=n_0}^{s-1} f_2(s,\tau)\Bigg) \\ & + \sum_{s=n_0}^{n-1} Ce^{\alpha}g_1(s)(1+|\theta(s)|)\sum_{\tau=n_0}^{s-1} f_2(s,\tau)\sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi), \\ \Phi_4(u) = {} & \int_1^u \frac{ds}{w_1(s)}, \quad u > 0, \\ \Phi_5(u) = {} & \int_1^u \frac{ds}{w_2(\Phi_1^{-1}(s))}, \quad u > 0, \\ \Phi_6(u) = {} & \int_1^u \frac{ds}{w_3(\Phi_1^{-1}(\Phi_2^{-1}(s)))}, \quad u > 0, \end{aligned} \tag{3.8}$$

Φ_i^{-1}, $i = 4, 5, 6$ *are the inverse functions of* Φ_i, $i = 4, 5, 6$, *respectively, and* M_4 *is the largest natural number such that*

$$U_4(M_4) \in \operatorname{Dom}\left(\Phi_6^{-1}\right), \qquad \Phi_6^{-1}(U_4(M_4)) \in \operatorname{Dom}\left(\Phi_5^{-1}\right),$$
$$\Phi_5^{-1}\left(\Phi_6^{-1}(U_4(M_4))\right) \in \operatorname{Dom}\left(\Phi_4^{-1}\right). \tag{3.9}$$

Proof. By using the variation of constants formula, any solution $x_\sigma(n, n_0, x_0)$ of (3.1) and (3.2) can be represented by

$$x_\sigma(n, n_0, x_0) = Y(n)Y^{-1}(n_0)x_0 + \sum_{s=n_0}^{n-1} Y(s)Y^{-1}(s+1)f(s, x_\sigma(s, n_0, x_0), \sigma(s)), \tag{3.10}$$

for all $n \in N_0$. Using the conditions (3.4) and (3.6) in (3.10), we have

$$|x_\sigma(n, n_0, x_0)| \leq |x_0| C\exp(-\alpha(n - n_0)) + \sum_{s=n_0}^{n-1} C\exp(-\alpha(n - s - 1))$$
$$\times g_1(s)e^{-\alpha s}w_1(|x_\sigma(s, n_0, x_0)|e^{\alpha s})(1 + |\sigma(s)|), \quad \forall n \in N_0. \tag{3.11}$$

Further, using the relationships (3.2), (3.5), and (3.11), we derive

$$|x_\sigma(n, n_0, x_0)| \leq |x_0| C\exp(-\alpha(n - n_0)) + \sum_{s=n_0}^{n-1} C\exp(-\alpha(n - 1)) \times g_1(s)w_1(|x_\sigma(s, n_0, x_0)|e^{\alpha s})$$
$$\left[1 + |\theta(s)| + \sum_{\tau=n_0}^{s-1} g_2(s, \tau)w_2(|x_\sigma(\tau, n_0, x_0)|e^{\alpha\tau})\right.$$
$$\left.\times\left(1 + \sum_{\tau=n_0}^{\tau-1} g_3(\tau, \xi)w_3\left(|x_\sigma(\xi, n_0, x_0)|e^{\alpha\xi}\right)\right)\right], \tag{3.12}$$

for all $n \in N_0$. Let $u(n) = |x_\sigma(n, n_0, x_0)|\exp(\alpha n)$, then, (3.12) can be rewritten as

$$u(n) \leq |x_0| C\exp(\alpha n_0) + \sum_{s=n_0}^{n-1} Ce^{\alpha}g_1(s)(1 + |\theta(s)|)w_1(u(s))$$
$$+ \sum_{s=n_0}^{n-1} Ce^{\alpha}g_1(s)(1 + |\theta(s)|)w_1(u(s)) \sum_{\tau=n_0}^{s-1} g_2(s, \tau)w_2(u(\tau))$$

$$+\sum_{s=n_0}^{n-1} Ce^{\alpha} g_1(s)(1+|\theta(s)|) w_1(u(s)) \sum_{\tau=n_0}^{s-1} g_2(s,\tau)$$
$$\times w_2(u(\tau)) \sum_{\xi=n_0}^{\tau-1} g_3(\tau,\xi) w_3(u(\xi)), \quad \forall n \in N_0. \tag{3.13}$$

Let $a(n) = |x_0| C \exp(\alpha n_0)$, $f_1(n,s) = C g_1(s) e^{\alpha}(1+|\theta(s)|)$, $f_2(n,s) = g_2(n,s)$, and $f_3(n,s) = g_3(n,s)$, then (3.13) can be further estimated as follows:

$$\begin{aligned} u(n) \le a(n) &+ \sum_{s=n_0}^{n-1} f_1(n,s) w_1(u(s)) + \sum_{s=n_0}^{n-1} f_1(n,s) w_1(u(s)) \sum_{\tau=n_0}^{s-1} f_2(s,\tau) w_2(u(\tau)) \\ &+ \sum_{s=n_0}^{n-1} f_1(n,s) w_1(u(s)) \sum_{\tau=n_0}^{s-1} f_2(s,\tau) w_2(u(\tau)) \sum_{\xi=n_0}^{\tau-1} f_3(\tau,\xi) w_3(u(\xi)), \end{aligned} \tag{3.14}$$

for all $n \in N_0$. Notice that, by our assumption, all functions in (3.14) satisfy the conditions of Theorem 2.2. Applying Theorem 2.2 to the inequality (3.14), (3.7) is immediately derived, where the relationship $u(n) = |x_\sigma(n, n_0, x_0)| \exp(\alpha n)$ is adopted. This completes the proof of Corollary 3.1. □

Based on Corollary 3.1 and one additional assumption, the next corollary gives the stability result of the control system (3.1) and (3.2).

Corollary 3.2. *Under the assumptions of Corollary 3.1, if there exists a positive constant B such that*

$$\left\{\Phi_4^{-1}\left[\Phi_5^{-1}\left(\Phi_6^{-1}(U_4(n))\right)\right]\right\} \le B, \quad \forall n \in \mathbf{N}, \tag{3.15}$$

then the perturbed system (3.1) *and* (3.2) *is exponentially asymptotically stable.*

Proof. Under condition (3.15), (3.7) can be further estimated as follows:

$$|x_\sigma(n, n_0, x_0)| \le B \exp(-\alpha n), \quad \forall n \in [n_0, \infty) \cap \mathbf{N}. \tag{3.16}$$

The exponentially asymptotic stability of system (3.1) and (3.2) is directly implied. □

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