

## Research Article

# A Positivity-Preserving Numerical Scheme for Nonlinear Option Pricing Models

Shengwu Zhou, Wei Li, Yu Wei, and Cui Wen

College of Sciences, China University of Mining and Technology, Jiangsu, Xuzhou 221116, China

Correspondence should be addressed to Wei Li, liwei.123@126.com

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A positivity-preserving numerical method for nonlinear Black-Scholes models is developed in this paper. The numerical method is based on a nonstandard approximation of the second partial derivative. The scheme is not only unconditionally stable and positive, but also allows us to solve the discrete equation explicitly. Monotone properties are studied in order to avoid unwanted oscillations of the numerical solution. The numerical results for European put option and European butterfly spread are compared to the standard finite difference scheme. It turns out that the proposed scheme is efficient and reliable.

## 1. Introduction

It is widely recognized that the value of a European option can be obtained by solving the linear Black-Scholes equation under quite restrictive assumptions (such as liquid, frictionless, and complete markets) [1, 2]. However, these restrictive assumptions are never fulfilled in reality. In order to conform the actual situation, many modified Black-Scholes models have been proposed in recent years, such as transaction costs (Leland [3], Palmer [4], Hoggard et al. [5], Barles and Soner [6], and Jandačka and Ševčovič [7]), illiquid market (Frey and Patie [8], Sircar and Papanicolaou [9], Liu and Yong [10], etc.), and volatility uncertainty (Avellaneda et al. [11]). These models result in quasilinear or fully nonlinear Black-Scholes equations.

In this paper, we are interested in the option pricing model with transaction costs proposed by Barles and Soner [6] that are motivated by Hodges and Neuberger [12]. In their model, the value  $V(S, t)$  of the option satisfies the following partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma(V_{SS})^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (1.1)$$

with the nonlinear volatility  $\sigma(V_{SS})$  that reads

$$\sigma(V_{SS})^2 = \sigma_0^2 \left( 1 + \Psi \left( e^{r(T-t)} a^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) \right) \quad (1.2)$$

and the terminal condition

$$V(S, T) = f(S), \quad (1.3)$$

where  $S$  is the price of the underlying asset,  $T$  is the maturity date,  $r$  is the risk-free interest rate,  $\sigma_0$  is the asset volatility, and  $a = \mu\sqrt{\gamma N}$ , with the proportional transaction cost  $\mu$ , the risk aversion factor  $\gamma$ , and the number  $N$  of options to be sold.  $\Psi(x)$  is the solution of the following ordinary differential equation:

$$\Psi'(x) = \frac{\Psi(x) + 1}{2\sqrt{x\Psi(x) - x}}, \quad x \neq 0, \quad \Psi(0) = 0. \quad (1.4)$$

The existence and uniqueness of the solution of (1.1)–(1.3) have been shown by the theory of stochastic optimal control in [6]. However, analytical solutions cannot be found because of fully nonlinear properties of (1.1); thus, we need to compute the option values numerically.

There have been rich achievements for the numerical method of linear Black-Scholes equations (e.g., see [13–17]). As for the nonlinear situation, only a few results can be found. There is a stable numerical scheme developed in [7] (see also [18] for application to a general class of nonlinear Black-Scholes equations) for the so-called gamma equation (a quasilinear parabolic equation for the  $SV_{SS}$ ). Recently Kútik and Mikula [19] did some progress in showing its stability and accuracy for nonlinear Black-Scholes equation. Company et al. [20–22] construct explicit finite difference schemes for (1.1)–(1.3), and consistency and stability are studied. However, they have the disadvantage that strictly restrictive conditions on the discretization parameters are needed to guarantee stability and positivity. The implicit schemes do not have this disadvantage, but they are quite time-consuming. Yousef et al. [23] develop a new second-order exponential time differencing (ETD) scheme to avoid unwanted oscillations near the non-smooth nodes for the Hoggard-Whalley-Wilmott (HWW) model [5] based on the Cox and Matthews approach [24] and partial fraction version of the matrix exponential, but the theoretical analysis of stability and convergence are not studied. Some authors (for instance, Düring et al. [25] for European options, Dremkova and Ehrhardt [26] for American options) construct high-order compact difference schemes with frozen values of the nonlinear coefficient of the nonlinear Black-Scholes equation to make the scheme linear and show that the resulted linearized problem is stable.

On the other hand, since the value of option is nonnegative, it is very important to make numerical schemes preserve the positivity of solution. Several authors have developed some schemes that guarantee the positivity of solutions for ordinary differential equations [27, 28] and parabolic equations [29]. In [30], Chen-Charpentier and Kojouharov propose an unconditionally positivity-preserving scheme for linear advection-diffusion reaction equations. They construct a nontraditional discretization of the advection and diffusion terms by the approximation of the spatial derivatives using values at different time levels.

Motivated by this work, we will develop the method to a nonlinear Black-Scholes equation, and some properties (such as stability, monotonicity, and consistency,) of numerical scheme are studied in this paper. The new numerical method unconditionally preserves the positivity of the solutions, the stability, and monotonicity of the scheme. In addition, the designed numerical approximations allow us to solve the discrete equation explicitly, which reduces the time of calculation and increases the efficiency of the methods.

The rest of the paper is organized as follows. In the next section the original problem (1.1)–(1.3) is transformed into a nonlinear diffusion problem by an appropriate change of variables and some properties of the function  $\Psi(x)$  are given. In Section 3, the discretization method is constructed. In Section 4, we prove the boundedness of coefficients, positivity, and monotonicity of the numerical scheme. Stability and consistency are studied in Section 5. In Section 6, numerical experiments for European put option and a European butterfly spread are presented to support these theoretical results. Finally, some conclusions are drawn in Section 7.

## 2. The Transformed Problem

For the convenience in the numerical processing and the study of the numerical analysis, we are going to transform the problem (1.1)–(1.3) into a nonlinear diffusion equation. Taking the variable transformation

$$x = \exp(r(T-t))S, \quad \tau = T-t, \quad u = \exp(r(T-t))V. \quad (2.1)$$

the original problem (1.1)–(1.3) is transformed into

$$\mathcal{L}(u) = \frac{\partial u}{\partial \tau} - \frac{1}{2}\sigma_0^2 \left( 1 + \Psi \left( a^2 x^2 \frac{\partial^2 u}{\partial x^2} \right) \right) x^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in [0, +\infty), \tau \in [0, T], \quad (2.2)$$

with the initial condition

$$u(x, 0) = f(x) \quad (2.3)$$

and the boundary condition

$$\begin{aligned} \text{Call option : } u(0, t) = 0, \quad \lim_{x \rightarrow \infty} \frac{u(x, \tau)}{x} = 1, \\ \text{Put option : } u(0, t) = K, \quad \lim_{x \rightarrow \infty} u(x, \tau) = 0. \end{aligned} \quad (2.4)$$

The following two lemmas give the properties of the function  $\Psi$  appearing in (1.2), which will play an important role in the numerical analysis and numerical calculation.

**Lemma 2.1** (see [20]). *The solution  $\Psi$  of ordinary differential equation (1.4) exists and is unique, and it satisfies,*

- (i)  $\Psi$  is an increasing function mapping the real line onto the interval  $(-1, +\infty)$ .

(ii)  $\Psi = \Psi(A)$  is implicitly defined by

$$\begin{aligned} A &= \left( -\frac{\operatorname{arc\,sinh} \sqrt{\Psi}}{\sqrt{\Psi+1}} + \sqrt{\Psi} \right)^2 \quad \text{if } \Psi > 0, \\ A &= -\left( \frac{\operatorname{arc\,sin} \sqrt{-\Psi}}{\sqrt{\Psi+1}} - \sqrt{-\Psi} \right)^2 \quad \text{if } -1 < \Psi < 0, \end{aligned} \quad (2.5)$$

(iii) if  $A > 0$ , then the function  $\Psi(A)$  is bounded and satisfies

$$0 < \Psi(A) < \Psi'(A_2)A + d_2, \quad (2.6)$$

where

$$\begin{aligned} A_2 &= \left( \sinh 2 - \frac{2}{\sqrt{(\sinh 2)^2 + 1}} \right)^2 \approx 9.58, \quad \Psi(A_2) = (\sinh 2)^2, \\ \Psi'(A_2) &= \frac{(e^8 + 2e^4 + 1)^2}{e^{16} - 66e^8 + 1} \approx 1.10, \quad d_2 = \Psi(A_2) - \Psi'(A_2)A_2 \approx 2.62. \end{aligned} \quad (2.7)$$

**Lemma 2.2** (see [20]). Let  $g(A) = A\Psi(A)$ , then  $g(A)$  is continuously differentiable at  $A = 0$  and satisfies

$$|g'(A)| \leq \max\{G, 2|A|\Psi'(A_2) + d_2\}, \quad A \in \mathbb{R}, \quad (2.8)$$

where  $A_2$  and  $d_2$  are given by (2.7), and

$$A_1 = -\frac{(4\pi - 3\sqrt{3})^2}{36}, \quad G = \max\{|g'(A)|; A_1 \leq A \leq A_2\}. \quad (2.9)$$

### 3. The Unconditionally Positivity-Preserving Scheme

Since the value of an option is nonnegative, it is important that numerical scheme is positivity preserving.

We see that the problem (2.2) is described in an infinite domain  $\mathbb{R}^+ \times [0, T]$ , which makes it difficult to construct scheme effectively. Let us consider the truncated numerical domain  $\Omega = [0, B] \times [0, T]$  and discretize it in the following form. We introduce a grid of mesh points  $(x, \tau) = (x_i, \tau_n)$ , where  $x_i = ih$ ,  $\tau_n = nk$ ,  $i = 0, 1, \dots, N$ ,  $n = 0, 1, \dots, L$ , and the spatial step size given by  $h = B/N$ , and the time step size is  $k = T/L$ . Let us denote the

approximation of  $u(x_i, \tau_n)$  by  $u_i^n$  and define the following finite difference approximations of derivatives:

$$\begin{aligned}\frac{\partial u}{\partial \tau}(x_i, \tau_n) &\approx \frac{u_i^{n+1} - u_i^n}{k}, \\ \frac{\partial^2 u}{\partial x^2}(x_i, \tau_n) &\approx \frac{u_{i+1}^n - 2u_i^{n+1} + u_{i-1}^n}{h^2} = \delta_i^n.\end{aligned}\tag{3.1}$$

Clearly, the approximation used to calculate the second partial derivative of  $u$  with respect to  $x$  is nonstandard. From (2.2), we can obtain the positivity-preserving finite-difference numerical scheme as follows:

$$\frac{u_i^{n+1} - u_i^n}{k} - \frac{1}{2}\sigma_0^2 \left(1 + \Psi\left(a^2 x_i^2 \delta_i^n\right)\right) x_i^2 \frac{u_{i+1}^n - 2u_i^{n+1} + u_{i-1}^n}{h^2} = 0.\tag{3.2}$$

Scheme (3.2) is equivalent to

$$u_i^{n+1} = \frac{\rho \alpha_i^n (u_{i+1}^n + u_{i-1}^n) + u_i^n}{1 + 2\rho \alpha_i^n},\tag{3.3}$$

where

$$\rho = \frac{k}{h^2}, \quad \alpha_i^n = \frac{1}{2}\sigma_0^2 \left(1 + \Psi\left(a^2 x_i^2 \delta_i^n\right)\right) x_i^2.\tag{3.4}$$

*Remark 3.1.* From property (i) of Lemma 2.1,  $\Psi$  takes values in the interval  $(-1, +\infty)$ , so the coefficients are nonnegative, that is,  $\alpha_i^n \geq 0$  for any  $i, n$ .

Obviously, numerical scheme (3.3) is unconditionally positive for a nonnegative payoff  $u_i^0$ . However, we cannot obtain the numerical solution explicitly since the nonlinear term  $\alpha_i^n$  involves the value  $u$  at the time level  $n + 1$ , which makes it quite difficult to prove the stability of the scheme presented previously. In fact, the numerical scheme (3.3) can only be solved by a nonlinear iteration in each time step which is quite time-consuming.

In order to obtain an efficient scheme, we correct the approximation of the nonlinear coefficients in (2.2) by using the standard second-order central difference. Thus the corrected numerical scheme is as follows:

$$u_i^{n+1} = \frac{\rho \beta_i^n (u_{i+1}^n + u_{i-1}^n) + u_i^n}{1 + 2\rho \beta_i^n},\tag{3.5}$$

where

$$\beta_i^n = \frac{1}{2}\sigma_0^2 \left(1 + \Psi\left(a^2 x_i^2 \Delta_i^n\right)\right) x_i^2, \quad \Delta_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}.\tag{3.6}$$

Since the calculation of (3.5) for  $i = 0$  and  $i = N$  requires us to know the fictitious values  $u_{-1}^n$  and  $u_{N+1}^n$ , we obtain them using the linear extrapolation as follows:

$$u_{-1}^n = 2u_0^n - u_1^n, \quad u_{N+1}^n = 2u_N^n - u_{N-1}^n. \quad (3.7)$$

Thus the values of boundary points turn out

$$u_0^{n+1} = u_0^n = \dots = u_0^0 = f(0), \quad u_N^{n+1} = u_N^n = \dots = u_N^0 = f(B). \quad (3.8)$$

The left boundary condition is not needed and in fact must not be prescribed in the case of a parabolic equation with degenerating diffusion term at  $x = 0$ . This is known in the literature as the Fichera condition [31] (see [18] for application in Black-Scholes equations, Chapter 8, (8.25)). The Fichera condition is just a comparison of the speed of degeneration versus. speed of advection at the boundary  $x = 0$ . Fortunately, the left boundary condition in (3.8) is a consequence of (3.5) with  $x_0 = 0$ ,  $i = 0$ ,  $u_0^0 = f(0)$ . The only boundary condition that can be prescribed is the right boundary condition at  $x = B$ . For the convenience in the numerical calculation, let us denote the vectors  $u^n = [u_0^n, u_1^n, \dots, u_N^n]^T$ , then numerical scheme (3.5), (3.8) can be written in matrix form

$$\begin{aligned} u^{n+1} &= C(n)u^n, \\ u^0 &= [f(0), f(x_1), \dots, f(B)]^T, \end{aligned} \quad (3.9)$$

where

$$C(n) = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ a_1 & b_1 & c_1 & 0 & \dots & 0 \\ 0 & a_2 & b_2 & c_2 & \dots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (3.10)$$

$$a_i = c_i = \frac{\rho\beta_i^n}{1 + 2\rho\beta_i^n}, \quad b_i = \frac{1}{1 + 2\rho\beta_i^n}, \quad i = 1, 2, \dots, N-1. \quad (3.11)$$

In order to study the related properties of numerical scheme (such as monotonicity and stability), we need to know the behaviour of  $\Delta_i^n$  appearing in the nonlinear term  $\beta_i^n$ .

**Lemma 3.2.** *With the previous notation,  $\Delta_i^n$  appearing in the nonlinear term  $\beta_i^n$  satisfies the scheme*

$$\begin{aligned} \Delta_i^{n+1} &= \frac{\rho\beta_{i+1}^n}{1 + 2\rho\beta_{i+1}^n} \Delta_{i+1}^n + \frac{\rho\beta_{i-1}^n}{1 + 2\rho\beta_{i-1}^n} \Delta_{i-1}^n + \frac{1}{1 + 2\rho\beta_i^n} \Delta_i^n, \quad 1 \leq i \leq N-1, \\ \Delta_0^n &= \Delta_N^n = 0, \quad n = 0, 1, 2, \dots, L. \end{aligned} \quad (3.12)$$

*Proof.* From (3.6) and (3.7) we can know for  $i = 0$  and  $i = N$  that

$$\Delta_0^n = \Delta_N^n = 0, \quad n = 0, 1, 2, \dots, L. \quad (3.13)$$

Putting (3.5) into the expression (3.6) of  $\Delta_i^n$ , and after a simple calculation we can obtain (3.12).

On the other hand, the most common option price sensitivities are the first and second derivatives with respect to the price of the underlying asset, that is, “delta” and “gamma”, respectively. These are important features in risk management and are challenging to compute numerically. From the transformation (2.1) we can obtain the approximations of Greeks of the option as follows:

$$\begin{aligned} \text{Gamma} &= \frac{\partial^2 V}{\partial S^2}(S_i, t_n) = \exp(r(T-t)) \Delta_i^n(u), \\ \text{Delta} &= \frac{\partial V}{\partial S}(S_i, t_n) = \exp(r(T-t)) \frac{u_{i+1}^n - u_{i-1}^n}{2h}. \end{aligned} \quad (3.14)$$

□

## 4. Properties of the Numerical Scheme

### 4.1. Boundedness of the Coefficients

For the sake of convenience, we introduce the following definition.

*Definition 4.1* (see [32]). If  $x = [x_1, x_2, \dots, x_N]$  is a vector in  $\mathbb{R}^N$ , then its 1-norm is denoted by  $\|x\|_1 = \sum_{i=1}^N |x_i|$ , and maximum-norm is denoted by  $\|x\|_\infty = \max_{1 \leq i \leq N} |x_i|$ .

The following theorem shows that the nonlinear term  $\beta_i^n$  appearing in (3.5) is bounded.

**Theorem 4.2.** *Let  $\Delta^n = [\Delta_0^n, \Delta_1^n, \dots, \Delta_N^n]^T$ , then the following properties hold true.*

- (i)  $\|\Delta^n\|_1$  is nonincreasing.
- (ii) The nonlinear term  $\beta_i^n$  appearing in (3.5) satisfies

$$0 \leq \beta_i^n \leq L(h), \quad i = 0, 1, \dots, N, \quad n = 0, 1, \dots, L, \quad (4.1)$$

where

$$L(h) = \frac{B^2 \sigma_0^2}{2} \left( 1 + d_2 + a^2 B^2 \left\| \Delta^0 \right\|_1 \Psi'(A_2) \right), \quad (4.2)$$

with  $A_2$ ,  $d_2$ , and  $\Psi'(A_2)$  given by (2.7).

*Proof.* Property (i) is proved using the induction principle over index  $n$ .

For  $n = 0$ , from Lemma 3.2 and  $\beta_i^n \geq 0$  by Remark 4.7, it follows that

$$|\Delta_i^1| \leq \frac{\rho\beta_{i+1}^0}{1+2\rho\beta_{i+1}^0} |\Delta_{i+1}^0| + \frac{\rho\beta_{i-1}^0}{1+2\rho\beta_{i-1}^0} |\Delta_{i-1}^0| + \frac{1}{1+2\rho\beta_i^0} |\Delta_i^0|, \quad 1 \leq i \leq N-1. \quad (4.3)$$

Taking into account (3.13) and (4.3), we have

$$\|\Delta^1\|_1 \leq \sum_{i=1}^{N-1} |\Delta_i^0| - \frac{\rho\beta_1^0}{1+2\rho\beta_1^0} |\Delta_1^0| - \frac{\rho\beta_{N-1}^0}{1+2\rho\beta_{N-1}^0} |\Delta_{N-1}^0| \leq \|\Delta^0\|_1. \quad (4.4)$$

Thus property (i) is proved for  $n = 0$ .

Now, let us assume that property (i) holds true up  $n$ , that is,

$$\|\Delta^n\|_1 \leq \|\Delta^{n-1}\|_1 \leq \|\Delta^{n-2}\|_1 \leq \dots \leq \|\Delta^0\|_1. \quad (4.5)$$

For  $1 \leq i \leq N-1$ , from Lemma 3.2, it follows that

$$|\Delta_i^{n+1}| \leq \frac{\rho\beta_{i+1}^n}{1+2\rho\beta_{i+1}^n} |\Delta_{i+1}^n| + \frac{\rho\beta_{i-1}^n}{1+2\rho\beta_{i-1}^n} |\Delta_{i-1}^n| + \left(1 - \frac{2\rho\beta_i^n}{1+2\rho\beta_i^n}\right) |\Delta_i^n|. \quad (4.6)$$

Taking into account  $\Delta_0^{n+1} = \Delta_N^{n+1} = 0$ , we have

$$\begin{aligned} \|\Delta^{n+1}\|_1 &\leq \sum_{i=2}^N \frac{\rho\beta_i^n}{1+2\rho\beta_i^n} |\Delta_i^n| + \sum_{i=0}^{N-2} \frac{\rho\beta_i^n}{1+2\rho\beta_i^n} |\Delta_i^n| + \sum_{i=1}^{N-1} \left(1 - \frac{2\rho\beta_i^n}{1+2\rho\beta_i^n}\right) |\Delta_i^n| \\ &= \sum_{i=1}^{N-1} |\Delta_i^n| - \frac{\rho\beta_1^n}{1+2\rho\beta_1^n} |\Delta_1^n| - \frac{\rho\beta_{N-1}^n}{1+2\rho\beta_{N-1}^n} |\Delta_{N-1}^n| \leq \|\Delta^n\|_1. \end{aligned} \quad (4.7)$$

Hence property (i) is proved completely.

On the other hand, from (3.6) and the monotonic increasing property of  $\Psi$  (see Lemma 2.1), it follows that

$$\begin{aligned} \beta_i^n &= \frac{1}{2} \sigma_0^2 \left(1 + \Psi\left(a^2 x_i^2 \Delta_i^n\right)\right) x_i^2 \\ &\leq \frac{1}{2} \sigma_0^2 \left(1 + \Psi\left(a^2 x_i^2 \|\Delta^n\|_1\right)\right) x_i^2 \\ &\leq \frac{1}{2} \sigma_0^2 \left(1 + \Psi\left(a^2 x_i^2 \|\Delta^0\|_1\right)\right) x_i^2. \end{aligned} \quad (4.8)$$

Since  $x_i \in [0, B]$  and from the property (iii) and property (i) of Lemma 2.2, we have

$$0 \leq \beta_i^n \leq L(h), \quad i = 0, 1, \dots, N, \quad n = 0, 1, \dots, L. \quad (4.9)$$

Thus the proof of theorem is complete.  $\square$



## 4.2. Positivity

Since the value of option is nonnegative, a nice property of the numerical scheme for the pricing equation is positivity-preserving.

Clearly, all the coefficients of the numerical scheme (3.5) are nonnegative (see Theorem 4.2). Hence, for a nonnegative payoff  $u_i^0$ , the following result has been established.

**Proposition 4.3.** *The numerical scheme (3.5), (3.8) is unconditionally positive.*

*Remark 4.4.*  $\Delta_i^n$  is also unconditionally positive, that is, if  $\Delta_i^0 \geq 0$  for  $0 \leq i \leq N$ , then  $\Delta_i^n \geq 0$  for  $0 \leq i \leq N$ ,  $0 \leq n \leq L$ .

## 4.3. Monotonicity

For the sake of convenience in the presentation, we introduce the following definition of a monotonicity preserving numerical scheme.

*Definition 4.5* (see [31]). In numerical scheme  $F(u_i^n) = 0$ , we say that it is monotonicity-preserving. If each time that  $u_i^n \leq u_{i+1}^n$  or  $u_i^n \geq u_{i+1}^n$  for all  $i$ , then it occurs that  $u_i^{n+1} \leq u_{i+1}^{n+1}$  or  $u_i^{n+1} \geq u_{i+1}^{n+1}$  for all  $i$ .

The next result shows the monotonicity of the numerical scheme.

**Theorem 4.6.** *The numerical scheme (3.5), (3.8) is unconditionally monotonicity-preserving with  $0 \leq i \leq N$ ,  $0 \leq n \leq L$ .*

*Proof.* Let us write

$$u_{i+1}^{n+1} - u_i^{n+1} = (u_{i+1}^{n+1} - u_{i+1}^n) + (u_{i+1}^n - u_i^n) - (u_i^{n+1} - u_i^n). \quad (4.10)$$

Assuming that  $u_{i+1}^n \geq u_i^n$  for  $0 \leq i \leq N-1$ ,  $0 \leq n \leq L-1$ , then from (3.5), it follows that

$$\begin{aligned} u_i^{n+1} - u_i^n &= \frac{\rho\beta_i^n (u_{i+1}^n + u_{i-1}^n - 2u_i^n)}{1 + 2\rho\beta_i^n} \\ &\leq \frac{\rho\beta_i^n (u_{i+1}^n - u_i^n)}{1 + 2\rho\beta_i^n}, \quad 1 \leq i \leq N-1, \end{aligned} \quad (4.11)$$

$$u_i^{n+1} - u_i^n \geq \frac{-\rho\beta_i^n (u_i^n - u_{i-1}^n)}{1 + 2\rho\beta_i^n}, \quad 1 \leq i \leq N-1, \quad (4.12)$$

that substituting  $i$  by  $i+1$  one gets

$$u_{i+1}^{n+1} - u_{i+1}^n \geq \frac{-\rho\beta_{i+1}^n (u_{i+1}^n - u_i^n)}{1 + 2\rho\beta_{i+1}^n}, \quad 0 \leq i \leq N-2. \quad (4.13)$$

From (4.10), (4.11), and (4.13), it follows that

$$\begin{aligned} u_{i+1}^{n+1} - u_i^{n+1} &\geq \frac{-\rho\beta_{i+1}^n (u_{i+1}^n - u_i^n)}{1 + 2\rho\beta_{i+1}^n} + (u_{i+1}^n - u_i^n) - \frac{\rho\beta_i^n (u_{i+1}^n - u_i^n)}{1 + 2\rho\beta_i^n} \\ &= \left(1 - \frac{\rho\beta_i^n}{1 + 2\rho\beta_i^n} - \frac{\rho\beta_{i+1}^n}{1 + 2\rho\beta_{i+1}^n}\right) (u_{i+1}^n - u_i^n). \end{aligned} \quad (4.14)$$

Taking into account (4.1) and (4.14) for  $i$  and  $i + 1$ , we have

$$u_{i+1}^{n+1} - u_i^{n+1} \geq \left(1 - \frac{2\rho L(h)}{1 + 2\rho L(h)}\right) (u_{i+1}^n - u_i^n) \geq 0, \quad 1 \leq i \leq N - 2. \quad (4.15)$$

The monotonicity of the scheme in the internal mesh points has been proved. In an analogous way, we can verify that

$$u_1^{n+1} \geq u_0^{n+1}, \quad u_N^{n+1} \geq u_{N-1}^{n+1}. \quad (4.16)$$

Similarly, we can prove that if  $u_i^n \geq u_{i+1}^n$ , then  $u_i^{n+1} \geq u_{i+1}^{n+1}$  for  $0 \leq i \leq N-1$ ,  $0 \leq n \leq L-1$ . Thus the proof of theorem is complete.  $\square$

*Remark 4.7.* If the payoff  $f(x)$  is nondecreasing with  $f(0) = 0$  (e.g.,  $f(x) = \max\{x - K, 0\}$ ), then  $0 = u_0^n \leq u_1^n \leq \dots \leq u_i^n \leq u_{i+1}^n \leq \dots \leq u_N^n$  for a fixed  $n$  with  $0 \leq n \leq L$ .

## 5. Stability and Consistency

### 5.1. Stability

**Theorem 5.1.** *The difference scheme (3.5), (3.8) is unconditionally  $\|\cdot\|_\infty$ -stable.*

*Proof.* From (3.5), let us write

$$u_i^{n+1} = \frac{\rho\beta_i^n}{1 + 2\rho\beta_i^n} (u_{i+1}^n + u_{i-1}^n) + \left(1 - \frac{2\rho\beta_i^n}{1 + 2\rho\beta_i^n}\right) u_i^n. \quad (5.1)$$

Since all the coefficients of (5.1) are nonnegative, then using triangle inequality, it follows that

$$|u_i^{n+1}| \leq \frac{\rho\beta_i^n}{1 + 2\rho\beta_i^n} (|u_{i+1}^n| + |u_{i-1}^n|) + \left(1 - \frac{2\rho\beta_i^n}{1 + 2\rho\beta_i^n}\right) |u_i^n| \leq \sup_{0 \leq i \leq N} |u_i^n|, \quad (5.2)$$

so that

$$\sup_{0 \leq i \leq N} |u_i^{n+1}| \leq \sup_{0 \leq i \leq N} |u_i^n|. \quad (5.3)$$

According to the definition of the maximum norm, it follows that

$$\|u^{n+1}\|_{\infty} \leq \|u^n\|_{\infty}. \quad (5.4)$$

Thus the result is established.  $\square$

## 5.2. Consistency

The proposed new difference scheme (3.5) is explicit, unconditionally positive, and unconditionally stable; however, it is not unconditionally consistent. There are extra truncation error terms since the approximations to second derivatives with respect to  $x$  are at different time levels.

The following theorem gives the consistency condition of the difference scheme (3.5).

**Theorem 5.2.** *With the previous notation, suppose that the exact solution  $u$  of (1.1)–(1.3) satisfies  $u \in C^{4,2}(\bar{\Omega})$ . Then the local truncation error is given by*

$$T_{h,k}(u_i^n) = O(h^2) + O(k) + O\left(\frac{k}{h^2}\right). \quad (5.5)$$

*Proof.* Let us write the scheme (3.5) in the form

$$F_{h,k}(u_i^n) = \frac{u_i^{n+1} - u_i^n}{k} - \beta_i^n \delta_i^n = 0. \quad (5.6)$$

Using Taylor's expansion about  $(x_i, \tau_n)$  and  $u \in C^{4,2}(\bar{\Omega})$ , it follows that

$$\begin{aligned} \delta_i^n &= \frac{\partial^2 u}{\partial x^2}(x_i, \tau_n) + \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(\eta, \tau_n) - \frac{2k}{h^2} \frac{\partial u}{\partial \tau}(x_i, \xi) \\ &= \frac{\partial^2 u}{\partial x^2}(x_i, \tau_n) + h^2 M_i^n(1) - \frac{k}{h^2} M_i^n(2), \end{aligned} \quad (5.7)$$

$$\Delta_i^n = \frac{u_{i+1}^n - 2u_i^{n+1} + u_{i-1}^n}{h^2} = \frac{\partial^2 u}{\partial x^2}(x_i, \tau_n) + \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(\eta, \tau_n), \quad (5.8)$$

$$\frac{u_i^{n+1} - u_i^n}{k} = \frac{\partial u}{\partial \tau}(x_i, \tau_n) + \frac{k}{2} \frac{\partial^2 u}{\partial \tau^2}(x_i, \xi) = \frac{\partial u}{\partial \tau}(x_i, \tau_n) + k M_i^n(3), \quad (5.9)$$

where

$$\begin{aligned}
 x_i - h < \eta < x_i + h, \quad \tau_n < \xi < \tau_n + k, \\
 |M_i^n(1)| &\leq \frac{1}{12} \max \left\{ \left| \frac{\partial^4 u}{\partial x^4}(x, \tau_n) \right|; 0 \leq x \leq B \right\} = |W_i^n(1)|_{\max}, \\
 |M_i^n(2)| &\leq 2 \max \left\{ \left| \frac{\partial u}{\partial \tau}(x_i, \tau) \right|; \tau_n \leq \tau \leq \tau_{n+1} \right\} = |W_i^n(2)|_{\max}, \\
 |M_i^n(3)| &\leq \frac{1}{2} \max \left\{ \left| \frac{\partial^2 u}{\partial \tau^2}(x_i, \tau) \right|; \tau_n \leq \tau \leq \tau_{n+1} \right\} = |W_i^n(3)|_{\max}.
 \end{aligned} \tag{5.10}$$

From (5.5)–(5.8), it follows that the local truncation error takes the form

$$\begin{aligned}
 T_{h,k}(u_i^n) &= F_{h,k}(u_i^n) - \mathcal{L}(u_i^n) \\
 &= -\frac{1}{2} \sigma_0^2 x_i^2 \left\{ \left( 1 + \Psi(a^2 x_i^2 \Delta_i^n) \right) \delta_i^n \right. \\
 &\quad \left. - \left( 1 + \Psi \left( a^2 x_i^2 \frac{\partial^2 u}{\partial x^2}(x_i, \tau_n) \right) \right) \frac{\partial^2 u}{\partial x^2}(x_i, \tau_n) \right\} + k M_i^n(3).
 \end{aligned} \tag{5.11}$$

Let us introduce the notation

$$\begin{aligned}
 A_i^n &= a^2 x_i^2 \frac{\partial^2 u}{\partial x^2}(x_i, \tau_n), \quad \delta_1 A_i^n = a^2 x_i^2 h^2 M_i^n(1), \\
 \delta_2 A_i^n &= -a^2 x_i^2 \frac{k}{h^2} M_i^n(2).
 \end{aligned} \tag{5.12}$$

Using function  $g(A)$  introduced in Lemma 2.2, it follows that

$$\begin{aligned}
 &\left( 1 + \Psi(a^2 x_i^2 \Delta_i^n) \right) \delta_i^n - \left( 1 + \Psi \left( a^2 x_i^2 \frac{\partial^2 u}{\partial x^2}(x_i, \tau_n) \right) \right) \frac{\partial^2 u}{\partial x^2}(x_i, \tau_n) \\
 &= a^{-2} x_i^{-2} \{ g(A_i^n + \delta_1 A_i^n) - g(A_i^n) + \delta_1 A_i^n + (1 + \Psi(A_i^n + \delta_1 A_i^n)) \delta_2 A_i^n \} \\
 &= g'(A_i^n + \theta \delta_1 A_i^n) h^2 M_i^n(1) + h^2 M_i^n(1) - \frac{k}{h^2} \beta_i^n, \quad 0 < \theta < 1.
 \end{aligned} \tag{5.13}$$

From Lemma 2.2, (5.11), and  $\beta_i^n \leq L(h)$ , it is easy to know that

$$|T_{h,k}(u_i^n)| \leq O(h^2) + O(k) + O\left(\frac{k}{h^2}\right). \tag{5.14}$$

Thus the result has been proved.  $\square$

From Theorem 5.2, we can see that the meshes should satisfy  $k/h^2 \rightarrow 0$  as  $k$  and  $h$  go to zero in order to ensure the consistency. Therefore, the key to the convergence of the scheme is the consistency rather than the stability. In actual calculation, we can choose the time step depending on the spatial size so that inconsistent terms go to zero.

## 6. Numerical Experiments

In this section, we implement the positivity-preserving scheme (3.5), (3.8) on European put option and European butterfly spread. We analyse the effectiveness of the method and compare it with the numerical scheme (SFD1) given in [20] and a standard forward Euler finite difference scheme (SFD2)

$$\frac{u_i^{n+1} - u_i^n}{k} - \beta_i^n \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} = 0, \quad (6.1)$$

which is analysed in [21]. The function  $\Psi$  is calculated with (2.5) using the “fsolve” function in Matlab. Both the experiments are performed for large values of  $\sigma_0$  to visualize the sensitivity of the methods towards high volatility.

### 6.1. European Put Option

A European put option is a contract where the owner of the option has the right to sell an underlying asset  $S(t)$  for a fixed amount, known as the strike price  $K$ , at the expiry date  $T$ . The payoff function  $f(S)$  is given by

$$f(S) = \max\{K - S, 0\}. \quad (6.2)$$

We choose the parameters as

$$K = 2, \quad T = 0.5, \quad \sigma_0 = 0.5, \quad B = 10, \quad r = 0.04. \quad (6.3)$$

Equations (1.1)–(1.3) give analytical solution for only  $a = 0$  (see [6]). Figure 1 (the left one:  $a = 0$ , the right one:  $a = 0.02$ ) gives the option value using scheme (3.5) and schemes (SFD1 and SFD2) in [20, 21], which shows that our scheme stable, monotonous, and is able to produce solution that is close to the exact solution, but numerical solution of scheme (6.1) appears as spurious oscillation for  $k = 0.0005$  and  $h = 0.1$ .

Moreover, Figure 2 presents the related hedging parameters of European put option using the two schemes. We can see that our proposed scheme produces smoother solutions than standard scheme for the delta and gamma. In addition, the gamma is positivity preserving and is maximal as it closes the strike price  $K = 2$ .

Next we consider the influence of transaction costs (parameter  $a$ ) on the value of European put option (Figure 3). The left one shows the change of the option value with the parameter  $a$ , and the right one presents an evolution profile of the difference  $V_{\text{nonlinear}}(S, t) - V_{\text{linear}}(S, t)$  between our proposed scheme with transaction costs and without transaction costs. We can see that the difference is not symmetric, but decreases towards the expiry date,

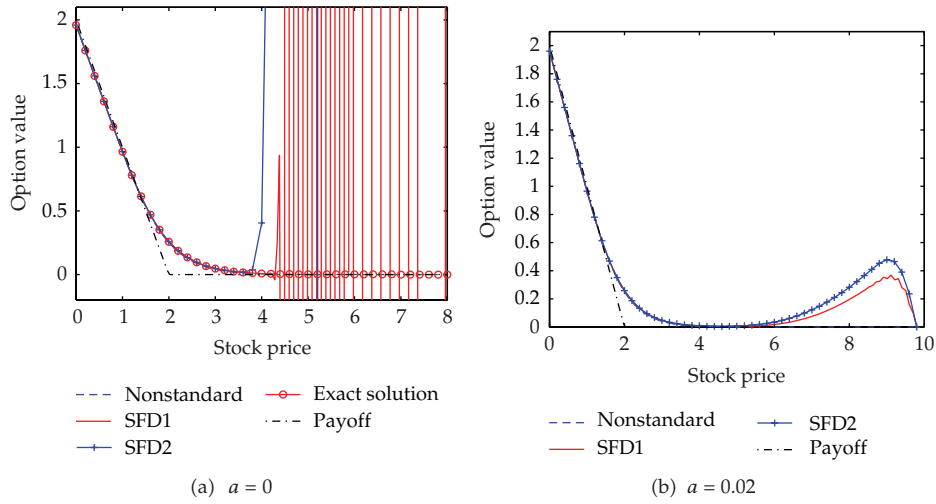


Figure 1: The values of European put option under different schemes with the computational parameters  $h = 0.1$  and  $k = 0.0005$ .

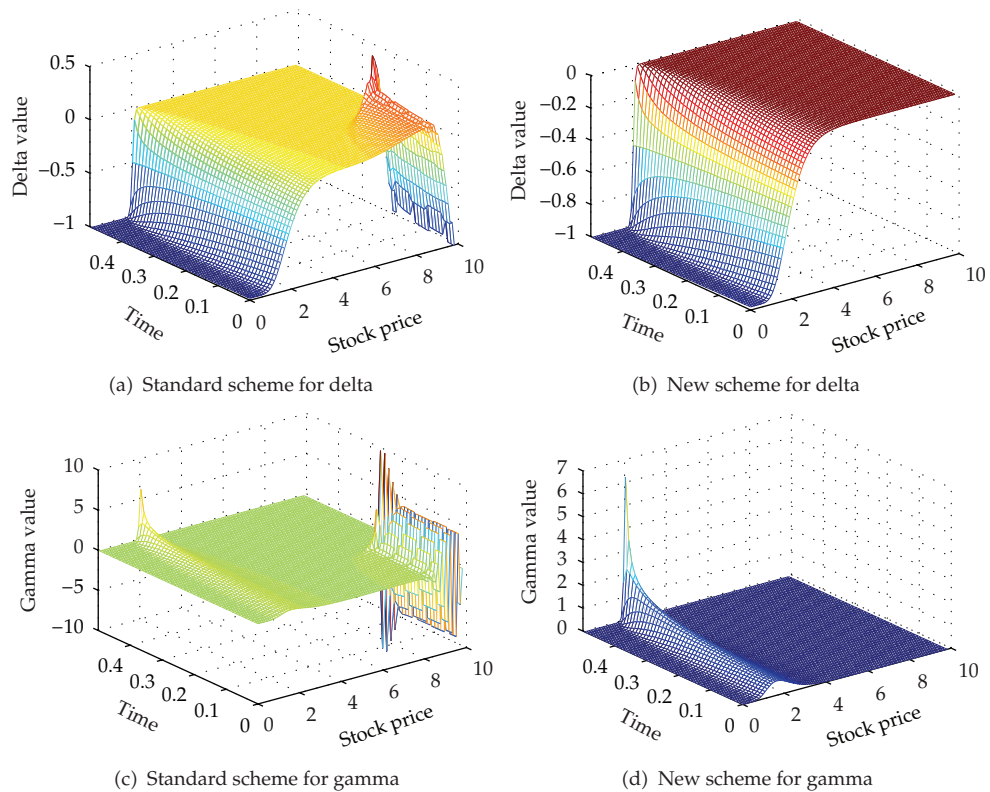
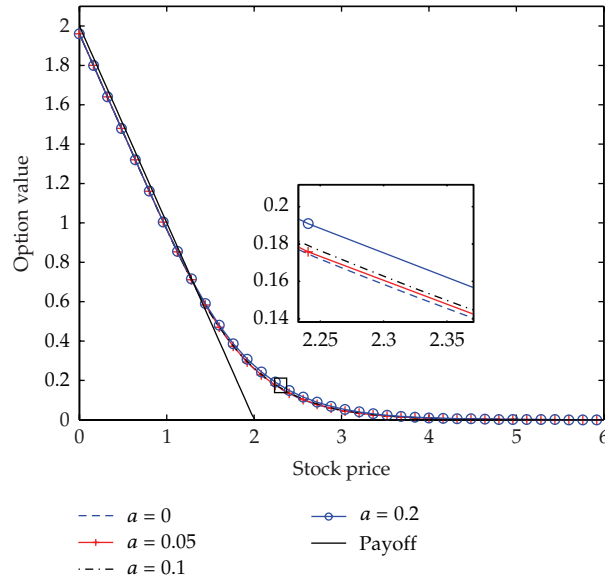
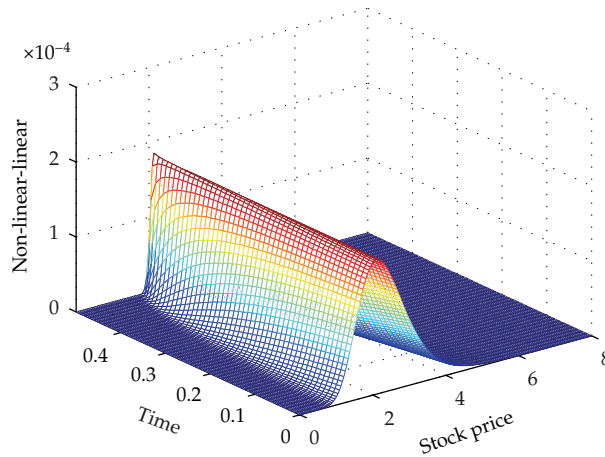


Figure 2: Time evolution profiles of the Greeks of European put option with the computational parameters  $h = 0.1$ ,  $k = 0.0005$ , and  $a = 0.02$ .



(a) Linear and nonlinear cases



(b) Nonlinear-linear ( $a = 0.02$ )

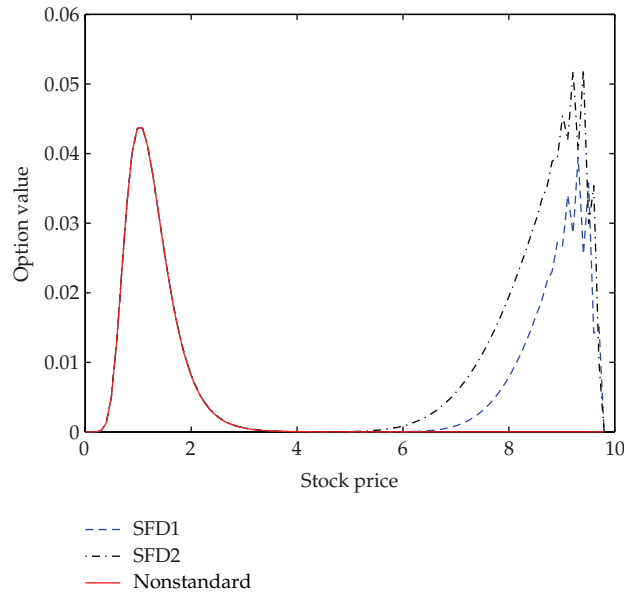
**Figure 3:** Influence of the transaction costs (parameter  $a$ ) on the value of European put option with the computational parameters  $h = 0.1$  and  $k = 0.0005$ .

and is maximal close to the strike price  $K = 2$ , where the nonlinear value is significantly higher than the linear value.

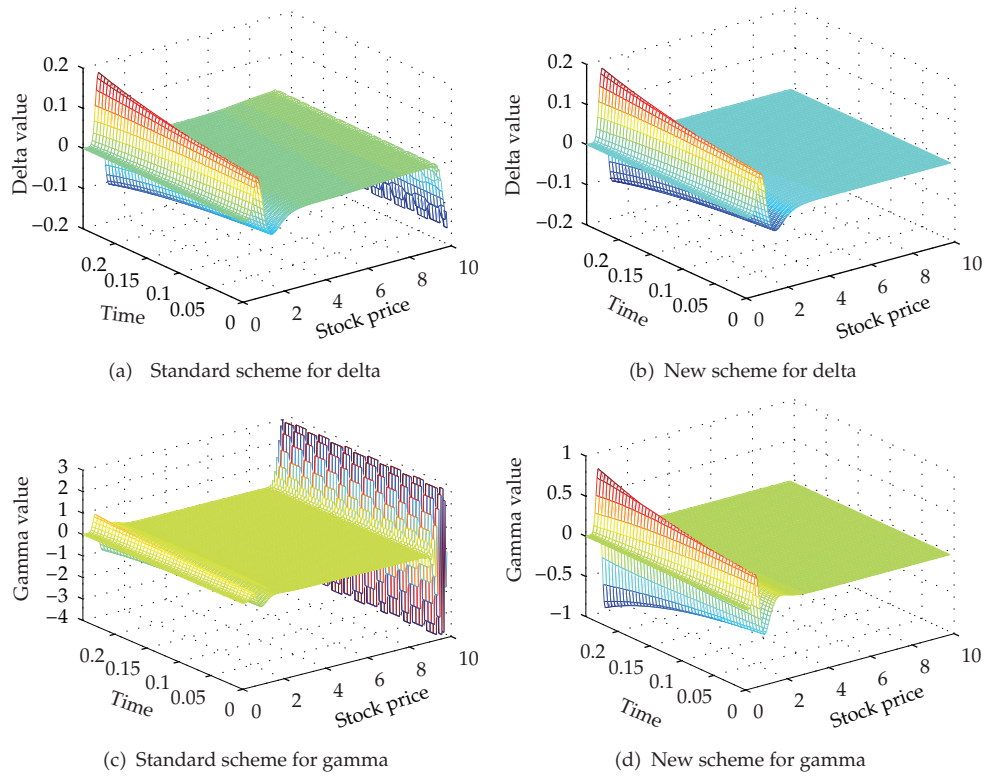
### 6.2. European Butterfly Spread

A butterfly spread is a combination of three-call options with three-strike prices, in which one contract is purchased with two outside strike prices and two contracts are sold at the middle strike price. The payoff function  $f(S)$  is given by

$$f(S) = \max\{S - K_1, 0\} - 2 \max\{S - K_2, 0\} + \max\{S - K_3, 0\}, \tag{6.4}$$

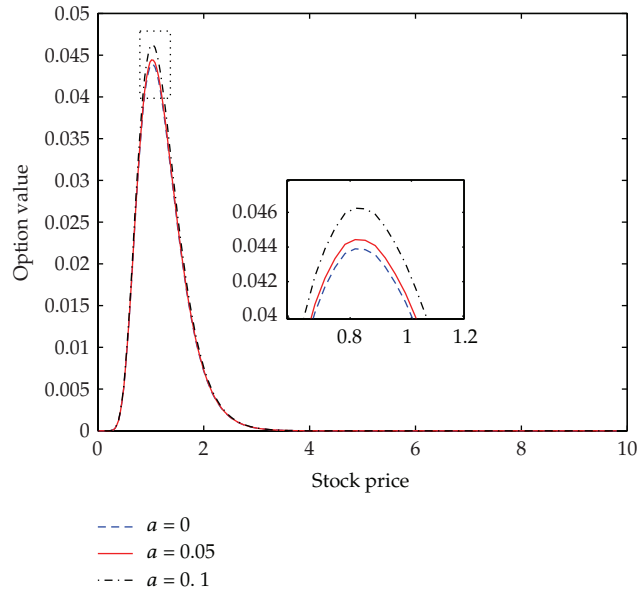


**Figure 4:** The value of European butterfly spread under different schemes with the computational parameters  $h = 0.1$ ,  $k = 0.000455$ , and  $a = 0.02$ .

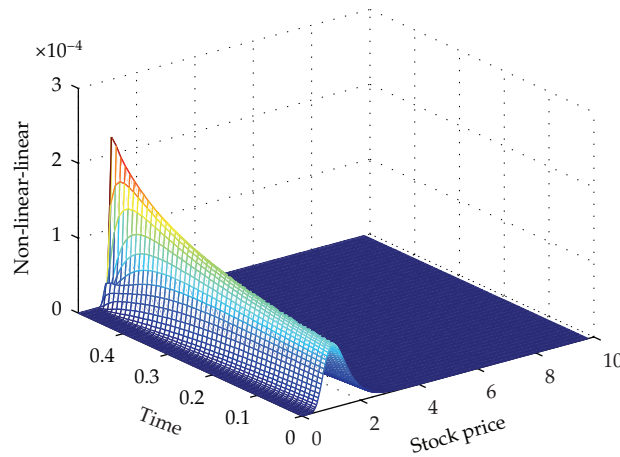


**Figure 5:** Time evolution profiles of the Greeks of European butterfly spread with the computational parameters  $h = 0.1$ ,  $k = 0.000455$ , and  $a = 0.02$ .





(a) Linear and nonlinear cases



(b) Nonlinear-linear ( $a = 0.02$ )

**Figure 6:** Influence of the transaction costs (parameter  $a$ ) on the value of European butterfly spread with the computational parameters  $h = 0.1$  and  $k = 0.000455$ .

where  $K_1, K_2,$  and  $K_3$  are the strike prices that satisfy  $K_1 < K_2 < K_3$  and  $K_2 = (K_1 + K_3)/2$ .

We choose the following parameters:

$$\begin{aligned}
 r = 0.04, \quad K_1 = 0.8, \quad K_2 = 1, \quad K_3 = 1.2, \quad T = 0.5, \\
 \sigma_0 = 0.5, \quad B = 10,
 \end{aligned}
 \tag{6.5}$$

the time step  $k = 0.000455$ , and the spatial step  $h = 0.1$ .

Figure 4 shows the different solutions of two schemes at  $t = 0$  for  $a = 0.02$ . We can see that our proposed scheme is smooth and stable at the same step sizes. Moreover, Figure 5 shows the Greeks of the numerical solution calculated with scheme (3.14), which is different from the vanilla option. The standard scheme (the left one) is unstable and produces unwanted oscillations which are not present while using our proposed scheme (the right one).

The last two figures show the value of European butterfly spread with different transaction costs (parameter  $a$ ). It is evident from Figure 6 that butterfly spread becomes more expensive in the presence of transaction cost and the difference is maximal as it closes the strike price  $K_2 = 1$ , which is similar with the vanilla option.

## 7. Discussions and Conclusions

In this paper, we have extended the numerical method in [30] to nonlinear situation and presented the numerical scheme for a nonlinear Black-Scholes equation in the presence of transaction costs. The numerical method is based on a nonstandard approximation of the second partial derivative  $\partial^2 u(x_i, \tau_n) / \partial x^2$  by  $(u_{i+1}^n - 2u_i^{n+1} + u_{i-1}^n) / h^2$ , and the nonlinear term is treated explicitly, which guarantees to solve the original problem without iteration. The scheme is unconditionally positive and stable, but it is conditionally consistent. In fact, as  $u_i^{n+1} \approx u_i^n + k(\partial u / \partial \tau)(x_i, \tau_n)$ , the scheme effectively solves the nonlinear parabolic equation

$$\left(1 + \frac{k}{h^2} \beta(x)\right) \frac{\partial u}{\partial \tau} - \frac{1}{2} \beta(x) \frac{\partial^2 u}{\partial x^2} = 0, \quad (7.1)$$

where

$$\beta(x) = \sigma_0^2 \left(1 + \Psi\left(a^2 x^2 \frac{\partial^2 u}{\partial x^2}\right)\right) x^2. \quad (7.2)$$

It means that the new scheme converges to a solution of (2.2) if  $k/h^2 \rightarrow 0$ . Otherwise, (if  $k/h^2$  is fixed) it converges to a solution of (2.2) in a different time scale. In fact, it can be seen from Theorem 5.2, where the truncation error really depends on the ratio  $k/h^2$ , which is also the reason that we consider the smaller ratio  $k/h^2$  in the experiment. The numerical results show that our method produces better numerical solutions than the schemes in [20, 21] with the same step sizes. In the future work, we will extend the method to the problem of American option pricing.

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