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Research Article

Bounds for the Kirchhoff Index of Bipartite Graphs

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A (m,n)-bipartite graph is a bipartite graph such that one bipartition has m vertices and the other bipartition has n vertices. The tree dumbbell $D(n,a,b)$ consists of the path P_{n-a-b} together with a independent vertices adjacent to one pendent vertex of P_{n-a-b} and b independent vertices adjacent to the other pendent vertex of P_{n-a-b}. In this paper, firstly, we show that, among (m,n)-bipartite graphs $(m \leq n)$, the complete bipartite graph $K_{m,n}$ has minimal Kirchhoff index and the tree dumbbell $D(m+n, \lfloor n-(m+1)/2 \rfloor, \lceil n-(m+1)/2 \rceil)$ has maximal Kirchhoff index. Then, we show that, among all bipartite graphs of order l, the complete bipartite graph $K_{\lfloor l/2 \rfloor, l-\lfloor l/2 \rfloor}$ has minimal Kirchhoff index and the path P_l has maximal Kirchhoff index, respectively. Finally, bonds for the Kirchhoff index of (m,n)-bipartite graphs and bipartite graphs of order l are obtained by computing the Kirchhoff index of these extremal graphs.

1. Introduction

Let G be a connected graph with vertices labeled as $v_1, v_2, \ldots, v_n$. The distance between vertices v_i and v_j, denoted by $d_G(v_i, v_j)$, is the length of a shortest path between them. The famous Wiener index $W(G)$ [1] is the sum of distances between all pairs of vertices, that is

$$W(G) = \sum_{i<j} d_G(v_i, v_j). \tag{1.1}$$

In 1993, Klein and Randić [2] introduced a new distance function named resistance distance on the basis of electrical network theory. They view G as an electrical network N such that each edge of G is assumed to be a unit resistor. Then, the resistance distance between

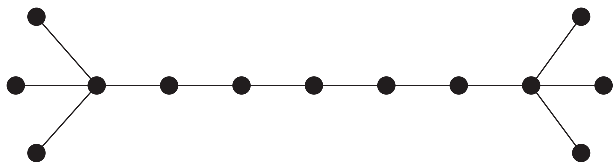

Figure 1: $D(13,3,3)$.

vertices v_i and v_j, denoted by $r_G(v_i,v_j)$, is defined to be the effective resistance between nodes v_i and v_j in N. Analogous to the Wiener index, the Kirchhoff index $Kf(G)$ [2, 3] is defined as

$$Kf(G)=\sum_{i<j} r_G(v_i,v_j). \tag{1.2}$$

As an analogy to the famous Wiener index, the Kirchhoff index is an important molecular structure descriptor [4], and thus it is well studied in both mathematical and chemical literatures. For more information on the Kirchhoff index, the readers are referred to recent papers [5–16] and references therein.

It is of interest to determine bounds for the Kirchhoff index of some classes of graphs and characterize extremal graphs as well. Along this line, much research work has been done. For a general graph G, Lukovits et al. [17] proved that $Kf(G) \geq n-1$ with equality if and only if G is a complete graph, and they also indicated that the maximal Kirchhoff index graph is the path P_n. Palacios [18] proved that $Kf(G) \leq 1/6(n^3-n)$ with equality if and only if G is a path. For a circulant graph, Zhang and Yang [19] showed that

$$n-1 \leq Kf(G) \leq \frac{n^3-n}{12}, \tag{1.3}$$

where the first equality holds if and only if G is a complete graph and the second does if and only if G is a cycle. Furthermore, tight bounds for the Kirchhoff index are also obtained for a special class of unicyclic graphs [20], bicyclic graphs [21, 22], and Cacti [23].

Bipartite graphs are perhaps the most basic of objects in graph theory, both from a theoretical and practical point of view. Let G be a bipartite graph with bipartition X and Y such that X is the set of white vertices and Y is the set of black vertices. Suppose that $|X|=m$ and $|Y|=n$. Such graph is also known as (m,n)-bipartite graph. Without loss of generality, we supposed that $m \leq n$. The tree dumbbell $D(n,a,b)$ consists of the path P_{n-a-b} together with a independent vertices adjacent to one pendent vertex of P_{n-a-b} and b independent vertices adjacent to the other pendent vertex of P_{n-a-b}. For instance, $D(13,3,3)$ is referred to Figure 1.

In the next section, we first obtain that $K_{m,n}$ has the minimal Kirchhoff index among all (m,n)-bipartite graphs according to strictly increasing property of the Kirchhoff index. Then we prove that tree dumbbell $D(m+n,\lfloor (n-m+1)/2\rfloor,\lceil (n-m+1)/2\rceil)$ has maximal Kirchhoff index among all (m,n)-bipartite graphs. Therefore, tight bounds for the Kirchhoff index of (m,n)-bipartite graphs are determined. In the last section, we discuss general bipartite graphs of order l. We obtain that, among all bipartite graphs of order l, complete bipartite graph $K_{\lfloor l/2\rfloor,l-\lfloor l/2\rfloor}$ and path P_l have minimal and maximal Kirchhoff index, respectively. Thus bounds for the Kirchhoff index of bipartite graphs of order l are also obtained.

2. (m, n)-Bipartite Graphs with Extremal Kirchhoff Index

Lemma 2.1 (see [19]). *Let G be a connected graph with n vertices and H a connected spanning subgraph of G. Then, $Kf(G) \le Kf(H)$ with equality if and only if $G = H$.*

By Lemma 2.1, the complete bipartite graph $K_{m,n}$ has minimal Kirchhoff index among all (m, n)-bipartite graphs. Now, we compute the Kirchhoff index of $K_{m,n}$.

Lemma 2.2.

$$Kf(K_{m,n}) = \frac{(m+n-1)(m^2+n^2) - mn}{mn}. \tag{2.1}$$

Proof. For $K_{m,n}$, Klein [24] obtained that the resistance distance between two vertices of different parts is $(m+n-1)/mn$, the resistance distance between two vertices of m-vertex part and n-vertex part is $2/n$ and $2/m$, respectively. Hence,

$$Kf(K_{m,n}) = mn\frac{m+n-1}{mn} + \binom{m}{2}\frac{2}{n} + \binom{n}{2}\frac{2}{m} = \frac{(m+n-1)(m^2+n^2) - mn}{mn}. \tag{2.2}$$

□

In the following, we search for (m, n)-bipartite graph with maximal Kirchhoff index. By Lemma 2.1, the graph possesses maximal Kirchhoff index must be a tree since otherwise any of its spanning tree has lager Kirchhoff index than it. It is well known that the Kirchhoff index and the Wiener index concise for trees. Hence, we only need to consider the Wiener index which has been extensively studied. Now, we introduce some well-known results on the Wiener index of trees.

Let P_n and S_n denote n-vertex path and n-vertex star, respectively. Then we have the following.

Lemma 2.3 (see [25]). *Let T be any n-vertex tree different from P_n and S_n. Then,*

$$W(S_n) < W(T) < W(P_n). \tag{2.3}$$

It is also obtained in [25] that

$$W(S_n) = (n-1)^2, \tag{2.4}$$

$$W(P_n) = \binom{n+1}{3} = \frac{n^3 - n}{6}. \tag{2.5}$$

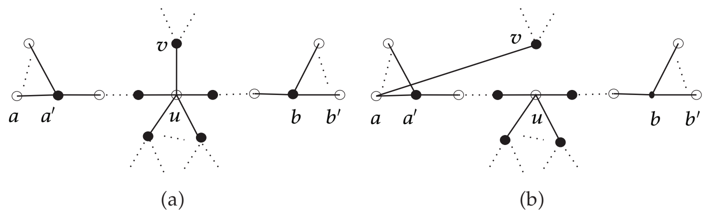

Figure 2: T (a) and T' (b) in the proof of Claim 1.

Let $e = (x, y)$ be an edge of T. Let $n_1(e)$ be the number of vertices of T lying closer to x than to y, and let $n_2(e)$ be the number of vertices of T lying closer to y than to x. That is,

$$\begin{aligned} n_1(e) &= \left|\{v \mid v \in V(T), d_T(v,x) < d_T(v,y)\}\right|, \\ n_2(e) &= \left|\{v \mid v \in V(T), d_T(v,y) < d_T(v,x)\}\right|. \end{aligned} \tag{2.6}$$

Theorem 2.4 (see [1]). *Let T be a n-vertex tree. Then,*

$$W(T) = \sum_{e \in E(T)} n_1(e) n_2(e). \tag{2.7}$$

In the following, we let $W_T(e) = n_1(e)n_2(e)$.

Theorem 2.5. *$D(m+n, \lfloor (n-m+1)/2 \rfloor, \lceil (n-m+1)/2 \rceil)$ has maximal Kirchhoff index among all (m,n)-bipartite graphs.*

Proof. Suppose that T is the tree possessing maximal Wiener (Kirchhoff) index among all (m,n)-bipartite graphs.

Case 1. $n = m$ or $n = m+1$. In this case, $D(m+n, \lfloor (n-m+1)/2 \rfloor, \lceil (n-m+1)/2 \rceil)$ is the path P_{m+n}. By Lemma 2.3, the result holds.

Case 2. $n > m+1$. Let P be a longest path in T with end vertices a and b. Suppose that a' and b' are neighbors of a and b in P, respectively.

Claim 1. The inner vertices of P all have degree 2 in T except for a' and b'.

Suppose to the contrary that there exists an inner point u of P different from a' and b' has degree lager than 2 and v is a neighbor of u such that $v \in P$. Suppose that the size of the component of $T - uv$ containing v is k. Suppose that e_1 and e_2 are edges in P incident to u. Let C_a, C_b, and C_u denote the components of $T - e_1 - e_2$ containing a, b, and u, respectively. We choose from C_a and C_b the one containing less vertices, say C_a. a and a' must have one that belongs to the part containing u, say a. Let $T' = T - uv + av$ (see Figure 2). Now we show that $W(T) < W(T')$ by considering the contributions of edges. Obviously, $W_T(uv) = W_{T'}(av) = k(m+n-k)$. Let E denote the edge set of $E(T) - uv = E(T') - av$, and let P' denote the path aPu. For $e \in E - E(P')$, $W_T(e) = W_{T'}(e)$. For $e \in E(P')$, suppose that $C_a(e)$ and $C_b(e)$

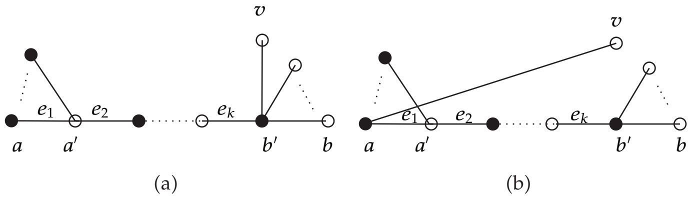

Figure 3: T (a) and T' (b) in the proof of Subcase 2 of Claim 2.

are components of $T - e$ containing a and b, respectively. Then, $W_T(e) = |C_a(e)||C_b(e)|$ and $W_{T'}(e) = (|C_a(e)| + k)(|C_b(e)| - k)$. Then,

$$W_{T'}(e) - W_T(e) = k(|C_b(e)| - |C_a(e)| - k) > k(|C_b| + k - |C_a(e)| - k) \geq k(|C_b| - |C_a|) \geq 0. \tag{2.8}$$

Hence,

$$\begin{aligned} W(T') &= W_{T'}(av) + \sum_{e \in E - E(P')} W_{T'}(e) + \sum_{e \in E(P')} W_{T'}(e) \\ &> W_T(uv) + \sum_{e \in E - E(P')} W_T(e) + \sum_{e \in E(P')} W_T(e) = W(T). \end{aligned} \tag{2.9}$$

This contradicts the choice of T.

Claim 2. Both a and b belong to Y.

Suppose not. Then, we can distinguish the following two cases.

Subcase 1. Both a and b belong to X. By Claim 1, the inner vertices of P all have degree 2 in T; hence, the vertices of Y all belong to P, that is, $n < m$, a contradiction.

Subcase 2. a and b belong to different parts. Suppose that a belongs to Y. By claim 1 and $n > m + 1$, we have $d_T(b') > d_T(a') + 1$. Let $T' = T - b'b + ab$ (see Figure 3). Now, we show that $W(T') < W(T)$. Obviously, $W_{T'}(ab) = W_T(b'b)$. Let E_1 denote the edge set of $E(T) - uv = E(T') - av$, and let P_1 denote the path aPb'. For $e \in E_1 - E(P_1)$, $W_T(e) = W_{T'}(e)$. Suppose that the edges of P_1 are $aa' = e_1, e_2, \ldots, e_l$ such that e_i is adjacent to e_{i+1} for $1 \leq i < k$. It is easy to see that $W_{T'}(e_1) = 2(m + n - 2) > W_T(e_1) = m + n - 1$ and $W_{T'}(e_i) = W_T(e_{i+1})$ for $2 \leq i < k$. What is left is to compare $W_{T'}(e_k)$ with $W_T(e_2)$. $W_{T'}(e_k) = (d_T(b') - 1)(m + n - d_T(b') + 1)$ and $W_T(e_2) = d_T(a')(m + n - d_T(a'))$. Then,

$$W_{T'}(e_k) - W_T(e_2) = \left(d_T(b') - 1 - d_T(a')\right)\left(m + n - d_T(a') - d_T(b') + 1\right) \geq 0 \tag{2.10}$$

since $d_T(b') - 1 - d_T(a') > 0$ and $m + n - d_T(a') - d_T(b') + 1 \geq 0$ with equality if and only if a' and b' are adjacent. Hence, $\sum_{e\in E(P_1)} W_{T'}(e) > \sum_{e\in E(P_1)} W_T(e)$. Thus,

$$\begin{aligned} W(T') &= W_{T'}(ab) + \sum_{e\in E_1 - E(P_1)} W_{T'}(e) + \sum_{e\in E(P_1)} W_{T'}(e) \\ &> W_T(b'b) + \sum_{e\in E_1 - E(P_1)} W_T(e) + \sum_{e\in E(P_1)} W_T(e) = W(T). \end{aligned} \tag{2.11}$$

As before, this contradicts the choice of T.

Claim 3. The length of P is $2m$.

By Claims 1 and 2, the vertices of X are all contained in P, the end vertices of P are both contained in Y. Hence the length of P is $2m$ as claimed.

Claim 4. $|d_T(a') - d_T(b')| \leq 1$. Suppose to the contrary that $|d_T(a') - d_T(b')| \geq 2$. Without less of generality, suppose that $d_T(b') - d_T(a') \geq 2$. Let $T' = T - b'b + a'b$. We can prove that $W(T') > W(T)$ by methods similar to the proof of Claim 2.

By Claims 1, 2, 3, and 4, we may conclude that $T = D(m + n, \lfloor (n - m + 1)/2 \rfloor, \lceil (n - m + 1)/2 \rceil)$, which implies Theorem 2.5.

□

Now, we compute the Kirchhoff (Wiener) index of $D(m + n, \lfloor (n - m + 1)/2 \rfloor, \lceil (n - m + 1)/2 \rceil)$. For convenience, in what follows, we denote $D(m+n, \lfloor (n-m+1)/2 \rfloor, \lceil (n-m+1)/2 \rceil)$ by D^*.

If $m = n$, D^* is the path P_{m+n}. Hence, by (2.5),

$$Kf(D^*) = Kf(P_{m+n}) = \binom{m+n+1}{3}. \tag{2.12}$$

Otherwise, let

$$\begin{aligned} E_1 &= \{e \in D^* \mid e \text{ is incident to a leaf of } D^*\}, \\ E_2 &= E(D^*) - E_1. \end{aligned} \tag{2.13}$$

For $e \in E_1$, obviously $W_{D^*}(e) = m + n - 1$. Noticing that D^* has $n - m + 1$ leaves,

$$\sum_{e\in E_1} W_{D^*}(e) = (n - m + 1)(m + n - 1) = n^2 - (m - 1)^2. \tag{2.14}$$

We can see that the induced subgraph of E_2 is the path P_{2m-1}, from which D^* can be obtained by adding $\lfloor (n - m + 1)/2 \rfloor$ pendant edges to one of its endpoint and $\lceil (n - m + 1)/2 \rceil$ pendant edges to the other endpoint. Hence, the degrees of endpoints of the path P_{2m-1} in D^* are $\lfloor (n - m + 1)/2 \rfloor + 1$ and $\lceil (n - m + 1)/2 \rceil + 1$, respectively. Therefore,

$$\sum_{e\in E_2} W_{D^*}(e) = \sum_{i=\lfloor (n-m+1)/2 \rfloor + 1}^{m+n-(\lceil (n-m+1)/2 \rceil + 1)} i(m + n - i). \tag{2.15}$$

Hence, the Kirchhoff index of D^* is

$$
\begin{aligned}
Kf(D^*) &= W(D^*) = \sum_{e\in E_1} W_{D^*}(e) + \sum_{e\in E_2} W_{D^*}(e) \\
&= n^2 - (m-1)^2 + \sum_{i=\lfloor n-m+1/2\rfloor+1}^{m+n-(\lceil n-m+1/2\rceil+1)} i(m+n-i) \\
&= \begin{cases} \frac{1}{6}\left(-2m+3m^2-m^3-6mn+6m^2n+3n^2+3mn^2\right) & (n-m)\equiv 0(\text{mod } 2), \\ \frac{1}{6}\left(-3+m+3m^2-m^3-6mn+6m^2n+3n^2+3mn^2\right) & (n-m)\equiv 1(\text{mod } 2). \end{cases}
\end{aligned}
\tag{2.16}
$$

In sum, we have our main result.

Theorem 2.6. *For (m,n)-bipartite graph $G(m\le n)$, we have*

$$
\begin{aligned}
&\frac{(m+n-1)\left(m^2+n^2\right)-mn}{mn} \\
&\quad\le Kf(G)\le \begin{cases} \frac{1}{6}\left(-2m+3m^2-m^3-6mn+6m^2n+3n^2+3mn^2\right) & (n-m)\equiv 0\,(\text{mod } 2), \\ \frac{1}{6}\left(-3+m+3m^2-m^3-6mn+6m^2n+3n^2+3mn^2\right) & (n-m)\equiv 1\,(\text{mod } 2). \end{cases}
\end{aligned}
\tag{2.17}
$$

The first equality holds if and only if $G=K_{m,n}$, and the second does if and only if $G=D(m+n,\lfloor(n-m+1)/2\rfloor,\lceil(n-m+1)/2\rceil)$.

3. Bipartite Graphs with Extremal Kirchhoff Index

In this section, we consider general bipartite graphs of order l. By Lemmas 2.1 and 2.3, one can see that the path P_l has maximal Kirchhoff index among all bipartite graphs of order l. The minimal bipartite graph of Kirchhoff index must be $\min_{1\le m\le\lfloor l/2\rfloor}\{K_{m,l-m}\}$. By Lemma 2.2,

$$
\begin{aligned}
Kf(K_{m,l-m}) &= \frac{(l-1)\left(m^2+(l-m)^2\right)-m(l-m)}{m(l-m)} \\
&= (l-1)\frac{2m^2-2ml+l^2}{m(l-m)} - 1 = (l-1)\frac{2m(m-l)+l^2}{m(l-m)} - 1 \\
&= -2l+1+\frac{l^2}{m(l-m)}.
\end{aligned}
\tag{3.1}
$$

Hence,

$$
\min_{1\le m\le\lfloor l/2\rfloor}\{K_{m,l-m}\} = K_{\lfloor l/2\rfloor,l-\lfloor l/2\rfloor}.
\tag{3.2}
$$

It is easy to compute that

$$Kf\left(K_{\lfloor l/2\rfloor,l-\lfloor l/2\rfloor}\right) = \frac{(l-1)\left(l^2 - 2l\lfloor l/2\rfloor + 2\lfloor l/2\rfloor^2\right)}{\lfloor l/2\rfloor(l-\lfloor l/2\rfloor)}. \tag{3.3}$$

Hence, we have the following result.

Theorem 3.1. *For bipartite graph G of order l, we have*

$$\frac{(l-1)\left(l^2 - 2l\lfloor l/2\rfloor + 2\lfloor l/2\rfloor^2\right)}{\lfloor l/2\rfloor(l-\lfloor l/2\rfloor)} \leq Kf(G) \leq \frac{l^3 - l}{6}. \tag{3.4}$$

The first equality holds if and only if $G = K_{\lfloor l/2\rfloor,l-\lfloor l/2\rfloor}$, and the second does if and only if $G = P_l$.

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References

[1] H. Wiener, "Structural determination of paraffin boiling points," *Journal of the American Chemical Society*, vol. 69, pp. 17–20, 1947.
[2] D. J. Klein and M. Randić, "Resistance distance," *Journal of Mathematical Chemistry*, vol. 12, no. 1–4, pp. 81–95, 1993.
[3] D. Bonchev, A. T. Balaban, X. Liu, and D. J. Klein, "Molecular cyclicity and centricity of polycyclic graphs—I. Cyclicity based on resistance distances or reciprocal distances," *International Journal of Quantum Chemistry*, vol. 50, pp. 1–20, 1994.
[4] W. Xiao and I. Gutman, "Resistance distance and Laplacian spectrum," *Theoretical Chemistry Accounts*, vol. 110, no. 4, pp. 284–289, 2003.
[5] W. Zhang and H. Deng, "The second maximal and minimal Kirchhoff indices of unicyclic graphs," *Communications in Mathematical and in Computer Chemistry*, vol. 61, no. 3, pp. 683–695, 2009.
[6] H. Zhang, Y. Yang, and C. Li, "Kirchhoff index of composite graphs," *Discrete Applied Mathematics*, vol. 157, no. 13, pp. 2918–2927, 2009.
[7] E. Bendito, A. Carmona, A. M. Encinas, J. M. Gesto, and M. Mitjana, "Kirchoff indexes of a network," *Linear Algebra and its Applications*, vol. 432, no. 9, pp. 2278–2292, 2010.
[8] Y. Wang and W. Zhang, "Kirchhoff index of linear pentagonal chains," *International Journal of Quantum Chemistry*, vol. 110, no. 9, pp. 1594–1604, 2010.
[9] Y. Wang and W. Zhang, "Kirchhoff index of cyclopolyacenes," *Zeitschrift fur Naturforschung*, vol. 65, no. 10, pp. 865–870, 2010.
[10] X. Gao, Y. Luo, and W. Liu, "Resistance distances and the Kirchhoff index in Cayley graphs," *Discrete Applied Mathematics*, vol. 159, no. 17, pp. 2050–2057, 2011.
[11] X. Gao, Y. Luo, and W. Liu, "Kirchhoff index in line, subdivision and total graphs of a regular graph," *Discrete Applied Mathematics*, vol. 160, no. 4-5, pp. 560–565, 2012.
[12] Y. Yang, H. Zhang, and D. J. Klein, "New Nordhaus-Gaddum-type results for the Kirchhoff index," *Journal of Mathematical Chemistry*, vol. 49, no. 8, pp. 1587–1598, 2011.
[13] L. Ye, "On the Kirchhoff index of some toroidal lattices," *Linear and Multilinear Algebra*, vol. 59, no. 6, pp. 645–650, 2011.
[14] B. Zhou and N. Trinajstić, "A note on Kirchhoff index," *Chemical Physics Letters*, vol. 455, no. 1–3, pp. 120–123, 2008.
[15] B. Zhou and N. Trinajstić, "On resistance-distance and Kirchhoff index," *Journal of Mathematical Chemistry*, vol. 46, no. 1, pp. 283–289, 2009.

[16] B. Zhou and N. Trinajstić, "The kirchhoff index and the matching number," *International Journal of Quantum Chemistry*, vol. 109, no. 13, pp. 2978–2981, 2009.
[17] I. Lukovits, S. Nikolić, and N. Trinajstić, "Resistance distance in regular graphs," *International Journal of Quantum Chemistry*, vol. 71, no. 3, pp. 217–225, 1999.
[18] J. L. Palacios, "Resistance distance in graphs and random walks," *International Journal of Quantum Chemistry*, vol. 81, no. 1, pp. 29–33, 2001.
[19] H. Zhang and Y. Yang, "Resistance distance and kirchhoff index in circulant graphs," *International Journal of Quantum Chemistry*, vol. 107, no. 2, pp. 330–339, 2007.
[20] Q. Guo, H. Deng, and D. Chen, "The extremal Kirchhoff index of a class of unicyclic graphs," *Communications in Mathematical and in Computer Chemistry*, vol. 61, no. 3, pp. 713–722, 2009.
[21] H. Zhang, X. Jiang, and Y. Yang, "Bicyclic graphs with extremal Kirchhoff index," *Communications in Mathematical and in Computer Chemistry*, vol. 61, no. 3, pp. 697–712, 2009.
[22] L. Feng, G. Yu, K. Xu, and Z. Jiang, "A note on the Kirchhoff index of bicyclic graphs," *Ars Combinatoria*. In press.
[23] H. Wang, H. Hua, and D. Wang, "Cacti with minimum, second-minimum, and third-minimum Kirchhoff indices," *Mathematical Communications*, vol. 15, no. 2, pp. 347–358, 2010.
[24] D. J. Klein, "Resistance-distance sum rules," *Croatica Chemica Acta*, vol. 75, no. 2, pp. 633–649, 2002.
[25] R. C. Entringer, D. E. Jackson, and D. A. Snyder, "Distance in graphs," *Czechoslovak Mathematical Journal*, vol. 26, no. 2, pp. 283–296, 1976.