Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 182905, 8 pages
doi:10.1155/2012/182905

Research Article

Optimal Inequalities for Power Means

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Received 9 December 2011; Revised 30 January 2012; Accepted 2 February 2012

Academic Editor: Hector Pomares

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We present the best possible power mean bounds for the product $M_p^{\alpha}(a,b)M_{-p}^{1-\alpha}(a,b)$ for any $p>0$, $\alpha\in(0,1)$, and all $a,b>0$ with $a\neq b$. Here, $M_p(a,b)$ is the pth power mean of two positive numbers a and b.

1. Introduction

For $p\in\mathbb{R}$, the pth power mean $M_p(a,b)$ of two positive numbers a and b is defined by

$$M_p(a,b)=\begin{cases}\left(\dfrac{a^p+b^p}{2}\right)^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0.\end{cases}\tag{1.1}$$

It is well known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$. Many classical means are special cases of the power mean, for example, $M_{-1}(a,b)=H(a,b)=2ab/(a+b)$, $M_0(a,b)=G(a,b)=\sqrt{ab}$ and $M_1(a,b)=A(a,b)=(a+b)/2$ are the harmonic, geometric and arithmetic means of a and b, respectively. Recently, the power mean has been the subject of intensive research. In particular, many remarkable inequalities and properties for the power mean can be found in literature [1–22].

Let $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/[4\arctan(\sqrt{a/b}) - \pi]$ and $I(a,b) = 1/e(a^a/b^b)^{1/(a-b)}$ be the logarithmic, Seiffert and identric means of two positive numbers a and b with $a \neq b$, respectively. Then it is well known that

$$\min\{a,b\} < H(a,b) < G(a,b) < L(a,b) < P(a,b) < I(a,b) < A(a,b) < \max\{a,b\}, \tag{1.2}$$

for all $a,b > 0$ with $a \neq b$.

In [23–29], the authors presented the sharp power mean bounds for L, I, $(IL)^{1/2}$ and $(L+I)/2$ as follows:

$$\begin{gathered} M_0(a,b) < L(a,b) < M_{1/3}(a,b), \qquad M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b), \\ M_0(a,b) < \sqrt{L(a,b)I(a,b)} < M_{1/2}(a,b), \qquad \frac{1}{2}(L(a,b) + I(a,b)) < M_{1/2}(a,b), \end{gathered} \tag{1.3}$$

for all $a,b > 0$ with $a \neq b$.

Alzer and Qiu [12] proved that the inequality

$$\frac{1}{2}(L(a,b) + I(a,b)) > M_p(a,b) \tag{1.4}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $p \leq (\log 2)/(1+\log 2) = 0.40938\ldots$.

The following sharp bounds for the sum $\alpha A(a,b) + (1-\alpha)L(a,b)$, and the products $A^{\alpha}(a,b)L^{1-\alpha}(a,b)$ and $G^{\alpha}(a,b)L^{1-\alpha}(a,b)$ in terms of power means were proved in [5, 8]:

$$\begin{gathered} M_{\log 2/(\log 2 - \log \alpha)}(a,b) < \alpha A(a,b) + (1-\alpha)L(a,b) < M_{(1+2\alpha)/3}(a,b), \\ M_0(a,b) < A^{\alpha}(a,b)L^{1-\alpha}(a,b) < M_{(1+2\alpha)/3}(a,b), \\ M_0(a,b) < G^{\alpha}(a,b)L^{1-\alpha}(a,b) < M_{(1-\alpha)/3}(a,b), \end{gathered} \tag{1.5}$$

for any $\alpha \in (0,1)$ and all $a,b > 0$ with $a \neq b$.

In [2, 7] the authors answered the questions: for any $\alpha \in (0,1)$, what are the greatest values $p_1 = p_1(\alpha)$, $p_2 = p_2(\alpha)$, $p_3 = p_3(\alpha)$, and $p_4 = p_4(\alpha)$, and the least values $q_1 = q_1(\alpha)$, $q_2 = q_2(\alpha)$, $q_3 = q_3(\alpha)$, and $q_4 = q_4(\alpha)$, such that the inequalities

$$\begin{gathered} M_{p_1}(a,b) < P^{\alpha}(a,b)L^{1-\alpha}(a,b) < M_{q_1}(a,b), \\ M_{p_2}(a,b) < A^{\alpha}(a,b)G^{1-\alpha}(a,b) < M_{q_2}(a,b), \\ M_{p_3}(a,b) < G^{\alpha}(a,b)H^{1-\alpha}(a,b) < M_{q_3}(a,b), \\ M_{p_4}(a,b) < A^{\alpha}(a,b)H^{1-\alpha}(a,b) < M_{q_4}(a,b), \end{gathered} \tag{1.6}$$

hold for all $a,b > 0$ with $a \neq b$?

It is the aim of this paper to present the best possible power mean bounds for the product $M_p^{\alpha}(a,b)M_{-p}^{1-\alpha}(a,b)$ for any $p > 0$, $\alpha \in (0,1)$ and all $a,b > 0$ with $a \neq b$.

2. Main Result

Theorem 2.1. *Let $p > 0$, $\alpha \in (0,1)$ and $a, b > 0$ with $a \neq b$. Then*

(1) $M_{(2\alpha-1)p}(a,b) = M_p^{\alpha}(a,b)M_{-p}^{1-\alpha}(a,b) = M_0(a,b)$ *for* $\alpha = 1/2$,

(2) $M_{(2\alpha-1)p}(a,b) > M_p^{\alpha}(a,b)M_{-p}^{1-\alpha}(a,b) > M_0(a,b)$ *for* $\alpha > 1/2$ *and* $M_{(2\alpha-1)p}(a,b) < M_p^{\alpha}(a,b)M_{-p}^{1-\alpha}(a,b) < M_0(a,b)$ *for* $\alpha < 1/2$, *and the bounds* $M_{(2\alpha-1)p}(a,b)$ *and* $M_0(a,b)$ *for the product* $M_p^{\alpha}(a,b)M_{-p}^{1-\alpha}(a,b)$ *in either case are best possible.*

Proof. From (1.1) we clearly see that $M_p(a,b)$ is symmetric and homogenous of degree 1. Without loss of generality, we assume that $b = 1$, $a = x > 1$.

(1) If $\alpha = 1/2$, then (1.1) leads to

$$\begin{aligned} M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1) &= \left(\frac{1+x^p}{2}\right)^{1/p}\left(\frac{1+x^{-p}}{2}\right)^{-1/p} \\ &= \left(\frac{1+x^p}{2}\right)^{1/p}\left(\frac{2x^p}{1+x^p}\right)^{1/p} = x = M_0^2(x,1) = M_{(2\alpha-1)p}^2(x,1). \end{aligned} \tag{2.1}$$

(2) Firstly, we compare the value of $M_{(2\alpha-1)p}(x,1)$ to the value of $M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1)$ for $\alpha \in (0,1/2) \cup (1/2,1)$. From (1.1) we have

$$\begin{aligned} &\log\left[M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1)\right] - \log M_{(2\alpha-1)p}(x,1) \\ &\quad = \frac{\alpha}{p}\log\frac{1+x^p}{2} - \frac{1-\alpha}{p}\log\frac{1+x^{-p}}{2} - \frac{1}{(2\alpha-1)p}\log\frac{1+x^{(2\alpha-1)p}}{2}. \end{aligned} \tag{2.2}$$

Let

$$f(x) = \frac{\alpha}{p}\log\frac{1+x^p}{2} - \frac{1-\alpha}{p}\log\frac{1+x^{-p}}{2} - \frac{1}{(2\alpha-1)p}\log\frac{1+x^{(2\alpha-1)p}}{2}, \tag{2.3}$$

then simple computations lead to

$$f(1) = 0, \tag{2.4}$$

$$f'(x) = \frac{g(x)}{x(1+x^p)\left(1+x^{(2\alpha-1)p}\right)}, \tag{2.5}$$

where

$$\begin{gathered} g(x) = (\alpha-1)x^{2\alpha p} + \alpha x^p - \alpha x^{(2\alpha-1)p} + 1 - \alpha, \\ g(1) = 0, \end{gathered} \tag{2.6}$$

$$g'(x) = \alpha p x^{p-1}h(x), \tag{2.7}$$

where

$$h(x) = 2(\alpha - 1)x^{(2\alpha-1)p} - (2\alpha - 1)x^{2(\alpha-1)p} + 1,$$
$$h(1) = 0, \tag{2.8}$$

$$h'(x) = -2p(1-\alpha)(2\alpha - 1)x^{2(\alpha-1)p-1}(x^p - 1). \tag{2.9}$$

If $\alpha \in (1/2, 1)$, then (2.9) implies that $h(x)$ is strictly decreasing in $[1, +\infty)$. Therefore, $M_{(2\alpha-1)p}(x,1) > M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1)$ follows easily from (2.2)–(2.8) and the monotonicity of $h(x)$.

If $\alpha \in (0, 1/2)$, then (2.9) leads to the conclusion that $h(x)$ is strictly increasing in $[1, +\infty)$. Therefore, $M_{(2\alpha-1)p}(x,1) < M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1)$ follows easily from (2.2)–(2.8) and the monotonicity of $h(x)$.

Secondly, we compare the value of $M_0(x,1)$ to the value of $M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1)$. It follows from (1.1) that

$$\log\left[M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1)\right] - \log M_0(x,1)$$
$$= \frac{\alpha}{p}\log\frac{1+x^p}{2} - \frac{1-\alpha}{p}\log\frac{1+x^{-p}}{2} - \frac{1}{2}\log x. \tag{2.10}$$

Let

$$F(x) = \frac{\alpha}{p}\log\frac{1+x^p}{2} - \frac{1-\alpha}{p}\log\frac{1+x^{-p}}{2} - \frac{1}{2}\log x, \tag{2.11}$$

then simple computations lead to

$$F(1) = 0, \tag{2.12}$$

$$F'(x) = \frac{(2\alpha-1)(x^p - 1)}{x(1+x^p)\left(1+x^{(2\alpha-1)p}\right)}. \tag{2.13}$$

If $\alpha \in (1/2, 1)$, then (2.13) implies that $F(x)$ is strictly increasing in $[1, +\infty)$. Therefore, $M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1) > M_0(x,1)$ follows easily from (2.10)–(2.12) and the monotonicity of $F(x)$.

If $\alpha \in (0, 1/2)$, then (2.13) leads to the conclusion that $F(x)$ is strictly decreasing in $[1, +\infty)$. Therefore, $M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1) < M_0(x,1)$ follows easily from (2.10)–(2.12) and the monotonicity of $F(x)$.

Next, we prove that the bound $M_{(2\alpha-1)p}(a,b)$ for the product $M_p^{\alpha}(a,b)M_{-p}^{1-\alpha}(a,b)$ in either case is best possible.

If $\alpha \in (0, 1/2)$, then for any $\epsilon \in (0, (1-2\alpha)p)$ and $x > 0$ we have

$$\begin{aligned} &M_p^{\alpha}(1+x,1)M_{-p}^{1-\alpha}(1+x,1) - M_{(2\alpha-1)p+\epsilon}(1+x,1) \\ &\quad = \left[\frac{1+(1+x)^p}{2}\right]^{\alpha/p}\left[\frac{1+(1+x)^{-p}}{2}\right]^{(\alpha-1)/p} \\ &\qquad - \left[\frac{1+(1+x)^{(2\alpha-1)p+\epsilon}}{2}\right]^{1/[(2\alpha-1)p+\epsilon]}. \end{aligned} \tag{2.14}$$

Letting $x \to 0$ and making use of Taylor's expansion, one has

$$\begin{aligned} &\left[\frac{1+(1+x)^p}{2}\right]^{\alpha/p}\left[\frac{1+(1+x)^{-p}}{2}\right]^{(\alpha-1)/p} - \left[\frac{1+(1+x)^{(2\alpha-1)p+\epsilon}}{2}\right]^{1/[(2\alpha-1)p+\epsilon]} \\ &\quad = \left[1+\frac{\alpha}{2}x + \frac{\alpha(p+\alpha-2)}{8}x^2 + o\left(x^2\right)\right] \\ &\qquad \times \left[1+\frac{1-\alpha}{2}x - \frac{(1-\alpha)(p+\alpha+1)}{8}x^2 + o\left(x^2\right)\right] \\ &\qquad - \left[1+\frac{1}{2}x + \frac{(2\alpha-1)p+\epsilon-1}{8}x^2 + o\left(x^2\right)\right] \\ &\quad = \left[1+\frac{1}{2}x + \frac{(2\alpha-1)p-1}{8}x^2 + o\left(x^2\right)\right] \\ &\qquad - \left[1+\frac{1}{2}x + \frac{(2\alpha-1)p+\epsilon-1}{8}x^2 + o\left(x^2\right)\right] \\ &\quad = -\frac{\epsilon}{8}x^2 + o\left(x^2\right). \end{aligned} \tag{2.15}$$

Equations (2.14) and (2.15) imply that for any $\alpha \in (0, 1/2)$ and $\epsilon \in (0, (1-2\alpha)p)$ there exists $\delta_1 = \delta_1(\epsilon) > 0$, such that $M_p^{\alpha}(1+x,1)M_{-p}^{1-\alpha}(1+x,1) < M_{(2\alpha-1)p+\epsilon}(1+x,1)$ for $x \in (0, \delta_1)$.

If $\alpha \in (1/2, 1)$, then for any $\epsilon \in (0, (2\alpha-1)p)$ and $x > 0$ we have

$$\begin{aligned} &M_p^{\alpha}(1+x,1)M_{-p}^{1-\alpha}(1+x,1) - M_{(2\alpha-1)p-\epsilon}(1+x,1) \\ &\quad = \left[\frac{1+(1+x)^p}{2}\right]^{\alpha/p}\left[\frac{1+(1+x)^{-p}}{2}\right]^{(\alpha-1)/p} \\ &\qquad - \left[\frac{1+(1+x)^{(2\alpha-1)p-\epsilon}}{2}\right]^{1/[(2\alpha-1)p-\epsilon]}. \end{aligned} \tag{2.16}$$

Letting $x \to 0$ and making use of Taylor's expansion, one has

$$\begin{aligned}
&\left[\frac{1+(1+x)^{p}}{2}\right]^{\alpha/p}\left[\frac{1+(1+x)^{-p}}{2}\right]^{(\alpha-1)/p}-\left[\frac{1+(1+x)^{(2\alpha-1)p-\epsilon}}{2}\right]^{1/[(2\alpha-1)p-\epsilon]}\\
&\quad=\left[1+\frac{\alpha}{2}x+\frac{\alpha(p+\alpha-2)}{8}x^{2}+o\left(x^{2}\right)\right]\\
&\qquad\times\left[1+\frac{1-\alpha}{2}x-\frac{(1-\alpha)(p+\alpha+1)}{8}x^{2}+o\left(x^{2}\right)\right]\\
&\qquad-\left[1+\frac{1}{2}x+\frac{(2\alpha-1)p-\epsilon-1}{8}x^{2}+o\left(x^{2}\right)\right]\\
&\quad=\left[1+\frac{1}{2}x+\frac{(2\alpha-1)p-1}{8}x^{2}+o\left(x^{2}\right)\right]\\
&\qquad-\left[1+\frac{1}{2}x+\frac{(2\alpha-1)p-\epsilon-1}{8}x^{2}+o\left(x^{2}\right)\right]\\
&\quad=\frac{\epsilon}{8}x^{2}+o\left(x^{2}\right).
\end{aligned} \tag{2.17}$$

Equations (2.16) and (2.17) imply that for any $\alpha \in (1/2, 1)$ and $\epsilon \in (0, (2\alpha-1)p)$ there exists $\delta_2 = \delta_2(\epsilon) > 0$, such that $M_p^{\alpha}(1+x,1)M_{-p}^{1-\alpha}(1+x,1) > M_{(2\alpha-1)p-\epsilon}(1+x,1)$ for $x \in (0, \delta_2)$.

Finally, we prove that the bound $M_0(a,b)$ for the product $M_p^{\alpha}(a,b)M_{-p}^{1-\alpha}(a,b)$ in either case is best possible.

If $\alpha \in (0, 1/2)$, then for any $\epsilon > 0$ we clearly see that

$$\lim_{x\to+\infty}\frac{M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1)}{M_{-\epsilon}(x,1)}=+\infty. \tag{2.18}$$

Equation (2.18) implies that for any $\alpha \in (0, 1/2)$ and $\epsilon > 0$ there exists $T_1 = T_1(\epsilon) > 1$, such that $M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1) > M_{-\epsilon}(x,1)$ for $x \in (T_1, +\infty)$.

If $\alpha \in (1/2, 1)$, then for any $\epsilon > 0$ we have

$$\lim_{x\to+\infty}\frac{M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1)}{M_{\epsilon}(x,1)}=0. \tag{2.19}$$

Equation (2.19) implies that for any $\alpha \in (1/2, 1)$ and $\epsilon > 0$ there exists $T_2 = T_2(\epsilon) > 1$, such that $M_p^{\alpha}(x,1)M_{-p}^{1-\alpha}(x,1) < M_{\epsilon}(x,1)$ for $x \in (T_2, +\infty)$. □

Acknowledgments

This paper was supported by the Natural Science Foundation of China under Grants 11071069 and 11171307, the Natural Science Foundation of Hunan Province under Grant 09JJ6003, and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant T200924.

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