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Research Article

Eisenstein Series Identities Involving the Borweins' Cubic Theta Functions

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Based on the theories of Ramanujan's elliptic functions and the (p, k)-parametrization of theta functions due to Alaca et al. (2006, 2007, 2006) we derive certain Eisenstein series identities involving the Borweins' cubic theta functions with the help of the computer. Some of these identities were proved by Liu based on the fundamental theory of elliptic functions and some of them may be new. One side of each identity involves Eisenstein series, the other products of the Borweins' cubic theta functions. As applications, we evaluate some convolution sums. These evaluations are different from the formulas given by Alaca et al.

1. Introduction

Let $\mathbb{N}$ and $\mathbb{C}$ denote the sets of positive integers and complex numbers, respectively. Throughout the paper, we always assume that $q \in \mathbb{C}$ and $|q| < 1$.

In their paper, J. M. Borwein and P. B. Borwein [1] introduced the following three functions:

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$
$$b(q) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \tag{1.1}$$
$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2},$$

where $\omega = e^{2\pi i/3}$. These functions are now called the Borweins' cubic theta functions. The Borwein brothers [1] derived representations for $b(q)$ and $c(q)$ in terms of infinite products, namely,

$$b(q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}, \tag{1.2}$$

$$c(q) = \frac{3q^{1/3}(q^3;q^3)_\infty^3}{(q;q)_\infty}, \tag{1.3}$$

where

$$(a;q)_\infty = \prod_{i=0}^{\infty}\left(1-aq^i\right). \tag{1.4}$$

The function $a(q)$ has the following representation derived by the Borwein brothers [1] and Berndt [2]:

$$a(q) = \left(-q;q^2\right)_\infty^2\left(q^2;q^2\right)_\infty\left(-q^3;q^6\right)_\infty^2\left(q^6;q^6\right)_\infty + 4q\frac{(q^4;q^4)_\infty(q^{12};q^{12})_\infty}{(q^2;q^4)_\infty(q^6;q^{12})_\infty}. \tag{1.5}$$

Elementary proofs of (1.2), (1.3), and (1.5) can be found in [3]. The Borwein brothers [1] also proved the following well-known relation for $a(q)$, $b(q)$ and $c(q)$, namely:

$$a^3(q) = b^3(q) + c^3(q). \tag{1.6}$$

In his second notebook [4], Ramanujan gave the definitions of the Eisenstein series $L(q)$, $M(q)$, and $N(q)$, namely,

$$L(q) := 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{1.7}$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \tag{1.8}$$

$$N(q) := 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}. \tag{1.9}$$

Utilizing Ramanujan's elliptic functions in the theory of signature 3, Berndt et al. [5] proved the following representations for $M(q)$, $M(q^3)$, $N(q)$, and $N(q^3)$, namely:

$$M(q) = a(q)\left(a^3(q) + 8c^3(q)\right), \tag{1.10}$$

$$M\left(q^3\right) = \frac{1}{9}a(q)\left(9a^3(q) - 8c^3(q)\right), \tag{1.11}$$

$$N(q) = a^6(q) - 20a^3(q)c^3(q) - 8c^3(q), \tag{1.12}$$

$$N\left(q^3\right) = a^6(q) - \frac{4}{3}a^3(q)c^3(q) + \frac{8}{27}c^3(q). \tag{1.13}$$

Chan [6] proved (1.10) and (1.11) by employing the classical theory of elliptic functions and modular equations of degree 3. Based on the fundamental theory of elliptic functions, Liu [7, 8] provided different proofs of (1.10), (1.11), (1.12), and (1.13). He also discovered some striking Eisenstein series identities.

In this paper, using the parameters p and k introduced by Alaca et al. [9–11] (see (2.1)), we deduce some Eisenstein series identities involving the Borweins' cubic theta functions with the help of the computer. These identities are examples of sum-to-product identities. Some of these identities were proved by Liu [7, 8] based on the fundamental theory of elliptic function and some of them may be new.

This paper is organized as follows. In Section 2, we gave the (p,k)-parametrization of $mL(q^m) - L(q)$, and $M(q^l)$ for $m \in \{2,3,4,6,12\}$, $l \in \{1,2,3,6\}$, which are due to Alaca et al. [9–11]. Section 3 is devoted to deriving some Eisenstein series identities involving the Borweins' cubic theta functions. As corollaries of our results, in Section 4, we derive new representations for the convolution sums $\sum_{i+6j=n}\sigma(i)\sigma(j)$ and $\sum_{i+12j=n}\sigma(i)\sigma(j)$ and contrast them with the known evaluations due to Alaca and Williams [11] and Alaca et al. [10].

2. Eisenstein Series $L(q)$, $M(q)$ and Parameters p, k

In this section, we gave the parametric representations for $mL(q^m) - L(q)$, $M(q^l)$, $b(q^i)$ and $c(q^i)$ for $m \in \{2,3,4,6,12\}$, $l \in \{1,2,3,6\}$ and $i \in \{1,2,4\}$ in terms of the parameters p and k first defined by Alaca and Williams [11], namely,

$$p = p(q) = \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, \qquad k = k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{2.1}$$

Alaca and Williams [11] derived the representations of $M(q)$, $M(q^i)$, and $iL(q^i) - L(q)$ $(i = 2,3,6)$ in terms of p and k. Equations (3.69), (3.70), (3.71), (3.72), (3.84), (3.87), and (3.89) in [11] are

$$M(q) = \left(1 + 124p + 964p^2 + 2788p^3 + 3910p^4 + 2788p^5 + 964p^6 + 124p^7 + p^8\right)k^4,$$

$$M\left(q^2\right) = \left(1 + 4p + 64p^2 + 178p^3 + 235p^4 + 178p^5 + 64p^6 + 4p^7 + p^8\right)k^4,$$

$$M\left(q^3\right) = \left(1 + 4p + 4p^2 + 28p^3 + 70p^4 + 28p^5 + 4p^6 + 4p^7 + p^8\right)k^4,$$
$$M\left(q^6\right) = \left(1 + 4p + 4p^2 - 2p^3 - 5p^4 - 2p^5 + 4p^6 + 4p^7 + p^8\right)k^4,$$
$$2L\left(q^2\right) - L(q) = \left(1 + 14p + 24p^2 + 14p^3 + p^4\right)k^2,$$
$$3L\left(q^3\right) - L(q) = \left(2 + 16p + 36p^2 + 16p^3 + 2p^4\right)k^2,$$
$$6L\left(q^6\right) - L(q) = \left(5 + 22p + 36p^2 + 22p^3 + 5p^4\right)k^2, \tag{2.2}$$

respectively. Alaca et al. [12, 13] also deduced the representations of $12L(q^{12}) - L(q)$, $M(q^{12})$ and $4L(q^4) - L(q)$ in terms of p and k. Equations (3.12), and (3.19) in [12] and (3.13) in [13] are

$$12L\left(q^{12}\right) - L(q) = \left(11 + 34p + 36p^2 + 16p^3 + 2p^4\right)k^2,$$
$$M\left(q^{12}\right) = \left(1 + 4p + 4p^2 - 2p^3 - 5p^4 - 2p^5 + \frac{1}{4}p^6 + \frac{1}{4}p^7 + \frac{1}{16}p^8\right)k^4, \tag{2.3}$$
$$4L\left(q^4\right) - L(q) = \left(24p^3 + 36p^2 + 18p + 3\right)k^2,$$

respectively. Alaca et al. [9] also derived the following parametric representations for $b(q)$ and $c(q)$ in terms of p and k. From Theorems 1, 2, and 4 in [9], we have

$$b(q) = \frac{(1-p)((1-p)(1+2p)(2+p))^{1/3}k}{2^{1/3}},$$
$$c(q) = \frac{3(1+p)(p(1+p))^{1/3}k}{2^{1/3}},$$
$$b\left(q^2\right) = \frac{((1-p)(1+2p)(2+p))^{2/3}k}{2^{2/3}},$$
$$c\left(q^2\right) = \frac{3(p(1+p))^{2/3}k}{2^{2/3}}, \tag{2.4}$$
$$b\left(q^4\right) = \frac{(2+p)((1-p)(1+2p)(2+p))^{1/3}k}{2^{4/3}},$$
$$c\left(q^4\right) = \frac{3p(p(1+p))^{1/3}k}{2^{4/3}}.$$

We now describe our approach. Let $R(e(q), e(q^2), e(q^4))$ be a function, where $e = b$ or $e = c$. Utilizing the representations for $b(q)$, $b(q^2)$, $b(q^4)$, $c(q)$, $c(q^2)$ and $c(q^4)$ in terms of p and k, we derive the representations $R(p, k)$ for $R(e(q), e(q^2), e(q^4))$ in terms of p and k. We select suitable $R(e(q), e(q^2), e(q^4))$ such that $R(p,k)$ is a polynomial in p and k. We want to show that

$$R(p,k) = \sum_{i=1}^{s} C_i T_i, \tag{2.5}$$

where each C_i is a rational number and T_i is a product involving $mL(q^m) - L(q)$ and $M(q^m)$. Substituting the representation for T_i in terms of p and k into (2.5), then both sides of (2.5) are functions in p and k. Equating the coefficients of $p^i k^j$ on both sides of (2.5), we obtain some linear equations in C_i. If these equations have a solution, then we can use computer to solve the equations and determine the values of the C_i. We then obtain some Eisenstein series identities involving the Borweins' cubic theta functions.

3. Some Eisenstein Series Identities

In this section, we derive some Eisenstein series identities. In fact, utilizing our method, we can obtain many identities, here we just list some of them. Our main theorem can be stated as follows.

Theorem 3.1. *One has*

$$\begin{aligned} 1 - 6\sum_{n=1}^{\infty}\left(\frac{nq^n}{1-q^n} - \frac{4nq^{2n}}{1-q^{2n}} - \frac{9nq^{3n}}{1-q^{3n}} + \frac{16nq^{4n}}{1-q^{4n}}\right) \\ = \frac{(q;q)_\infty^6 (q^2;q^2)_\infty^3 (q^{12};q^{12})_\infty}{(q^4;q^4)_\infty^3 (q^3;q^3)_\infty^2 (q^6;q^6)_\infty} = \frac{b^2(q)b(q^2)}{b(q^4)}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} 1 + 3\sum_{n=1}^{\infty}\left(\frac{nq^n}{1-q^n} - \frac{nq^{2n}}{1-q^{2n}} + \frac{nq^{4n}}{1-q^{4n}} - \frac{9nq^{12n}}{1-q^{12n}}\right) \\ = \frac{(q^4;q^4)_\infty^6 (q^2;q^2)_\infty^3 (q^3;q^3)_\infty}{(q^{12};q^{12})_\infty^2 (q^6;q^6)_\infty (q;q)_\infty^3} = \frac{b^2(q^4)b(q^2)}{b(q)}, \end{aligned} \tag{3.2}$$

$$1 - 3\sum_{n=0}^{\infty}\left(\frac{nq^n}{1-q^n} - \frac{2nq^{2n}}{1-q^{2n}} - \frac{9nq^{3n}}{1-q^{3n}} + \frac{18nq^{6n}}{1-q^{6n}}\right) = \frac{(q;q)_\infty^3 (q^2;q^2)_\infty^3}{(q^3;q^3)_\infty (q^6;q^6)_\infty} = b(q)b\left(q^2\right), \tag{3.3}$$

$$1 - 12\sum_{n=1}^{\infty}\left(\frac{nq^n}{1-q^n} - \frac{8nq^{2n}}{1-q^{2n}} + \frac{9nq^{3n}}{1-q^{3n}}\right) = \frac{(q;q)_\infty^{12} (q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty^4} = \frac{b^4(q)}{b^2(q^2)}, \tag{3.4}$$

$$1 + 3\sum_{n=1}^{\infty}\left(\frac{2nq^n}{1-q^n} - \frac{nq^{2n}}{1-q^{2n}} - \frac{9nq^{6n}}{1-q^{6n}}\right) = \frac{(q^2;q^2)_\infty^{12} (q^3;q^3)_\infty^2}{(q;q)_\infty^6 (q^6;q^6)_\infty^4} = \frac{b^4(q^2)}{b^2(q)}, \tag{3.5}$$

$$\begin{aligned} 1 + 3\sum_{n=1}^{\infty}\left(\frac{nq^n}{1-q^n} - 4\frac{nq^{2n}}{1-q^{2n}} - 9\frac{nq^{3n}}{1-q^{3n}} + 4\frac{nq^{4n}}{1-q^{4n}} + 36\frac{nq^{6n}}{1-q^{6n}} - 36\frac{nq^{12n}}{1-q^{12n}}\right) \\ = \frac{(q^2;q^2)_\infty^{12} (q^3;q^3)_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty^3 (q^4;q^4)_\infty^3 (q^6;q^6)_\infty^4} = \frac{b^4(q^2)}{b(q)b(q^4)}, \end{aligned} \tag{3.6}$$

$$1+3\sum_{n=0}^{\infty}\left(\frac{5nq^{2n}}{1-q^{2n}}-\frac{nq^{n}}{1-q^{n}}-\frac{3nq^{4n}}{1-q^{4n}}-9\frac{nq^{12n}}{1-q^{12n}}\right)$$
$$=\frac{(q;q)_\infty^3(q^4;q^4)_\infty^9(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty^3(q^2;q^2)_\infty^6}=\frac{b(q)b^3(q^4)}{b^2(q^2)}, \quad (3.7)$$

$$1+3\sum_{n=1}^{\infty}\left(20\frac{nq^{2n}}{1-q^{2n}}-3\frac{nq^{n}}{1-q^{n}}-9\frac{nq^{3n}}{1-q^{3n}}-16\frac{nq^{4n}}{1-q^{4n}}\right)$$
$$=\frac{(q;q)_\infty^9(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty^3(q^{12};q^{12})_\infty(q^2;q^2)_\infty^6}=\frac{b^3(q)b(q^4)}{b^2(q^2)}, \quad (3.8)$$

$$\frac{107}{32}+\frac{3}{2}\sum_{n=1}^{\infty}\left(\frac{11n^3q^{n}}{1-q^{n}}-\frac{16n^3q^{2n}}{1-q^{2n}}+\frac{540n^3q^{6n}}{1-q^{6n}}\right)$$
$$-\frac{3}{32}\left(5+24\sum_{n=1}^{\infty}\left(\frac{nq^{n}}{1-q^{n}}-\frac{6nq^{6n}}{1-q^{6n}}\right)\right)^2=\frac{(q;q)_\infty^6(q^2;q^2)_\infty^6}{(q^3;q^3)_\infty^2(q^6;q^6)_\infty^2}=b^2(q)b^2\left(q^2\right), \quad (3.9)$$

$$-\frac{11}{64}+\frac{3}{4}\sum_{n=1}^{\infty}\left(\frac{n^3q^{n}}{1-q^{n}}-\frac{2n^3q^{2n}}{1-q^{2n}}-\frac{54n^3q^{6n}}{1-q^{6n}}\right)$$
$$+\frac{3}{64}\left(5+24\sum_{n=1}^{\infty}\left(\frac{nq^{n}}{1-q^{n}}-\frac{6nq^{6n}}{1-q^{6n}}\right)\right)^2=\frac{(q^2;q^2)_\infty^{24}(q^3;q^3)_\infty^4}{(q;q)_\infty^{12}(q^6;q^6)_\infty^8}=\frac{b^8(q^2)}{b^4(q)}, \quad (3.10)$$

$$-74-24\sum_{n=1}^{\infty}\left(\frac{31n^3q^{n}}{1-q^{n}}-\frac{128n^3q^{2n}}{1-q^{2n}}-\frac{243n^3q^{3n}}{1-q^{3n}}+\frac{1080n^3q^{6n}}{1-q^{6n}}\right)$$
$$+3\left(5+24\sum_{n=1}^{\infty}\left(\frac{nq^{n}}{1-q^{n}}-\frac{6nq^{6n}}{1-q^{6n}}\right)\right)^2=\frac{(q;q)_\infty^{24}(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^{12}(q^3;q^3)_\infty^8}=\frac{b^8(q)}{b^4(q^2)}, \quad (3.11)$$

$$1+2\sum_{n=1}^{\infty}\left(\frac{nq^{n}}{1-q^{n}}-\frac{nq^{3n}}{1-q^{3n}}+\frac{4nq^{6n}}{1-q^{6n}}-\frac{16nq^{12n}}{1-q^{12n}}\right)$$
$$=\frac{(q^3;q^3)_\infty^6(q^6;q^6)_\infty^3(q^4;q^4)_\infty}{(q;q)_\infty^2(q^2;q^2)_\infty(q^{12};q^{12})_\infty^3}=\frac{c^2(q)c(q^2)}{9c(q^4)}, \quad (3.12)$$

$$\sum_{n=1}^{\infty}\left(\frac{nq^{3n}}{1-q^{3n}}-\frac{nq^{4n}}{1-q^{4n}}-\frac{nq^{6n}}{1-q^{6n}}+\frac{nq^{12n}}{1-q^{12n}}\right)$$
$$=\frac{q^3(q^{12};q^{12})_\infty^6(q^6;q^6)_\infty^3(q;q)_\infty}{(q^4;q^4)_\infty^2(q^2;q^2)_\infty(q^3;q^3)_\infty^3}=\frac{c^2(q^4)c(q^2)}{9c(q)}, \quad (3.13)$$

$$\sum_{n=1}^{\infty}\left(\frac{nq^{n}}{1-q^{n}}-\frac{2nq^{2n}}{1-q^{2n}}-\frac{nq^{3n}}{1-q^{3n}}+\frac{2nq^{6n}}{1-q^{6n}}\right)=\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty}=\frac{c(q)c(q^2)}{9}, \quad (3.14)$$

$$\sum_{n=1}^{\infty}\left(\frac{nq^{2n}}{1-q^{2n}}-\frac{2nq^{3n}}{1-q^{3n}}+\frac{nq^{6n}}{1-q^{6n}}\right)=\frac{q^2(q^6;q^6)_\infty^{12}(q;q)_\infty^2}{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^6}=\frac{c^4(q^2)}{9c^2(q)}, \tag{3.15}$$

$$1+4\sum_{n=1}^{\infty}\left(\frac{nq^{n}}{1-q^{n}}+\frac{nq^{3n}}{1-q^{3n}}-\frac{8nq^{6n}}{1-q^{6n}}\right)=\frac{(q^3;q^3)_\infty^{12}(q^2;q^2)_\infty^2}{(q;q)_\infty^4(q^6;q^6)_\infty^6}=\frac{c^4(q)}{9c^2(q^2)}, \tag{3.16}$$

$$\begin{aligned}\sum_{n=1}^{\infty}&\left(\frac{nq^{n}}{1-q^{n}}-4\frac{nq^{2n}}{1-q^{2n}}-\frac{nq^{3n}}{1-q^{3n}}+4\frac{nq^{4n}}{1-q^{4n}}+4\frac{nq^{6n}}{1-q^{6n}}-4\frac{nq^{12n}}{1-q^{12n}}\right)\\&=q\frac{(q^6;q^6)_\infty^{12}(q;q)_\infty(q^4;q^4)_\infty}{(q^2;q^2)_\infty(q^3;q^3)_\infty^3(q^{12};q^{12})_\infty^3}=\frac{c^4(q^2)}{9c(q)c(q^4)},\end{aligned} \tag{3.17}$$

$$\begin{aligned}\sum_{n=1}^{\infty}&\left(\frac{nq^{3n}}{1-q^{3n}}+\frac{nq^{4n}}{1-q^{4n}}-5\frac{nq^{6n}}{1-q^{6n}}+3\frac{nq^{12n}}{1-q^{12n}}\right)\\&=q^3\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^3(q^{12};q^{12})_\infty^9}{(q;q)_\infty(q^4;q^4)_\infty^3(q^6;q^6)_\infty^6}=\frac{c(q)c^3(q^4)}{9c^2(q^2)},\end{aligned} \tag{3.18}$$

$$\begin{aligned}\sum_{n=1}^{\infty}&\left(3\frac{nq^{3n}}{1-q^{3n}}+\frac{nq^{n}}{1-q^{n}}-20\frac{nq^{6n}}{1-q^{6n}}+16\frac{nq^{12n}}{1-q^{12n}}\right)\\&=q\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^9(q^{12};q^{12})_\infty^3}{(q;q)_\infty^3(q^4;q^4)_\infty(q^6;q^6)_\infty^6}=\frac{c^3(q)c(q^4)}{9c^2(q^2)},\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\frac{1}{18}\sum_{n=1}^{\infty}\left(\frac{5n^3q^{n}}{1-q^{n}}-\frac{12n^3q^{3n}}{1-q^{3n}}+\frac{132n^3q^{6n}}{1-q^{6n}}\right)-\frac{1}{864}\left(5+24\sum_{n=1}^{\infty}\left(\frac{nq^{n}}{1-q^{n}}-\frac{6nq^{6n}}{1-q^{6n}}\right)\right)^2\\&\quad+\frac{25}{864}=\frac{q^2(q^3;q^3)_\infty^6(q^6;q^6)_\infty^6}{(q;q)_\infty^2(q^2;q^2)_\infty^2}=\frac{c^2(q)c^2(q^2)}{81},\end{aligned} \tag{3.20}$$

$$\begin{aligned}&-\frac{25}{1728}-\frac{1}{36}\sum_{n=1}^{\infty}\left(\frac{5n^3q^{n}}{1-q^{n}}-\frac{18n^3q^{2n}}{1-q^{2n}}-\frac{48n^3q^{3n}}{1-q^{3n}}+\frac{186n^3q^{6n}}{1-q^{6n}}\right)\\&\quad+\frac{1}{1728}\left(5+24\sum_{n=1}^{\infty}\left(\frac{nq^{n}}{1-q^{n}}-\frac{6nq^{6n}}{1-q^{6n}}\right)\right)^2=\frac{q^4(q^6;q^6)_\infty^{24}(q;q)_\infty^4}{(q^2;q^2)_\infty^8(q^3;q^3)_\infty^{12}}=\frac{c^8(q^2)}{81c^4(q)},\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\frac{2}{27}-\frac{8}{9}\sum_{n=1}^{\infty}\left(\frac{n^3q^{n}}{1-q^{n}}+\frac{3n^3q^{3n}}{1-q^{3n}}-\frac{24n^3q^{6n}}{1-q^{6n}}\right)\\&\quad+\frac{1}{27}\left(5+24\sum_{n=1}^{\infty}\left(\frac{nq^{n}}{1-q^{n}}-\frac{6nq^{6n}}{1-q^{6n}}\right)\right)^2=\frac{(q^3;q^3)_\infty^{24}(q^2;q^2)_\infty^4}{(q;q)_\infty^8(q^6;q^6)_\infty^{12}}=\frac{c^8(q)}{81c^4(q^2)},\end{aligned} \tag{3.22}$$

$$\frac{1}{108}\left(11+24\sum_{n=1}^{\infty}\left(\frac{nq^n}{1-q^n}-\frac{12nq^{12n}}{1-q^{12n}}\right)\right)^2-\frac{1}{216}\left(5+24\sum_{n=1}^{\infty}\left(\frac{nq^n}{1-q^n}-\frac{6nq^{6n}}{1-q^{6n}}\right)\right)^2$$
$$+\frac{2}{9}\sum_{n=1}^{\infty}\left(\frac{n^3q^n}{1-q^n}-\frac{6n^3q^{3n}}{1-q^{3n}}+\frac{96n^3q^{6n}}{1-q^{6n}}-\frac{96n^3q^{12n}}{1-q^{12n}}\right)-\frac{1}{216} \tag{3.23}$$
$$=\frac{(q^3;q^3)_\infty^{12}(q^6;q^6)_\infty^6(q^4;q^4)_\infty^2}{(q;q)_\infty^4(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^6}=\frac{c^4(q)c^2(q^2)}{81c^2(q^4)}.$$

Remark 3.2. The identities (1.16) and (1.17) in [8] contain typos, they should be (3.1) and (3.5), respectively.

Proof. We first prove the formula (3.1) by our method. We assume that

$$C_1\left(2L\left(q^2\right)-L(q)\right)+C_2\left(3L\left(q^3\right)-L(q)\right)+C_3\left(4L\left(q^4\right)-L(q)\right)$$
$$+C_4\left(6L\left(q^6\right)-L(q)\right)+C_5\left(12L\left(q^{12}\right)-L(q)\right)=\frac{b^2(q)b(q^2)}{b(q^4)}. \tag{3.24}$$

Equating the coefficients of k^2, pk^2, p^2k^2, p^3k^2, and p^4k^2 on both sides of (3.24), we obtain the following five equations:

$$\begin{gathered}C_1+2C_2+3C_3+5C_4+11C_5=1,\\ 14C_1+16C_2+18C_3+22C_4+34C_5=-1,\\ 24C_1+36C_2+36C_3+36C_4+36C_5=-3,\\ 14C_1+16C_2+24C_3+22C_4+16C_5=5,\\ C_1+2C_2+5C_4+2C_5=-2.\end{gathered} \tag{3.25}$$

Solving the above five equations, we obtain

$$C_1=-\frac{1}{2},\qquad C_2=-\frac{3}{4},\qquad C_3=1,\qquad C_4=0,\qquad C_5=0. \tag{3.26}$$

Substituting the above values into (3.24), from (1.2) and (1.7), we obtain (3.1).

Similarly, utilizing the same method, we also derive the following formulas:

$$\frac{b^2(q^4)b(q^2)}{b(q)}=\frac{1}{16}\left(2L\left(q^2\right)-L(q)\right)-\frac{1}{32}\left(4L\left(q^4\right)-L(q)\right)+\frac{3}{32}\left(12L\left(q^{12}\right)-L(q)\right),$$
$$b(q)b\left(q^2\right)=-\frac{1}{8}\left(2L\left(q^2\right)-L(q)\right)-\frac{3}{8}\left(3L\left(q^3\right)-L(q)\right)+\frac{3}{8}\left(6L\left(q^6\right)-L(q)\right),$$
$$\frac{b^4(q)}{b^2(q^2)}=-2\left(2L\left(q^2\right)-L(q)\right)+\frac{3}{2}\left(3L\left(q^3\right)-L(q)\right),$$

$$\frac{b^4(q^2)}{b^2(q)} = \frac{1}{16}\left(2L\left(q^2\right) - L(q)\right) + \frac{3}{16}\left(6L\left(q^6\right) - L(q)\right),$$

$$\begin{aligned}\frac{b^4(q^2)}{b(q)b(q^4)} &= \frac{1}{4}(2L(q^2) - L(q)) + \frac{3}{8}(3L(q^3) - L(q)) - \frac{1}{8}(4L(q^4) - L(q)) \\ &\quad - \frac{3}{4}(6L(q^6) - L(q)) + \frac{3}{8}(12L(q^{12}) - L(q)),\end{aligned}$$

$$\frac{b(q)b^3(q^4)}{b^2(q^2)} = -\frac{5}{16}\left(2L\left(q^2\right) - L(q)\right) + \frac{3}{32}\left(4L\left(q^4\right) - L(q)\right) + \frac{3}{32}\left(12L\left(q^{12}\right) - L(q)\right),$$

$$\frac{b^3(q)b(q^4)}{b^2(q^2)} = -\frac{5}{4}\left(2L\left(q^2\right) - L(q)\right) + \frac{3}{8}\left(3L\left(q^3\right) - L(q)\right) + \frac{1}{2}\left(4L\left(q^4\right) - L(q)\right),$$

$$b^2(q)b^2\left(q^2\right) = \frac{11}{160}M(q) - \frac{1}{10}M\left(q^2\right) + \frac{27}{8}M\left(q^6\right) - \frac{3}{32}\left(6L\left(q^6\right) - L(q)\right)^2,$$

$$\frac{b^8(q^2)}{b^4(q)} = \frac{1}{320}M(q) - \frac{1}{160}M\left(q^2\right) - \frac{27}{160}M\left(q^6\right) + \frac{3}{64}\left(6L\left(q^6\right) - L(q)\right)^2,$$

$$\frac{b^8(q)}{b^4(q^2)} = -\frac{31}{10}M(q) + \frac{64}{5}M\left(q^2\right) + \frac{243}{10}M\left(q^3\right) - 108M\left(q^6\right) + 3\left(6L\left(q^6\right) - L(q)\right)^2,$$

$$\frac{c^2(q)c(q^2)}{9c(q^4)} = \frac{1}{36}\left(3L\left(q^3\right) - L(q)\right) - \frac{1}{18}\left(6L\left(q^6\right) - L(q)\right) + \frac{1}{9}\left(12L\left(q^{12} - L(q)\right)\right),$$

$$\begin{aligned}\frac{c^2(q^4)c(q^2)}{9c(q)} &= -\frac{1}{72}(3L(q^3) - L(q)) + \frac{1}{96}(4L(q^4) - L(q)) \\ &\quad + \frac{1}{144}(6L(q^6) - L(q)) - \frac{1}{288}(12L(q^{12}) - L(q)),\end{aligned}$$

$$\frac{c(q)c(q^2)}{9} = \frac{1}{24}\left(2L\left(q^2\right) - L(q)\right) + \frac{1}{72}\left(3L\left(q^3\right) - L(q)\right) - \frac{1}{72}\left(6L\left(q^6\right) - L(q)\right),$$

$$\frac{c^4(q^2)}{9c^2(q)} = -\frac{1}{48}\left(2L\left(q^2\right) - L(q)\right) + \frac{1}{36}\left(3L\left(q^3\right) - L(q)\right) - \frac{1}{144}\left(6L\left(q^6\right) - L(q)\right),$$

$$\frac{c^4(q)}{9c^2(q^2)} = -\frac{1}{18}\left(3L\left(q^3\right) - L(q)\right) + \frac{2}{9}\left(6L\left(q^6\right) - L(q)\right),$$

$$\begin{aligned}\frac{c^4(q^2)}{9c(q)c(q^4)} &= \frac{3}{4}(2L(q^2) - L(q)) + \frac{1}{8}(3L(q^3) - L(q)) - \frac{3}{8}(4L(q^4) - L(q)) \\ &\quad - \frac{1}{4}(6L(q^6) - L(q)) + \frac{1}{8}(12L(q^{12}) - L(q)),\end{aligned}$$

$$\begin{aligned}\frac{c(q)c^3(q^4)}{9c^2(q^2)} &= -\frac{1}{8}(3L(q^3) - L(q)) - \frac{3}{32}(4L(q^4) - L(q)) \\ &\quad + \frac{5}{16}(6L(q^6) - L(q)) - \frac{3}{32}(12L(q^{12}) - L(q)),\end{aligned}$$

$$\frac{c^3(q)c(q^4)}{9c^2(q^2)} = -\frac{3}{8}\left(3L\left(q^3\right) - L(q)\right) + \frac{5}{4}\left(6L\left(q^6\right) - L(q)\right) - \frac{1}{2}\left(12L\left(q^{12}\right) - L(q)\right),$$

$$\frac{c^2(q)c^2(q^2)}{81} = \frac{3}{32}M(q) - \frac{9}{40}M\left(q^3\right) + \frac{99}{40}M\left(q^6\right) - \frac{3}{32}\left(6L\left(q^6\right) - L(q)\right)^2,$$

$$\frac{c^8(q^2)}{81c^4(q)} = \frac{27}{160}M\left(q^2\right) - \frac{3}{64}M(q) + \frac{9}{20}M\left(q^3\right) - \frac{279}{160}M\left(q^6\right) + \frac{3}{64}\left(6L\left(q^6\right) - L(q)\right)^2,$$

$$\frac{c^8(q)}{81c^4(q^2)} = -\frac{3}{10}M(q) - \frac{9}{10}M\left(q^3\right) + \frac{36}{5}M\left(q^6\right) + 3\left(6L\left(q^6\right) - L(q)\right)^2,$$

$$\begin{aligned}\frac{c^4(q)c^2(q^2)}{81c^2(q^4)} = {} & \frac{3}{40}M(q) - \frac{9}{20}M(q^3) + \frac{36}{5}M(q^6) - \frac{36}{5}M(q^{12}) \\ & -\frac{3}{8}(6L(q^6) - L(q))^2 + \frac{3}{4}(12L(q^{12}) - L(q))^2.\end{aligned} \tag{3.27}$$

From the above identities and (1.2), (1.3), (1.7) and (1.8), we may derive other formulas. □

4. Convolution Sums $\sum_{i+6j=n}\sigma(i)\sigma(j)$ and $\sum_{i+12j=n}\sigma(i)\sigma(j)$

Let $n, k \in \mathbb{N}$. The divisor function $\sigma_k(n)$ is defined by

$$\sigma_k(n) = \sum_{d|n} d^k, \tag{4.1}$$

where d runs through the positive divisors of n. If n is not a positive integer, set $\sigma_i(n) = 0$. As usual, we write $\sigma(n)$ for $\sigma_1(n)$. For all n, the convolution $\sum_{i+kj=n}\sigma(i)\sigma(j)$ has been evaluated explicitly for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 16, 18$, and 24, see [10–19]. In this section, we also derive the representations for the convolution sums $\sum_{i+6j=n}\sigma(i)\sigma(j)$ and $\sum_{i+12j=n}\sigma(i)\sigma(j)$ from the identities in Theorem 3.1. Our representations are different from those derived in [11, 12]. In fact, we can derive many formulas for the convolution sums $\sum_{i+6j=n}\sigma(i)\sigma(j)$ and $\sum_{i+12j=n}\sigma(i)\sigma(j)$, and here we just list two of them.

Theorem 4.1. *Let n be a positive integer. One has*

$$\sum_{i+6j=n}\sigma(i)\sigma(j) = \frac{1}{108}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{2}\right) + \frac{1-n}{24}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{6}\right) + \frac{A(n)}{648}, \tag{4.2}$$

$$\begin{aligned}\sum_{i+12j=n}\sigma(i)\sigma(j) = {} & \frac{11}{864}\sigma_3(n) + \frac{1}{108}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{96}\sigma_3\left(\frac{n}{3}\right) - \frac{7}{48}\sigma_3\left(\frac{n}{6}\right) \\ & + \frac{7}{3}\sigma_3\left(\frac{n}{12}\right) + \frac{2-n}{48}\sigma(n) - \frac{6n-1}{24}\sigma\left(\frac{n}{12}\right) + \frac{A(n)}{2592} - \frac{B(n)}{128},\end{aligned} \tag{4.3}$$

where

$$1+\sum_{n=1}^{\infty}A(n)q^n=\frac{(q;q)_\infty^6(q^2;q^2)_\infty^6}{(q^3;q^3)_\infty^2(q^6;q^6)_\infty^2}, \tag{4.4}$$

$$1+\sum_{n=1}^{\infty}B(n)q^n=\frac{(q^3;q^3)_\infty^{12}(q^6;q^6)_\infty^6(q^4;q^4)_\infty^2}{(q;q)_\infty^4(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^6}. \tag{4.5}$$

Remark 4.2. Alaca and Williams [11] derived the representation for $\sum_{i+6j=n}\sigma(i)\sigma(j)$

$$\begin{aligned}\sum_{i+6j=n}\sigma(i)\sigma(j)&=\frac{\sigma_3(n)}{120}+\frac{\sigma_3(n/2)}{30}+\frac{3\sigma_3(n/3)}{40}+\frac{3\sigma_3(n/6)}{10}\\&\quad+\frac{1-n}{24}\sigma(n)+\frac{1-6n}{24}\sigma\left(\frac{n}{6}\right)-\frac{c_6(n)}{120},\end{aligned} \tag{4.6}$$

where

$$\sum_{n=1}^{\infty}c_6(n)q^n=q(q;q)_\infty^2\left(q^2;q^2\right)_\infty^2\left(q^3;q^3\right)_\infty^2\left(q^6;q^6\right)_\infty^2. \tag{4.7}$$

Remark. Alaca et al. [12] also derived the representation for $\sum_{i+12j=n}\sigma(i)\sigma(j)$

$$\begin{aligned}\sum_{i+12j=n}\sigma(i)\sigma(j)&=\frac{\sigma_3(n)}{480}+\frac{\sigma_3(n/2)}{160}+\frac{3\sigma_3(n/3)}{160}+\frac{\sigma_3(n/4)}{30}+\frac{9\sigma_3(n/6)}{160}\\&\quad+\frac{3\sigma_3(n/12)}{10}+\frac{2-n}{48}\sigma(n)+\frac{1-6n}{24}\sigma\left(\frac{n}{12}\right)-\frac{11}{480}c_{1,12}(n),\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}11\sum_{n=1}^{\infty}c_{1,12}(n)q^n&=10q\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^3(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q;q)_\infty(q^{12};q^{12})_\infty}\\&\quad+q\frac{(q^2;q^2)_\infty^8(q^6;q^6)_\infty^8}{(q;q)_\infty^2(q^3;q^3)_\infty^2(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}.\end{aligned} \tag{4.9}$$

Proof. From (3.9), we have

$$\frac{(q;q)_\infty^6(q^2;q^2)_\infty^6}{(q^3;q^3)_\infty^2(q^6;q^6)_\infty^2} = 1+\frac{33}{2}\sum_{n=1}^\infty\sigma_3(n)q^n-24\sum_{n=1}^\infty\sigma_3(n)q^{2n}+810\sum_{n=1}^\infty\sigma_3(n)q^{6n}$$
$$+135\sum_{n=1}^\infty\sigma(n)q^{6n}-\frac{45}{2}\sum_{n=1}^\infty\sigma(n)q^n-1944\left(\sum_{n=1}^\infty\sigma(n)q^{6n}\right)^2 \tag{4.10}$$
$$+648\left(\sum_{n=1}^\infty\sigma(n)q^{6n}\right)\left(\sum_{n=1}^\infty\sigma(n)q^n\right)-54\left(\sum_{n=1}^\infty\sigma(n)q^n\right)^2.$$

For $n\in\mathbb{N}$, equating the coefficients of q^n on both sides of (4.10), we obtain

$$A(n)=\frac{33}{2}\sigma_3(n)-24\sigma_3\left(\frac{n}{2}\right)+810\sigma_3\left(\frac{n}{6}\right)+135\sigma\left(\frac{n}{6}\right)-\frac{45}{2}\sigma(n)$$
$$-1944\sum_{i+j=n/6}\sigma(i)\sigma(j)+648\sum_{i+6j=n}\sigma(i)\sigma(j)-54\sum_{i+j=n}\sigma(i)\sigma(j), \tag{4.11}$$

where $A(n)$ is defined by (4.4). From (4.11), utilizing the convolution sum

$$\sum_{i+j=n}\sigma(i)\sigma(j)=\frac{5}{12}\sigma_3(n)-\frac{n}{2}\sigma(n)+\frac{\sigma(n)}{12}, \tag{4.12}$$

we can derive the formula (4.2).

From (3.23), we have

$$\frac{(q^3;q^3)_\infty^{12}(q^6;q^6)_\infty^6(q^4;q^4)_\infty^2}{(q;q)_\infty^4(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^6}$$
$$=1+\frac{2}{9}\sum_{n=1}^\infty\frac{n^3q^n}{1-q^n}-\frac{4}{3}\sum_{n=1}^\infty\frac{n^3q^{3n}}{1-q^{3n}}+\frac{64}{3}\sum_{n=1}^\infty\frac{n^3q^{6n}}{1-q^{6n}}$$
$$-\frac{64}{3}\sum_{n=1}^\infty\frac{n^3q^{12n}}{1-q^{12n}}+\frac{20}{3}\sum_{n=1}^\infty\frac{nq^{6n}}{1-q^{6n}}+\frac{34}{9}\sum_{n=1}^\infty\frac{nq^n}{1-q^n}-96\left(\sum_{n=1}^\infty\frac{nq^{6n}}{1-q^{6n}}\right)^2 \tag{4.13}$$
$$+32\left(\sum_{n=1}^\infty\frac{nq^{6n}}{1-q^{6n}}\right)\left(\sum_{n=1}^\infty\frac{nq^n}{1-q^n}\right)+\frac{8}{3}\left(\sum_{n=1}^\infty\frac{nq^{6n}}{1-q^{6n}}\right)^2-\frac{176}{3}\sum_{n=1}^\infty\frac{nq^{12n}}{1-q^{12n}}$$
$$+768\left(\sum_{n=1}^\infty\frac{nq^{12n}}{1-q^{12n}}\right)^2-128\left(\sum_{n=1}^\infty\frac{nq^{12n}}{1-q^{12n}}\right)\left(\sum_{n=1}^\infty\frac{nq^n}{1-q^n}\right).$$

For $n \in \mathbb{N}$, equating the coefficients of q^n on both sides of (4.13), we obtain

$$\begin{aligned} B(n) = {} & \frac{2}{9}\sigma_3(n) - \frac{4}{3}\sigma_3\left(\frac{n}{3}\right) + \frac{64}{3}\sigma_3\left(\frac{n}{6}\right) - \frac{64}{3}\sigma_3\left(\frac{n}{12}\right) + \frac{20}{3}\sigma\left(\frac{n}{6}\right) + \frac{34}{9}\sigma(n) \\ & - \frac{176}{3}\sigma\left(\frac{n}{12}\right) - 96 \sum_{i+j=n/6} \sigma(i)\sigma(j) + 32 \sum_{i+6j=n} \sigma(i)\sigma(j) \\ & + \frac{8}{3}\sum_{i+j=n} \sigma(i)\sigma(j) + 768 \sum_{i+j=n/12} \sigma(i)\sigma(j) - 128 \sum_{i+12j=n} \sigma(i)\sigma(j), \end{aligned} \tag{4.14}$$

where $B(n)$ is defined by (4.5). From (4.2), (4.12), and (4.14), we derive the formula (4.3). □

The advantage of the formulas of Alaca et al. is that the values of $c_6(n)$ and $c_{1,12}(n)$ are often very small. Numerical evidence suggests that $|c_6(n)/120| < |A(n)/648|$ and $|11c_{1,12}(n)/480| < |A(n)/2592 - B(n)/128|$. The advantage of our formulas is that the signs of $A(n)$ and $A(n)/2592 - B(n)/128$ appear to have periodicity. Numerical evidence suggests that for $n \geq 3$, $A(n) > 0$ when $3 \mid n$ and $A(n) < 0$ when, $3 \nmid n$ and $(A(n)/2592) - (B(n)/128) < 0$. Therefore, we conjecture that, for $n \geq 3$, we have

$$\begin{aligned} \sum_{i+6j=n} \sigma(i)\sigma(j) &> \frac{1}{108}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{2}\right) + \frac{1-n}{24}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{6}\right), \quad 3 \mid n, \\ \sum_{i+6j=n} \sigma(i)\sigma(j) &< \frac{1}{108}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{2}\right) + \frac{1-n}{24}\sigma(n), \quad 3 \nmid n. \end{aligned} \tag{4.15}$$

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