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Research Article

He-Laplace Method for Linear and Nonlinear Partial Differential Equations

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A new treatment for homotopy perturbation method is introduced. The new treatment is called He-Laplace method which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. The nonlinear terms can be easily handled by the use of He's polynomials. The method is implemented on linear and nonlinear partial differential equations. It is found that the proposed scheme provides the solution without any discretization or restrictive assumptions and avoids the round-off errors.

1. Introduction

Many important phenomena occurring in various field of engineering and science are frequently modeled through linear and nonlinear differential equations. However, it is still very difficult to obtain closed-form solutions for most models of real-life problems. A broad class of analytical methods and numerical methods were used to handle such problems. In recent years, various methods have been proposed such as finite difference method [1, 2] adomian decomposition method [3–8], variational iteration method [9–12], integral transform [13], weighted finite difference techniques [14], Laplace decomposition method [15–17], but all these methods have some limitations.

The homotopy perturbation method was first introduced by Chinese mathematician He [18–25]. The essential idea of this method is to introduce a homotopy parameter p, say which takes the values from 0 to 1. When $p = 0$, the system of equations usually reduces to a simplified form which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of deformation and the solution of each of which is close to that at the previous stage of deformation. Eventually at $p = 1$, the system takes the original form of equation, and final stage of deformation gives the desired result. One of the most remarkable features of the HPM is that only a few perturbation terms are sufficient to obtain a reasonably accurate solution.

The HPM has been employed to solve a large variety of linear and nonlinear problems. This technique was used by He [23, 25] to find the solution of nonlinear boundary value problems, the Blasius differential equation. Sharma and Methi [26] apply HPM for solution of equation to unsteady flow of a polytropic gas. Ganji and Rafei [27] implemented HPM for solution of nonlinear Hirota-Satsuma coupled KdV partial differential equations. Biazer and Ghazvini [13] presented solution of systems of Volterra integral equations. Abbasbandy [28] employed He's homotopy perturbation technique to solve functional integral equations, and obtained results were compared with the Lagrange interpolation formula. Ganji and Sadighi [29] considered the nonlinear coupled system of reaction-diffusion equations using HPM. They reported that the HPM is a powerful and efficient scheme to find analytical solutions for a wide class of nonlinear engineering problems and presents a rapid convergence for the solutions. The solution obtained by HPM shows that the results are in excellent agreement with those obtained by Adomian decomposition method. A comparison between the HPM and the Adomian decomposition shows that the former is more effective than the latter as the HPM can overcome the difficulties arising in calculating Adomian polynomials.

In the present paper, we use the homotopy perturbation method coupled with the Laplace transformation for solving the linear and nonlinear PDEs. It is worth mentioning that the proposed method is an elegant combination of the Laplace transformation, the homotopy perturbation method, and He's polynomials. The use of He's polynomials in the nonlinear term was first introduced by Ghorbani and Saberi-Nadjafi [30], Ghorbani [31]. The proposed algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for linear and nonlinear partial differential equations.

2. Basic Idea of Homotopy Perturbation Method

Consider the following nonlinear differential equation

$$A(y) - f(r) = 0, \quad r \in \Omega, \tag{2.1}$$

with the boundary conditions of

$$B\left(y, \frac{\partial y}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{2.2}$$

where A, B, $f(r)$, and Γ are a general differential operator, a boundary operator, a known analytic function and the boundary of the domain Ω, respectively.

The operator A can generally be divided into a linear part L and a nonlinear part N. Equation (2.1) may therefore be written as.

$$L(y) + N(y) - f(r) = 0, \tag{2.3}$$

By the homotopy technique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \rightarrow R$ which satisfies

$$H(v,p) = (1-p)\left[L(v) - L(y_0)\right] + p\left[A(v) - f(r)\right] = 0 \tag{2.4}$$

or

$$H(v,p) = L(v) - L(y_0) + pL(y_0) + p\left[N(v) - f(r)\right] = 0, \tag{2.5}$$

where $p \in [0,1]$ is an embedding parameter, while y_0 is an initial approximation of (2.1), which satisfies the boundary conditions. Obviously, from (2.4) and (2.5) we will have

$$\begin{aligned} H(v,0) &= L(v) - L(y_0) = 0, \\ H(v,1) &= A(v) - f(r) = 0. \end{aligned} \tag{2.6}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from y_0 to $y(r)$. In topology, this is called deformation, while $L(v) - L(y_0)$ and $A(v) - f(r)$ are called homotopy. If the embedding parameter p is considered as a small parameter, applying the classical perturbation technique, we can assume that the solution of (2.4) and (2.5) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots \infty. \tag{2.7}$$

Setting $p = 1$ in (2.7), we have

$$y = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{2.8}$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages. The series (2.8) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$. Moreover, He [21] made the following suggestions.

(1) The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large; that is, $p \rightarrow 1$.

(2) The norm of $L^{-1}(\partial N/\partial v)$ must be smaller than one so that the series converges.

3. He-Laplace Method

To illustrate the basic idea of this method, we consider a general nonlinear nonhomogeneous partial differential equation with initial conditions of the form

$$\begin{gathered} \frac{\partial^2 y}{\partial t^2} + R_1 y(x,t) + R_2 y(x,t) + Ny(x,t) = f(x,t), \\ y(x,0) = \alpha(x), \qquad \frac{\partial y}{\partial t}(x,0) = \beta(x), \end{gathered} \tag{3.1}$$

where $R_1 = \partial^2/\partial x^2$ and $R_2 = \partial/\partial x$ are the linear differential operators, N represents the general nonlinear differential operator and $f(x,t)$ is the source term. Taking the Laplace transform (denoted by L) on both sides of (3.1)

$$L\left[\frac{\partial^2 y}{\partial t^2}\right] + L\left[R_1 y(x,t) + R_2 y(x,t)\right] + L\left[Ny(x,t)\right] = L\left[f(x,t)\right]$$
$$\Longrightarrow s^2 L\left[y(x,t)\right] - sy(x,0) - \frac{\partial y}{\partial t}(x,0)$$
$$= -L\left[R_1 y(x,t) + R_2 y(x,t)\right] - L\left[Ny(x,t)\right] + L\left[f(x,t)\right]. \tag{3.2}$$

Applying the initial conditions given in (3.1), we have

$$L\left[y(x,t)\right] = \frac{\alpha(x)}{s} + \frac{\beta(x)}{s^2} - \frac{1}{s^2}\left(L\left[R_1 y(x,t) + R_2 y(x,t)\right] + L\left[Ny(x,t)\right]\right) + \frac{1}{s^2}\left(L\left[f(x,t)\right]\right). \tag{3.3}$$

Operating the inverse Laplace transform on both sides of (3.3), we have

$$y(x,t) = F(x,t) - L^{-1}\left[\frac{1}{s^2}\left(L\left[R_1 y(x,t) + R_2 y(x,t)\right] + L\left[Ny(x,t)\right]\right)\right], \tag{3.4}$$

where $F(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method:

$$y(x,t) = \sum_{n=0}^{\infty} p^n y_n(x,t), \tag{3.5}$$

and the nonlinear term can be decomposed as

$$Ny(x,t) = \sum_{n=0}^{\infty} p^n H_n(y). \tag{3.6}$$

For some He's polynomials H_n (see [31, 32]) with the coupling of the Laplace transform and the homotopy perturbation method are given by

$$\sum_{n=0}^{\infty} p^n y_n(x,t) = F(x,t) - p\left(L^{-1}\left[\frac{1}{s^2} L\left[(R_1 + R_2)\sum_{n=0}^{\infty} p^n y_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(y)\right]\right]\right). \tag{3.7}$$

Comparing the coefficients of like powers of p, we have the following approximations:

$$\begin{aligned}
p^0 &: \ y_0(x,t) = F(x,t),\\
p^1 &: \ y_1(x,t) = -L^{-1}\left(\frac{1}{s^2}L\left[(R_1+R_2)y_0(x,t)+H_0(y)\right]\right),\\
p^2 &: \ y_2(x,t) = -L^{-1}\left(\frac{1}{s^2}L\left[(R_1+R_2)y_1(x,t)+H_1(y)\right]\right),\\
p^3 &: \ y_3(x,t) = -L^{-1}\left(\frac{1}{s^2}L\left[(R_1+R_2)y_2(x,t)+H_2(y)\right]\right),\\
&\vdots
\end{aligned} \tag{3.8}$$

4. Application

To demonstrate the applicability of the above-presented method, we have applied it to two linear and three nonlinear partial differential equations. These examples have been chosen because they have been widely discussed in literature.

Example 4.1. Consider the following homogeneous linear PDE [33]:

$$\frac{\partial y}{\partial t}+\frac{\partial y}{\partial x}-\frac{\partial^2 y}{\partial x^2}=0 \tag{4.1}$$

with the following conditions:

$$y(x,0)=e^x-x, \qquad y(0,t)=1+t, \qquad \frac{\partial y}{\partial x}(1,t)=e-1. \tag{4.2}$$

By applying the aforesaid method subject to the initial condition, we have

$$y(x,s)=\frac{e^x-x}{s}-\frac{1}{s}L\left[\frac{\partial y}{\partial x}-\frac{\partial^2 y}{\partial x^2}\right] \tag{4.3}$$

The inverse of the Laplace transform implies that

$$y(x,t)=e^x-x-L^{-1}\left[\frac{1}{s}L\left[\frac{\partial y}{\partial x}-\frac{\partial^2 y}{\partial x^2}\right]\right] \tag{4.4}$$

Now, we apply the homotopy perturbation method; we have

$$\sum_{n=0}^{\infty}p^n y_n(x,t)=e^x-x-p\left(L^{-1}\left[\frac{1}{s}L\left[\frac{\partial y}{\partial x}-\frac{\partial^2 y}{\partial x^2}\right]\right]\right), \tag{4.5}$$

Comparing the coefficient of like powers of p, we have

$$\begin{aligned} p^0 &: \ y_0(x,t) = e^x - x, \\ p^1 &: \ y_1(x,t) = -L^{-1}\left[\frac{1}{s}L\left[\frac{\partial y_0}{\partial x} - \frac{\partial^2 y_0}{\partial x^2}\right]\right] = t, \\ p^2 &: \ y_2(x,t) = -L^{-1}\left[\frac{1}{s}L\left[\frac{\partial y_1}{\partial x} - \frac{\partial^2 y}{\partial x^2}\right]\right] = 0. \end{aligned} \tag{4.6}$$

Proceeding in a similar manner, we have

$$\begin{aligned} p^3 &: \ y_3(x,t) = 0, \\ p^4 &: \ y_4(x,t) = 0, \\ &\vdots \end{aligned} \tag{4.7}$$

so the solution $y(x,t)$ is given by

$$\begin{aligned} y(x,t) &= e^x - x + t + 0 + 0 \cdots \\ &= e^x - x + t, \end{aligned} \tag{4.8}$$

which is the exact solution of the problem.

Example 4.2. Consider the following homogeneous linear PDE (Klein-Gordon equation) [33]:

$$\frac{\partial^2 y}{\partial t^2} + y - \frac{\partial^2 y}{\partial x^2} = 0, \tag{4.9}$$

with the following conditions:

$$y(x,0) = e^{-x} + x, \qquad \frac{\partial y}{\partial t}(x,0) = 0. \tag{4.10}$$

By applying the aforesaid method subject to the initial condition, we have

$$y(x,s) = \frac{e^{-x} + x}{s} - \frac{1}{s^2}L\left[y - \frac{\partial^2 y}{\partial x^2}\right] \tag{4.11}$$

The inverse of the Laplace transform implies that

$$y(x,t) = e^{-x} + x - L^{-1}\left[\frac{1}{s^2}L\left[y - \frac{\partial^2 y}{\partial x^2}\right]\right]. \tag{4.12}$$

Now, we apply the homotopy perturbation method; we have

$$\sum_{n=0}^{\infty} p^n y_n(x,t) = e^{-x} + x - p\left(L^{-1}\left[\frac{1}{s^2} L\left[y - \frac{\partial^2 y}{\partial x^2}\right]\right]\right). \tag{4.13}$$

Comparing the coefficient of like powers of p, we have

$$\begin{aligned}
p^0 &: \; y_0(x,t) = e^{-x} + x, \\
p^1 &: \; y_1(x,t) = -L^{-1}\left[\frac{1}{s^2} L\left[y_0 - \frac{\partial^2 y_0}{\partial x^2}\right]\right] = \frac{-xt^2}{2!}, \\
p^2 &: \; y_2(x,t) = -L^{-1}\left[\frac{1}{s^2} L\left[y_1 - \frac{\partial^2 y_1}{\partial x^2}\right]\right] = \frac{xt^4}{4!}.
\end{aligned} \tag{4.14}$$

Proceeding in a similar manner, we have

$$\begin{aligned}
p^3 &: \; y_3(x,t) = \frac{-xt^6}{6!}, \\
p^4 &: \; y_4(x,t) = \frac{xt^8}{8!}, \\
p^n &: \; y_n(x,t) = \frac{(-1)^n xt^{2n}}{2n!},
\end{aligned} \tag{4.15}$$

so that the solution $y(x,t)$ is given by

$$\begin{aligned}
y(x,t) &= y_0 + y_1 + y_2 + y_3 + \cdots \\
&= e^{-x} + x - \frac{xt^2}{2!} + \frac{xt^4}{4!} - \frac{xt^6}{6!} + \cdots + \frac{(-1)^n xt^{2n}}{2n!} \\
&= e^{-x} + x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots + \frac{(-1)^n t^{2n}}{2n!} + \cdots\right) \\
&= e^{-x} + x\,\mathrm{Cos}(t),
\end{aligned} \tag{4.16}$$

which is the exact solution of the problem.

Example 4.3. Consider the following homogeneous nonlinear PDE (Burger equation) [33]:

$$\frac{\partial y}{\partial t} - y\frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2} = 0, \tag{4.17}$$

with the following conditions:

$$y(x,0) = 1 - x, \qquad y(0,t) = \frac{1}{(1+t)}, \qquad y(1,t) = 0. \tag{4.18}$$

By applying the aforesaid method subject to the initial condition, we have

$$y(x,s) = \frac{1-x}{s} + \frac{1}{s}L\left[\frac{\partial^2 y}{\partial x^2} + y\frac{\partial y}{\partial x}\right]. \tag{4.19}$$

The inverse of the Laplace transform implies that

$$y(x,t) = 1 - x + L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 y}{\partial x^2} + y\frac{\partial y}{\partial x}\right]\right]. \tag{4.20}$$

Now, we apply the homotopy perturbation method; we have

$$\sum_{n=0}^{\infty} p^n y_n(x,t) = 1 - x + p\left(L^{-1}\left[\frac{1}{s}\left\{L\left[\frac{\partial^2 y}{\partial x^2}\right] + L\left[\sum_{n=0}^{\infty} p^n H_n(y)\right]\right\}\right]\right), \tag{4.21}$$

where $H_n(y)$ are He's polynomials. The first few components of He's polynomials are given by

$$\begin{aligned}
H_0(y) &= y_0\frac{\partial y_0}{\partial x} = -(1-x), \\
H_1(y) &= y_0\frac{\partial y_1}{\partial x} + y_1\frac{\partial y_0}{\partial x} = 2(1-x)t, \\
H_2(y) &= y_0\frac{\partial y_2}{\partial x} + y_1\frac{\partial y_1}{\partial x} + y_2\frac{\partial y_0}{\partial x} = -3(1-x)t^2. \\
&\vdots
\end{aligned} \tag{4.22}$$

Comparing the coefficient of like powers of p, we have

$$\begin{aligned}
p^0 &: \; y_0(x,t) = 1 - x, \\
p^1 &: \; y_1(x,t) = L^{-1}\left[\frac{1}{s}\left\{L\left[\frac{\partial^2 y_0}{\partial x^2}\right] + L[H_0(y)]\right\}\right] = -(1-x)t, \\
p^2 &: \; y_2(x,t) = L^{-1}\left[\frac{1}{s}\left\{L\left[\frac{\partial^2 y_1}{\partial x^2}\right] + L[H_1(y)]\right\}\right] = (1-x)t^2,
\end{aligned} \tag{4.23}$$

Proceeding in a similar manner, we have

$$\begin{aligned} p^3:\ & y_3(x,t) = -(1-x)t^3, \\ p^4:\ & y_4(x,t) = (1-x)t^4, \\ & \vdots \end{aligned} \tag{4.24}$$

so that the solution $y(x,t)$ is given by

$$\begin{aligned} y(x,t) &= y_0 + y_1 + y_2 + y_3 + \cdots \\ &= (1-x) - (1-x)t + (1-x)t^2 - (1-x)t^3 + \cdots \\ &= (1-x)\left[1 - t + t^2 - t^3 + t^4 - \cdots\right] \\ &= (1-x)(1+t)^{-1} = \frac{(1-x)}{(1+t)}, \end{aligned} \tag{4.25}$$

which is the exact solution of the problem.

Example 4.4. Consider the following homogeneous nonlinear PDE [33]:

$$\frac{\partial y}{\partial t} - y - y\frac{\partial^2 y}{\partial x^2} - \left(\frac{\partial y}{\partial x}\right)^2 = 0, \tag{4.26}$$

with the following conditions:

$$y(x,0) = \sqrt{x}, \qquad y(0,t) = 0, \qquad y(1,t) = e^t. \tag{4.27}$$

By applying the aforesaid method subject to the initial condition, we have

$$y(x,s) = \frac{\sqrt{x}}{s} + \frac{1}{s}L\left[y + y\frac{\partial^2 y}{\partial x^2} + \left(\frac{\partial y}{\partial x}\right)^2\right]. \tag{4.28}$$

The inverse of the Laplace transform implies that

$$y(x,t) = \sqrt{x} + L^{-1}\left[\frac{1}{s}L\left[y + y\frac{\partial^2 y}{\partial x^2} + \left(\frac{\partial y}{\partial x}\right)^2\right]\right]. \tag{4.29}$$

Now, we apply the homotopy perturbation method; we have

$$\sum_{n=0}^{\infty} p^n y_n(x,t) = \sqrt{x} + p\left(L^{-1}\left[\frac{1}{s}\left\{L[y] + L\left[\sum_{n=0}^{\infty} p^n H_n(y)\right]\right\}\right]\right), \tag{4.30}$$

where $H_n(y)$ are He's polynomials. The first few components of He's polynomials are given by

$$\begin{aligned}
H_0(y) &= y_0\frac{\partial^2 y_0}{\partial x^2} + \left(\frac{\partial y_0}{\partial x}\right)^2 = 0,\\
H_1(y) &= y_0\frac{\partial^2 y_1}{\partial x^2} + y_1\frac{\partial^2 y_0}{\partial x^2} + 2\frac{\partial y_0}{\partial x}\frac{\partial y_1}{\partial x} = 0,\\
H_2(y) &= y_0\frac{\partial^2 y_2}{\partial x^2} + y_1\frac{\partial^2 y_1}{\partial x^2} + y_2\frac{\partial^2 y_0}{\partial x^2} + \left(\frac{\partial y_1}{\partial x}\right)^2 + 2\frac{\partial y_0}{\partial x}\frac{\partial y_2}{\partial x} = 0,\\
&\vdots
\end{aligned} \tag{4.31}$$

Comparing the coefficient of like powers of p, we have

$$\begin{aligned}
p^0 &: \; y_0(x,t) = \sqrt{x},\\
p^1 &: \; y_1(x,t) = L^{-1}\left[\frac{1}{s}\{L[y_0] + L[H_0(y)]\}\right] = \sqrt{x}\,t,\\
p^2 &: \; y_2(x,t) = L^{-1}\left[\frac{1}{s}\{L[y_1] + L[H_1(y)]\}\right] = \frac{\sqrt{x}t^2}{2!}.
\end{aligned} \tag{4.32}$$

Proceeding in a similar manner, we have

$$\begin{aligned}
p^3 &: \; y_3(x,t) = \frac{\sqrt{x}t^3}{3!},\\
p^4 &: \; y_4(x,t) = \frac{\sqrt{x}t^4}{4!},\\
&\vdots
\end{aligned} \tag{4.33}$$

so the solution $y(x,t)$ is given by

$$\begin{aligned}
y(x,t) &= y_0 + y_1 + y_2 + y_3 + \cdots\\
&= \sqrt{x} + \frac{\sqrt{x}t}{1!} + \frac{\sqrt{x}t^2}{2!} + \frac{\sqrt{x}t^3}{3!} + \cdots\\
&= \sqrt{x}\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!} + \cdots\right)\\
&= \sqrt{x}e^t,
\end{aligned} \tag{4.34}$$

which is the exact solution of the problem.

5. Comparison of Rate of Convergence of HPM and He-Laplace Method

Example 5.1. Consider the following nonhomogeneous nonlinear PDE:

$$\frac{\partial^2 y}{\partial t^2}+\frac{\partial^2 y}{\partial x^2}+\left(\frac{\partial y}{\partial x}\right)^2=2x+t^4, \tag{5.1}$$

with the following conditions:

$$y(x,0)=0, \qquad \frac{\partial y}{\partial t}(x,0)=a, \qquad y(0,t)=at, \qquad \frac{\partial y}{\partial x}(0,t)=t^2. \tag{5.2}$$

According to the homotopy perturbation method, we have

$$\begin{aligned} H(v,p)=&\ \frac{\partial^2}{\partial t^2}\left(v_0+pv_1+p^2v_2+p^3v_3+\cdots\right)-(1-p)\frac{\partial^2 y_0}{\partial t^2}\\ &+p\left[\frac{\partial^2}{\partial x^2}\left(v_0+pv_1+p^2v_2+p^3v_3+\cdots\right)+\left(\frac{\partial}{\partial x}\left(v_0+pv_1+p^2v_2+p^3v_3+\cdots\right)\right)^2-2x-t^4\right]=0. \end{aligned} \tag{5.3}$$

The initial approximation is chosen $y_0=at$. By equating the coefficients of p to zero, we obtain

$$\begin{aligned} &\text{Coefficient of } p^0: \frac{\partial^2 v_0}{\partial t^2}-\frac{\partial^2 y_0}{\partial t^2}=0, \Longrightarrow v_0=y_0=at,\\ &\text{Coefficient of } p^1: \frac{\partial^2 v_1}{\partial t^2}+\frac{\partial^2 u_0}{\partial t^2}+\frac{\partial^2 v_0}{\partial x^2}+\left(\frac{\partial v_0}{\partial x}\right)^2-2x-t^4=0, \Longrightarrow v_1=xt^2+\frac{1}{30}t^6,\\ &\text{Coefficient of } p^2: \frac{\partial^2 v_2}{\partial t^2}+\frac{\partial^2 v_1}{\partial x^2}+\left(\frac{\partial}{\partial x}2v_0v_1\right)^2=0, \Longrightarrow v_2=0; \end{aligned} \tag{5.4}$$

similarly

$$\begin{aligned} v_3&=-\frac{1}{30}t^6,\\ v_4&=0,\\ v_5&=0,\\ &\vdots\\ v_n&=0. \end{aligned} \tag{5.5}$$

Therefore, we obtain

$$\begin{aligned} y(x,t) &= v_0 + v_1 + v_2 + v_3 + \cdots \\ &= at + xt^2. \end{aligned} \tag{5.6}$$

Note. Now we solve the same problem using the He-Laplace method.

Example 5.2. Consider the following non-homogeneous nonlinear PDE:

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x^2} + \left(\frac{\partial y}{\partial x}\right)^2 = 2x + t^4, \tag{5.7}$$

with the following conditions:

$$y(x,0) = 0, \qquad \frac{\partial y}{\partial t}(x,0) = a, \qquad y(0,t) = at, \qquad \frac{\partial y}{\partial x}(0,t) = t^2. \tag{5.8}$$

By applying the HE-Laplace method subject to the initial condition, we have

$$\begin{aligned} y(x,s) &= \frac{a}{s^2} - \frac{1}{s^2} L\left[\frac{\partial^2 y}{\partial x^2} + \left(\frac{\partial y}{\partial x}\right)^2\right] + \frac{1}{s^2} L\left[2x + t^4\right] \\ &= \frac{a}{s^2} + \frac{2x}{s^3} + \frac{4!}{s^7} - \frac{1}{s^2} L\left[\frac{\partial^2 y}{\partial x^2} + \left(\frac{\partial y}{\partial x}\right)^2\right]. \end{aligned} \tag{5.9}$$

The inverse of the Laplace transform implies that

$$y(x,t) = at + xt^2 + \frac{t^6}{30} - L^{-1}\left[\frac{1}{s^2} L\left[\frac{\partial^2 y}{\partial x^2} + \left(\frac{\partial y}{\partial x}\right)^2\right]\right]. \tag{5.10}$$

Now, we apply the homotopy perturbation method, we have

$$\sum_{n=0}^{\infty} p^n y_n(x,t) = at + xt^2 + \frac{t^6}{30} - p\left(L^{-1}\left[\frac{1}{s^2}\left\{L\left[\frac{\partial^2 y}{\partial x^2}\right] + L\left[\sum_{n=0}^{\infty} p^n H_n(y)\right]\right\}\right]\right), \tag{5.11}$$

where $H_n(y)$ are He's polynomials. The first few components of He's polynomials are given by

$$
\begin{aligned}
H_0(y) &= \left(\frac{\partial y_0}{\partial x}\right)^2 = t^4, \\
H_1(y) &= 2\left(\frac{\partial y_0}{\partial x}\right) \times \left(\frac{\partial y_1}{\partial x}\right) = 0, \\
H_2(y) &= \left(\frac{\partial y_1}{\partial x}\right)^2 + 2\frac{\partial y_0}{\partial x}\frac{\partial y_2}{\partial x} = 0, \\
&\vdots
\end{aligned}
\tag{5.12}
$$

Comparing the coefficient of like powers of p, we have

$$
\begin{aligned}
p^0 &: \ y_0(x,t) = at + xt^2 + \frac{t^6}{30}, \\
p^1 &: \ y_1(x,t) = -L^{-1}\left[\frac{1}{s^2}\left\{L\left[\frac{\partial^2 y_0}{\partial x^2}\right] + L[H_0(y)]\right\}\right] = -\frac{t^6}{30}, \\
p^2 &: \ y_2(x,t) = -L^{-1}\left[\frac{1}{s^2}\left\{L\left[\frac{\partial^2 y_1}{\partial x^2}\right] + L[H_1(y)]\right\}\right] = 0.
\end{aligned}
\tag{5.13}
$$

Proceeding in a similar manner, we have

$$
\begin{aligned}
p^3 &: \ y_3(x,t) = 0, \\
p^4 &: \ y_4(x,t) = 0, \\
&\vdots
\end{aligned}
\tag{5.14}
$$

so that the solution $y(x,t)$ is given by

$$
\begin{aligned}
y(x,t) &= y_0 + y_1 + y_2 + y_3 + \cdots \\
&= at + xt^2 + \frac{t^6}{30} - \frac{t^6}{30} + 0 + 0 + 0 + \cdots \\
&= at + xt^2,
\end{aligned}
\tag{5.15}
$$

which is the exact solution of the problem.

Remark 5.3. From comparison, it is clear that the rate of convergence of He-Laplace method is faster than homotopy perturbation method (HPM). Also it can be seen the following demerits in the HPM.

(1) Choice of initial approximation is compulsory.

(2) According to the steps of homotopy, perturbation procedure operator L should be "easy to handle." We mean that it must be chosen in such a way that one has no difficulty in subsequently solving systems of resulting equations. It should be noted that this condition does not restrict L to be linear.In some cases, as was done by He to solve the Lighthill equation, a nonlinear choice of L may be more suitable, but its strongly recommended for beginners to take a linear operator as L.

6. Conclusions and Discussions

In this paper, the He-Laplace method is employed for solving linear and nonlinear partial differential equations, that is, heat and wave equations. In previous papers [6, 15, 34–38] many authors have already used Adomian polynomials to decompose the nonlinear terms in equations. The solution procedure is simple, but the calculation of Adomian polynomials is complex. To overcome this shortcoming, we proposed a He-Laplace method using He's polynomials [31, 32, 39]. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical results.

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