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Research Article

Lie and Riccati Linearization of a Class of Liénard Type Equations

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We construct a linearizing Riccati transformation by using an ansatz and a linearizing point transformation utilizing the Lie point symmetry generators for a three-parameter class of Liénard type nonlinear second-order ordinary differential equations. Since the class of equations also admits an eight-parameter Lie group of point transformations, we utilize the Lie-Tresse linearization theorem to obtain linearizing point transformations as well. The linearizing transformations are used to transform the underlying class of equations to linear third- and second-order ordinary differential equations, respectively. The general solution of this class of equations can then easily be obtained by integrating the linearized equations resulting from both of the linearization approaches. A comparison of the results deduced in this paper is made with the ones obtained by utilizing an approach of mapping the class of equations by a complex point transformation into the free particle equation. Moreover, we utilize the linearizing Riccati transformation to extend the underlying class of equations, and the Lie-Tresse linearization theorem is also used to verify the conditions of linearizability of this new class of equations.

1. Introduction

A three-parameter class of Liénard type nonlinear second-order ordinary differential equations (ODEs) of the form

$$x'' + (b + 3kx)x' + k^2x^3 + bkx^2 + \lambda x = 0, \tag{1.1}$$

where $x = x(t)$, $x' = dx/dt$ and so on with b, k, and λ arbitrary constants, has been a subject of great interest as the equations belonging to this class are widely applied in nonlinear oscillations.

A particular class of (1.1), when $b = \lambda = 0$, was studied in [1] for the exact solutions using the Lie symmetry group method. For $b = 0$, the authors in [2] have obtained the general solution of (1.1) indirectly through the use of nonlocal transformations associated with the modified Prelle-Singer method given in [3]. In [4], Bluman et al. investigated the class of (1.1) and have obtained the general solution of (1.1) by mapping (1.1) to the free particle equation by an explicit complex point transformation by utilizing the symmetry properties of the determining equations leading to the transformation.

In this paper, we revisit (1.1) from the viewpoint of linearization and derive the general solution of the class of (1.1) in a simpler manner than given previously. In particular, we construct a linearizing Riccati transformation of (1.1) by using an ansatz and a linearizing point transformations utilizing the Lie point symmetries admitted by (1.1) to transform the underlying equation to linear ones and hence obtain the general solution of (1.1) by integrating the linearized equations arising in both cases. We show that the general solutions obtained by the linearization of (1.1) by these two approaches coincide with the results obtained by Bluman et al. in [4] for certain nonzero constants. However, our approach is much simpler. Furthermore, we also make use of the linearizing Riccati transformation to extend the underlying class of Liénard type equations and then the Lie-Tresse linearization theorem is utilized to determine the conditions under which this new class of equations is linearizable.

The outline of the paper is as follows. In Section 2, we present a Riccati transformation which linearizes (1.1) to a linear third-order equation and the use of the conditions of Lie-Tresse (see, e.g., Mahomed [5] and the references therein) for the linearization by point transformation of the class of (1.1) to a linear equation as well. We also show how one can utilize these linearizing transformations arising in the two approaches to construct exact general solutions to (1.1). We provide, in Section 3, extension of the class of (1.1) and the conditions that determine linearizability of this new class of equations by Lie point transformations. Finally, in Section 4, concluding remarks are made.

2. Linearization and General Solution of (1.1)

In this section, we consider the linearizability of (1.1), firstly by finding a Riccati transformation and then by deriving a point transformation by the Lie-Tresse theorem. We derive the Riccati transformation by using an ansatz. We then also use the invariant criteria of Lie-Tresse [5] to obtain the linearizing point transformations. The general solution of (1.1) is easily constructed making use of these linearizing transformations.

2.1. Linearization by an Ansatz Method

We use the following ansatz of the form:

$$x = \alpha(t)\ \frac{u'}{u}, \tag{2.1}$$

where $u' = du/dt$, to linearize (1.1). Using (2.1), substituting for x and the derivatives x' and x'' into (1.1), we find that $\alpha(t) = 1/k$. Hence, we find that the transformation $x = u'/(ku)$ transforms (1.1) into the following linear third-order ODE:

$$u''' + bu'' + \lambda u' = 0. \tag{2.2}$$

Now letting $z = u'$, (2.2) becomes a linear second-order ODE:

$$z'' + bz' + \lambda z = 0. \tag{2.3}$$

The characteristic equation of (2.3) has roots given by

$$m = \frac{-b \pm \sqrt{b^2 - 4\lambda}}{2}. \tag{2.4}$$

Therefore, there arise three cases. It should be mentioned that our analysis of (1.1) is thus much simpler than that of [4].

Case 1 ($b^2 - 4\lambda > 0$). In this case, the general solution of (2.3) is given by

$$z = c_1 \exp\left[\left(\frac{-b + \sqrt{b^2 - 4\lambda}}{2}\right)t\right] + c_2 \exp\left[\left(\frac{-b - \sqrt{b^2 - 4\lambda}}{2}\right)t\right], \tag{2.5}$$

where c_1 and c_2 are arbitrary constants. Thus, the general solution of (2.2) takes the form

$$u = \frac{2c_1}{\left(-b + \sqrt{b^2 - 4\lambda}\right)} \exp\left[\left(\frac{-b + \sqrt{b^2 - 4\lambda}}{2}\right)t\right] - \frac{2c_2}{\left(b + \sqrt{b^2 - 4\lambda}\right)} \exp\left[\left(\frac{-b - \sqrt{b^2 - 4\lambda}}{2}\right)t\right] + c_3, \tag{2.6}$$

where c_3 is a constant of integration. Hence, the general solution of (1.1) using (2.1) yields

$$x(t) = \frac{1}{k} \frac{c_1 \exp\left[\sqrt{b^2 - 4\lambda}\; t\right] + c_2}{\left(2c_1/\left(-b + \sqrt{b^2 - 4\lambda}\right)\right) \exp\left[\sqrt{b^2 - 4\lambda}\; t\right] - 2c_2/\left(b + \sqrt{b^2 - 4\lambda}\right) + \mathcal{A}}, \tag{2.7}$$

where $\mathcal{A}$ denotes $c_3 \exp[((b + \sqrt{b^2 - 4\lambda})/2)t]$. Note that not all the constants c_i are arbitrary and only two are.

Case 2 ($b^2 - 4\lambda = 0$). Here we obtain the following general solution to (1.1) given by:

$$x(t) = \frac{1}{k} \frac{c_1 + c_2\, t}{-2c_1/b + c_2(-2t/b - 4/b^2) + c_3 \exp\,[bt/2]}, \tag{2.8}$$

where c_1, c_2, and c_3 are arbitrary constants of which only any two are arbitrary.

Case 3 ($b^2 - 4\lambda < 0$). The solution of (2.3) in this case is given by

$$z = \exp\left[-\frac{bt}{2}\right]\left[c_1 \cos\left(\frac{\sqrt{4\lambda - b^2}t}{2}\right) + c_2 \sin\left(\frac{\sqrt{4\lambda - b^2}t}{2}\right)\right], \tag{2.9}$$

where c_1 and c_2 are arbitrary constants. Hence, the general solution of (1.1) is given by

$$x(t) = \frac{2\lambda}{k}\,\frac{\exp[-bt/2]\left[c_1\cos\left(\sqrt{4\lambda - b^2}t/2\right) + c_2\sin\left(\sqrt{4\lambda - b^2}t/2\right)\right]}{\exp[-bt/2]\left[\left(-bc_1 - \sqrt{4\lambda - b^2}c_2\right)\cos\left(\sqrt{4\lambda - b^2}t/2\right) + \mathcal{B}\right] + 2\lambda c_3}, \tag{2.10}$$

where $\mathcal{B}$ denotes $(-bc_2 + \sqrt{4\lambda - b^2}c_1)\sin(\sqrt{4\lambda - b^2}t/2)$, and c_3 is an arbitrary constant. The same remark as in Case 1 applies regarding the constants.

Remark 2.1. In the general solution (2.7) in Case 1, where $b^2 - 4\lambda > 0$, by setting $\Omega = \sqrt{b^2 - 4\lambda}/2$, $A = c_3(b - \sqrt{b^2 - 4\lambda})(b + \sqrt{b^2 - 4\lambda})/2c_1$, and $B = -c_2/c_1$ if $c_1 \neq 0$, we obtain the general solution given in [4] for this case. In the same manner, in the general solution (2.8) in Case 2, where $b^2 - 4\lambda = 0$, letting $D_1 = -c_1/c_2$ and $D_2 = c_3 b^2/2c_2$ if $c_2 \neq 0$, we derive the general solution corresponding to this case given in [4].

2.2. Linearization by Lie-Tresse Method

Here we consider the linearizability of (1.1) with $b^2 - 4\lambda > 0$ by point transformation utilizing the Lie-Tresse result. Equation (1.1) has the maximal eight-dimensional Lie algebra. Moreover, one can readily verify that the linearizing conditions 4 and 5 as given in [5, Theorem 8] are satisfied. The two noncommuting symmetry generators are

$$\begin{aligned}
\mathrm{X}_1 &= 2x\exp\left[\left(\frac{b - \sqrt{b^2 - 4\lambda}}{2}\right)t\right]\partial_t - \left[\left(b + \sqrt{b^2 - 4\lambda}\right)x^2 + 2kx^3\right]\exp\left[\left(\frac{b - \sqrt{b^2 - 4\lambda}}{2}\right)t\right]\partial_x,\\
\mathrm{X}_2 &= \frac{-2}{\left(b - \sqrt{b^2 - 4\lambda}\right)}\partial_t.
\end{aligned} \tag{2.11}$$

By condition 9 in [5, Theorem 8], we find the linearizing transformation $X = X(t, x), Y = Y(t, x)$ that will reduce X_1 and X_2 to their canonical form:

$$\mathrm{X}_1 = \partial_Y, \qquad \mathrm{X}_2 = X\partial_X + Y\partial_Y. \tag{2.12}$$

Thus, by solving the system of partial differential equations (PDEs):

$$\begin{aligned}
\mathrm{X}_1(X) &= 0, \qquad \mathrm{X}_2(X) = X,\\
\mathrm{X}_1(Y) &= 1, \qquad \mathrm{X}_2(Y) = Y,
\end{aligned} \tag{2.13}$$

which involve some tedious calculations, we obtain the following linearizing point transformation:

$$X(t,x) = \left(\frac{x}{2kx + b + \sqrt{b^2 - 4\lambda}}\right)^{((\sqrt{b^2-4\lambda}-b)/(\sqrt{b^2-4\lambda}+b))} \exp\left[\left(\frac{-b + \sqrt{b^2 - 4\lambda}}{2}\right)t\right],$$

$$Y(t,x) = \frac{\left(2kx + b + \sqrt{b^2 - 4\lambda}\right)}{\left(2b^2 - 8\lambda + 2b\sqrt{b^2 - 4\lambda}\right)x} \exp\left[\left(\frac{\sqrt{b^2 - 4\lambda} - b}{2}\right)t\right] + \frac{k}{2\lambda} \exp\left[\left(\frac{\sqrt{b^2 - 4\lambda} - b}{2}\right)t\right], \tag{2.14}$$

which linearizes (1.1) to the linear second-order ODE:

$$XY'' = \frac{2\sqrt{b^2 - 4\lambda}}{\left(b - \sqrt{b^2 - 4\lambda}\right)} Y'. \tag{2.15}$$

The general solution of (2.15) is given by

$$Y = \frac{c_1\left(b - \sqrt{b^2 - 4\lambda}\right)}{\left(b + \sqrt{b^2 - 4\lambda}\right)} X^{((b+\sqrt{b^2-4\lambda})/(b-\sqrt{b^2-4\lambda}))} + c_2, \tag{2.16}$$

where c_1 and c_2 are arbitrary constants. Hence, by using the transformation (2.14), we obtain the following general solution for (1.1) given by

$$x(t) = \frac{\exp\left[\left(\left(\sqrt{b^2 - 4\lambda} - b\right)/2\right)t\right] - 2c_1\sqrt{b^2-4\lambda}\left(b - \sqrt{b^2 - 4\lambda}\right)\exp\left[\left(\left(-b - \sqrt{b^2 - 4\lambda}\right)/2\right)t\right]}{\left(4kc_1\sqrt{b^2 - 4\lambda}\left(b - \sqrt{b^2 - 4\lambda}\right)/\left(b + \sqrt{b^2 - 4\lambda}\right)\right)\exp\left[\left(\left(-b - \sqrt{b^2 - 4\lambda}\right)/2\right)t\right] - C}, \tag{2.17}$$

where C denotes $(2k/(b-\sqrt{b^2 - 4\lambda}))\exp\ [((\sqrt{b^2 - 4\lambda}-b)/2)t] + 2c_2\sqrt{b^2 - 4\lambda}$. One can similarly achieve linearization by point transformation for the other cases.

Remark 2.2. In the general solution (2.17), by setting $\Omega = \sqrt{b^2 - 4\lambda}/2$, $A = c_2/k\ \sqrt{b^2 - 4\lambda}\ (b - \sqrt{b^2 - 4\lambda})(b + \sqrt{b^2 - 4\lambda})$, and $B = 2c_1\sqrt{b^2 - 4\lambda}\ (b - \sqrt{b^2 - 4\lambda})$, we obtain the general solution for the case $b^2 - 4\lambda > 0$ given in [4].

3. Generalization of (1.1)

In this section, we extend the linearizable equation (1.1). That is, we obtain a bigger class of nonlinear second-order ODEs than (1.1) which is linearizable by a Riccati transformation to a class of linear variable coefficient third-order ODEs. We also determine the conditions of linearizability of this new class of nonlinear second-order ODEs using the Lie-Tresse linearization theorem.

3.1. Extension of (1.1) *by Riccati Transformation*

Let

$$u''' + \beta(t)u'' + \gamma(t)u' + \delta(t)u + \theta(t) = 0 \tag{3.1}$$

be the general linearized form of the class of linear variable coefficient third-order ODEs. By the Riccati transformation

$$\frac{u'}{u} = \frac{x(t)}{\alpha(t)}, \tag{3.2}$$

where $u' = du/dt$, we obtain

$$u = \exp\left(\int \frac{x}{\alpha(t)} dt\right). \tag{3.3}$$

Then, we deduce from (3.3), u', u'', and u''' as follows:

$$u' = \frac{x}{\alpha(t)} \exp\left(\int \frac{x}{\alpha(t)} dt\right),$$
$$u'' = \left(\frac{x'}{\alpha(t)} + \frac{x^2}{\alpha^2(t)} - \frac{x\alpha'(t)}{\alpha^2(t)}\right) \exp\left(\int \frac{x}{\alpha(t)} dt\right),$$
$$u''' = \left(\frac{x''}{\alpha(t)} - \frac{2x'\alpha'(t)}{\alpha^2(t)} - \frac{x\alpha''(t)}{\alpha^2(t)} - \frac{2x\alpha'(t)^2}{\alpha^3(t)} + \frac{3xx'}{\alpha^2(t)} - \frac{3x^2\alpha'(t)}{\alpha^3(t)} + \frac{x^3}{\alpha^3(t)}\right) \exp\left(\int \frac{x}{\alpha(t)}\, dt\right). \tag{3.4}$$

Substituting (3.3)-(3.4) into (3.1), we get the following nonlinear second-order ODE:

$$\begin{aligned} x'' &+ \left(\beta(t) + \frac{3x}{\alpha(t)} - \frac{2\alpha'(t)}{\alpha(t)}\right)x' + \frac{x^3}{\alpha^2(t)} + \left(\frac{\beta(t)}{\alpha(t)} - \frac{3\alpha'(t)}{\alpha^2(t)}\right)x^2 \\ &+ \left(\gamma(t) - \frac{\beta(t)\alpha'(t)}{\alpha(t)} + \frac{2\alpha'(t)^2}{\alpha^2(t)} - \frac{\alpha''(t)}{\alpha(t)}\right)x + \alpha(t)\delta(t) + \alpha(t)\theta(t)\exp\left(-\int \frac{x}{\alpha(t)} dt\right) = 0. \end{aligned} \tag{3.5}$$

Thus, the extended form of (1.1) is the class of (3.5) which is linearizable by the Riccati transformation (3.2) to (3.1).

Remark 3.1. If we substitute in (3.5), $\alpha = 1/k$, $\beta(t) = b$, $\gamma(t) = \lambda$, and $\delta(t) = \theta(t) = 0$, then (3.5) reduces to the class of (1.1).

3.2. Linearization of (3.5) *by Lie-Tresse Theorem*

If (3.5) is linearizable by Lie point transformations, then we must have $\theta(t) = 0$ by the linearizing condition 4 in [5, Theorem 8]. Therefore, with $\theta(t) = 0$, (3.5) takes the following form:

$$x'' = C(t,x)x' + D(t,x), \tag{3.6}$$

where

$$\begin{aligned} C(t,x) &= -\left(\beta(t) + \frac{3x}{\alpha(t)} - \frac{2\alpha'(t)}{\alpha(t)}\right), \\ D(t,x) &= -\left[\frac{x^3}{\alpha^2(t)} + \left(\frac{\beta(t)}{\alpha(t)} - \frac{3\alpha'(t)}{\alpha^2(t)}\right)x^2 + \left(\gamma(t) - \frac{\beta(t)\alpha'(t)}{\alpha(t)} + \frac{2\alpha'(t)^2}{\alpha^2(t)} - \frac{\alpha''(t)}{\alpha(t)}\right)x + \mathfrak{D}\right], \end{aligned} \tag{3.7}$$

where $\mathfrak{D}$ denotes $\alpha(t)\delta(t)$. One can readily verify that the coefficients $C(t,x)$ and $D(t,x)$ in (3.7) satisfy the system of PDEs in condition 5 in [5, Theorem 8]. Hence by the Lie-Tresse linearization Theorem 8 in [5], the class of (3.6) is linearizable by Lie point transformations.

4. Concluding Remarks

We revisited the three-parameter class of Liénard type nonlinear ODEs (1.1) from the viewpoint of straightforward linearization to obtain in an easier fashion the explicit general solutions for this class. For (1.1), we found a linearizing Riccati transformation by using an ansatz and a linearizing point transformation utilizing the Lie point symmetry generators admitted by (1.1). Since the underlying class of (1.1) satisfies the Lie-Tresse linearization theorem, it admits an eight-dimensional Lie algebra. We utilized two symmetries to obtain a linearizing point transformations. In the first approach to the linearization of (1.1) by a Riccati transformation, we transformed (1.1) into a linear third-order ODE, and in the latter case the linearization resulted in reducing (1.1) to a linear second-order ODE. We easily obtained the general solutions of the class of (1.1) by integrating the linearized equations resulting from both of the linearization approaches. A comparison of the results obtained in this paper is made with the ones obtained by using an approach of mapping the class of (1.1) by a complex point transformation into the free particle equation. We have shown that the general solutions of (1.1) obtained in [4] are equivalent to the general solutions of (1.1) obtained in this paper by utilizing the two linearization approaches. Furthermore, the two approaches to linearization and constructing the general solutions of (1.1), namely, linearization by a Riccati transformation and use of the Lie-Tresse linearization theorem, presented in this paper have not been reported in the earlier literature. Moreover, we have shown how one can make use of the linearizing Riccati transformation to extend the underlying class of Liénard type equations, and thus it is shown that a bigger class than the Liénard system can be linearized by Riccati transformations than given in [4]. By using the Lie-Tresse linearization theorem, we also obtained the conditions for linearization of this new class of equations by Lie point transformations.

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