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Research Article

Strong Convergence of Hybrid Algorithm for Asymptotically Nonexpansive Mappings in Hilbert Spaces

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The hybrid algorithms for constructing fixed points of nonlinear mappings have been studied extensively in recent years. The advantage of this methods is that one can prove strong convergence theorems while the traditional iteration methods just have weak convergence. In this paper, we propose two types of hybrid algorithm to find a common fixed point of a finite family of asymptotically nonexpansive mappings in Hilbert spaces. One is cyclic Mann's iteration scheme, and the other is cyclic Halpern's iteration scheme. We prove the strong convergence theorems for both iteration schemes.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and norm in H, respectively. Let T be a self-mapping of C. Then, T is said to be a Lipschitzian mapping if for each $n \geq 1$ there exists an nonnegative real number k_n such that

$$\left\|T^n x - T^n y\right\| \leq k_n \left\|x - y\right\|, \tag{1.1}$$

for all $x, y \in C$. A Lipschitzian mapping is said to be nonexpansive mapping if $k_n = 1$ for all $n \geq 1$ and asymptotically nonexpansive mapping [1] if $\lim_{n\to\infty} k_n = 1$, respectively. We use $F(T)$ to denote the set of fixed points of T (i.e., $F(T) = \{x \in C : Tx = x\}$). It is well known that if T is asymptotically nonexpansive mapping with $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.

Iterative methods for finding fixed points of nonexpansive mappings are an important topic in the theory of nonexpansive mappings and have wide applications in a number of applied areas, such as the convex feasibility problem [2–4], the split feasibility problem [5–7] and image recovery and signal processing [8–10]. The Mann's iteration is defined by the following:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.2}$$

where $x_0 \in C$ is chosen arbitrarily and $\{\alpha_n\} \subseteq [0, 1]$. Reich [11] proved that if X is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.2) converges weakly to a fixed point of nonexpansive mapping T. However, we highlight that the Mann's iterations have only weak convergence even in a Hilbert space (see e.g., [12]).

In order to obtain the strong convergence theorem for the Mann iteration method (1.2) to nonexpansive mappings, in 2003, Nakajo and Takahashi [13] proved the following theorem in a Hilbert space by using an idea of the hybrid method in mathematical programming.

Theorem 1.1 (see [13]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let P be the metric projection of H onto $F(T)$. Let $x_0 \in C$ and*

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\} \subseteq [0, 1]$ satisfies $\sup_{n \geq 0} \alpha_n < 1$ and $P_{C_n \cap Q_n} x_0$ is the metric projection of H onto $C_n \cap Q_n$. Then $\{x_n\}$ converges strongly to $P x_0 \in F(T)$.

The iterative algorithm (1.3) is often referred to as hybrid algorithm or CQ algorithm in the literature. We call it hybrid algorithm. Since then, the hybrid algorithm has been studied extensively by many authors (see, e.g., [14–18]). Specifically, Kim and Xu [19] extended the results of Nakajo and Takahashi [13] from nonexpansive mapping to asymptotically nonexpansive mapping; they proposed the following hybrid algorithm:

$$\begin{aligned} &x_0 \in C \text{ is chosen arbitrarily}, \\ &y_n = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \\ &C_n = \left\{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\right\}, \\ &Q_n = \{z \in C : \langle x_n - z, x_0 - z \rangle \geq 0\}, \\ &x_{n+1} = P_{C_n \cap Q_n} x_0, \end{aligned} \tag{1.4}$$

where $\theta_n = (1-\alpha_n)(k_n^2-1)(\operatorname{diam} C)^2$. Zhang and Chen in [20], studied the following hybrid algorithm of Halpern's type for asymptotically nonexpansive mappings:

$$\begin{aligned}
&x_0 \in C \text{ be chosen arbitrarily,}\\
&y_n = \alpha_n x_0 + (1-\alpha_n)T^n x_n,\\
&C_n = \left\{ z \in C : \left\|y_n - z\right\|^2 \le \|x_n - z\|^2 + \theta_n \right\},\\
&Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{aligned} \tag{1.5}$$

where $\theta_n = (1-\alpha_n)(k_n^2-1)(\operatorname{diam} C)^2$. Some other related works can be found in [21–25].

The hybrid algorithm of (1.3)–(1.5) just considered a single nonexpansive and asymptotically nonexpansive mapping. In order to extend them to a finite family of mappings. Recall that in 1996, Bauschke [26] investigated the following cyclic Halpern's type algorithm for a finite family of nonexpansive mappings $\{T_j\}_{j=0}^{N-1}$:

$$\begin{aligned}
C \ni x_0 &\longmapsto x_1 := \alpha_0 u + (1-\alpha_0)T_0 x_0 \longmapsto \cdots\\
&\longmapsto x_N := \alpha_{N-1} u + (1-\alpha_{N-1})T_{N-1} x_{N-1}\\
&\longmapsto x_{N+1} := \alpha_N u + (1-\alpha_N)T_0 x_N \longmapsto \cdots,
\end{aligned} \tag{1.6}$$

or, more compactly,

$$\begin{aligned}
&u, x_0 \in C,\\
&x_{n+1} = \alpha_n u + (1-\alpha_n)T_{[n]} x_n, \quad n \ge 0,
\end{aligned} \tag{1.7}$$

where $T_{[n]} := T_{n \bmod N}$, and the mod N function takes values in $\{0, 1, \ldots, N-1\}$.

If $\alpha_n = 0$ and each nonexpansive mapping $\{T_j\}_{j=0}^{N-1}$ is a projection onto a closed convex set, then (1.7) reduces to the famous Algebraic Reconstruction Technique (ART), which has numerous applications from computer tomograph to image reconstruction.

For the cyclic Mann's type algorithm, a finite family of asymptotically nonexpansive mappings was introduced by Qin et al. [17] and Osilike and Shehu [14], independently. Let $\{T_j\}_{j=0}^{N-1}$ be a finite family of asymptotically nonexpansive self-mappings of C. For a given $x_0 \in C$, and a real sequence $\{\alpha_n\}_{n=0}^{\infty} \subseteq (0,1)$, the sequence $\{x_n\}_{n=0}^{\infty}$ is generated as follows:

$$\begin{aligned}
x_1 &= \alpha_0 x_0 + (1-\alpha_0)T_0 x_0,\\
x_2 &= \alpha_1 x_1 + (1-\alpha_1)T_1 x_1,\\
&\ \vdots\\
x_N &= \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})T_{N-1} x_{N-1},
\end{aligned}$$

$$
\begin{aligned}
x_{N+1} &= \alpha_N x_N + (1-\alpha_N)T_0^2 x_N,\\
x_{N+2} &= \alpha_{N+1} x_{N+1} + (1-\alpha_{N+1})T_1^2 x_{N+1},\\
&\ \vdots\\
x_{2N} &= \alpha_{2N-1} x_{2N-1} + (1-\alpha_{2N-1})T_{N-1}^2 x_{2N-1},\\
x_{2N+1} &= \alpha_{2N} x_{2N} + (1-\alpha_{2N})T_0^3 x_{2N},\\
x_{2N+2} &= \alpha_{2N+1} x_{2N+1} + (1-\alpha_{2N+1})T_1^3 x_{2N+1},\\
&\ \vdots
\end{aligned}
\tag{1.8}
$$

The algorithm can be expressed in a compact form as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{i(n)}^{k(n)} x_n, \quad n \geq 0, \tag{1.9}$$

where $n = (k-1)N + i$, $i = i(n) \in J = \{0, 1, 2, \ldots, N-1\}$, $k = k(n) \geq 1$ positive integer and $\lim_{n\to\infty} k(n) = \infty$. Similarly, we can define the cyclic Halpern's type algorithm for asymptotically nonexpansive mappings as follows:

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)T_{i(n)}^{k(n)} x_n, \quad n \geq 0. \tag{1.10}$$

The purpose of this paper is to extend the hybrid algorithms (1.4) and (1.5) to the cyclic Mann's type (1.9) and the cyclic Halpern's type (1.10). Our results generalize the corresponding results of Kim and Xu [19] and Zhang and Chen [20] from a single asymptotically nonexpansive mapping to a finite family of asymptotically nonexpansive mappings, respectively.

2. Preliminaries

In this section, we collect some useful results which will be used in the following section.

We use the following notations:

(i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;

(ii) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

It is well known that a Hilbert space H satisfies the Opial's condition [27]; that is, for each sequence $\{x_n\}$ in H which converges weakly to a point $x \in H$, we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \tag{2.1}$$

for all $y \in H$, $y \neq x$.

Recall that given a closed convex subset of C of a real Hilbert space H, the nearest point projection P_C form H onto C assigns to each $x \in C$ its nearest point denoted $P_C x$ in C from x to C; that is, $P_C x$ is the unique point in X with the property

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{2.2}$$

The following Lemmas 2.1 and 2.2 are well known.

Lemma 2.1. *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z \rangle \le 0, \quad \forall y \in C. \tag{2.3}$$

Lemma 2.2. *Let H be a real Hilbert space, then for all $x, y \in H$*

$$\|x - y\|^2 = \|x\|^2 - \left\|y^2\right\| - 2\langle x - y, y \rangle. \tag{2.4}$$

Lemma 2.3 (see [28]). *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping. Then $(I - T)$ is demiclosed at zero, that is, if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in F(T)$.*

Lemma 2.4 (see [22]). *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be sequences in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \le \|u - q\|, \quad \forall n \ge 1, \tag{2.5}$$

then $\{x_n\}$ converges strongly to q.

Lemma 2.5 (see [22]). *Let C be a closed convex subset of a real Hilbert space H. For any $x, y, z \in H$ and real number $a \in \mathbb{R}$, the set*

$$\left\{ v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v \rangle + a \right\} \tag{2.6}$$

is convex and closed.

3. Main Results

In this section, we consider a finite family of asymptotically nonexpansive mappings $\{T_j\}_{j=0}^{N-1}$; that is, there exists $\{u_{jn}\} \subseteq [0, \infty)$, $j \in J := \{0, 1, 2, \ldots, N-1\}$ with $\lim_{n\to\infty} u_{jn} = 0$, for all $j \in J$ such that

$$\left\|T_j^n x - T_j^n y\right\| \le (1 + u_{jn}) \|x - y\|, \tag{3.1}$$

for all $n \geq 1$ and $x, y \in C$. Let $u_n := \max_{j \in J}\{u_{jn}\}$, then $\lim_{n\to\infty} u_n = 0$, and

$$\left\|T_j^n x - T_j^n y\right\| \leq (1+u_n)\|x-y\|, \tag{3.2}$$

for all $n \geq 1$, and for all $x, y \in C$ and $j \in J$.

We prove the following theorems.

Theorem 3.1. *Let C be a bounded closed convex subset of a Hilbert space H, and let $\{T_j\}_{j=0}^{N-1} : C \to C$ be a finite family of asymptotically nonexpansive mappings with $F := \bigcap_{j=0}^{N-1} F(T_j) \neq \emptyset$. Assume that $\{\alpha_n\} \subseteq (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Suppose the sequence $\{x_n\}$ generated by*

$$\begin{gathered} x_0 \in C \text{ is chosen arbitrary}, \\ y_n = \alpha_n x_0 + (1-\alpha_n) T_{i(n)}^{k(n)} x_n, \\ C_n = \left\{z \in C : \left\|y_n - z\right\|^2 \leq \|x_n - z\|^2 + \alpha_n\left(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle\right) + \theta_n\right\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{gathered} \tag{3.3}$$

where $\theta_n = (2+u_n)u_n(1-\alpha_n)(\operatorname{diam} C)^2 \to 0$, as $n \to \infty$. Then $\{x_n\}$ converges strongly to $P_F x_0$.

Proof. By Lemma 2.5, we conclude that C_n is closed and convex. It is obvious that Q_n and F are closed and convex. Then, the projection mappings $P_{C_n \cap Q_n} x_0$ and $P_F x_0$ are well defined. We divide the proof into several steps.

Step 1. We show that $F \subseteq C_n \cap Q_n$, for all n. Let $p \in F$. By the hybrid algorithm (3.3) and that $\|\cdot\|^2$ is convex, we have

$$\begin{aligned} \left\|y_n - p\right\|^2 &\leq \alpha_n\left\|x_0 - p\right\|^2 + (1-\alpha_n)\left\|T_{i(n)}^{k(n)} - p\right\|^2 \\ &\leq \alpha_n\left\|x_0 - p\right\|^2 + (1-\alpha_n)(1+u_n)^2\left\|x_n - p\right\|^2 \\ &= \left\|x_n - p\right\|^2 + \alpha_n\left(\left\|x_0 - p\right\|^2 - \left\|x_n - p\right\|^2\right) + (1-\alpha_n)(2+u_n)u_n\left\|x_n - p\right\|^2 \\ &\leq \left\|x_n - p\right\|^2 + \alpha_n\left(\|x_0\|^2 + 2\langle x_n - x_0, p\rangle\right) + \theta_n, \end{aligned} \tag{3.4}$$

where $\theta_n = (2+u_n)u_n(1-\alpha_n)(\operatorname{diam} C)^2$. Hence, $p \in C_n$, that is, F $\subseteq C_n$, for all n.

Next, we prove that $F \subseteq Q_n$, for all $n \geq 0$. Indeed, for $n = 0$, $Q_0 = C$, then $F \subseteq Q_0$. Assuming that $F \subseteq Q_m$, we show that $F \subseteq Q_{m+1}$. Since x_{m+1} is the projection of x_0 onto $C_m \cap Q_m$, it follows from Lemma 2.1 that

$$\langle x_{m+1} - z, x_0 - x_{m+1}\rangle \geq 0, \quad \forall z \in C_m \cap Q_m. \tag{3.5}$$

As $F \subseteq C_m \cap Q_m$, in particular, we have

$$\left\langle x_{n+1} - p, x_0 - x_{m+1} \right\rangle \geq 0, \quad \forall p \in F. \tag{3.6}$$

Thus, $F \subseteq Q_{m+1}$. Therefore, $F \subseteq C_n \cap Q_n$, for all $n \geq 0$.

Step 2. We prove that

$$\left\| x_{n+j} - x_n \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \forall j = 0, 1, 2, \ldots, N-1. \tag{3.7}$$

Since the definition of Q_n implies that $x_n = P_{Q_n} x_0$, we have

$$\|x_n - x_0\| \leq \left\| x_0 - y \right\|, \quad \forall y \in Q_n. \tag{3.8}$$

By Step 1, $F \subseteq Q_n$, we have

$$\|x_n - x_0\| \leq \left\| x_0 - p \right\|, \quad \forall p \in F. \tag{3.9}$$

In particular,

$$\|x_n - x_0\| \leq \left\| x_0 - q \right\|, \quad q = P_F x_0. \tag{3.10}$$

Since $x_{n+1} \in Q_n$, we have $\langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0$ and $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$. The second inequality shows that the sequence $\{\|x_n - x_0\|\}$ is nondecreasing. Since C is bounded, we obtain that the $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

With the help of Lemma 2.2, we obtain

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.11}$$

Consequently,

$$\left\| x_{n+j} - x_n \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \forall j \in J. \tag{3.12}$$

Step 3. We now claim that $\|x_n - T_j x_n\| \to 0$, as $n \to \infty$, for all $j \in J$. Notice that for all $n > N$, $n = (n-N)(\bmod N)$, since $n = (k(n)-1)N + i(n)$, we obtain $n-N = (k(n)-1)N + i(n) - N = (k(n-N)-1)N + i(n-N)$. So that $n-N = [(k(n)-1)-1]N + i(n) = (k(n-N)-1)N + i(n-N)$. Hence $k(n) - 1 = k(n-N)$ and $i(n) = i(n-N)$.

By the hybrid algorithm (3.3) and the condition $\lim_{n\to\infty}\alpha_n = 0$, we get

$$\left\| y_n - T_{i(n)}^{k(n)} x_n \right\| = \alpha_n \left\| x_0 - T_{i(n)}^{k(n)} x_n \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.13}$$

It follows from the fact $x_{n+1} \in C_n$ that we have

$$\begin{aligned}\left\|y_n - x_{n+1}\right\|^2 &\leq \|x_n - x_{n+1}\|^2 + \alpha_n\left(\|x_0\|^2 + 2\langle x_n - x_0, x_{n+1}\rangle\right) + \theta_n \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\ \left\|T_{i(n)}^{k(n)} x_n - x_n\right\| &\leq \left\|T_{i(n)}^{k(n)} x_n - y_n\right\| + \left\|y_n - x_{n+1}\right\| + \|x_{n+1} - x_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.14}$$

Putting $L = \sup_{n\geq 0}\{1 + u_n\}$, we deduce that

$$\begin{aligned}\left\|x_{n+1} - T_{i(n)}x_n\right\| &\leq \left\|x_{n+1} - T_{i(n)}^{k(n)} x_n\right\| + \left\|T_{i(n)}^{k(n)} x_n - T_{i(n)}x_n\right\| \\ &\leq \|x_{n+1} - x_n\| + \left\|x_n - T_{i(n)}^{k(n)} x_n\right\| + L\left\|T_{i(n)}^{k(n)-1} x_n - x_n\right\| \\ &\leq \|x_{n+1} - x_n\| + \left\|x_n - T_{i(n)}^{k(n)} x_n\right\| \\ &\quad + L\left(\left\|T_{i(n)}^{k(n)-1} x_n - T_{i(n-N)}^{k(n)-1} x_{n-N}\right\| + \left\|T_{i(n-N)}^{k(n)-1} x_{n-N} - x_{n-N-1}\right\| + \|x_{n-N-1} - x_n\|\right) \\ &\leq \|x_{n+1} - x_n\| + \left\|x_n - T_{i(n)}^{k(n)} x_n\right\| + L^2\|x_n - x_{n-N}\| \\ &\quad + L\left\|T_{i(n-N)}^{k(n)-1} x_{n-N} - x_{n-N-1}\right\| + L\|x_{n-N-1} - x_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.15}$$

Hence,

$$\left\|x_n - T_{i(n)}x_n\right\| \leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_{i(n)}x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.16}$$

Consequently, for all $j = 0, 1, \ldots, N-1$, we have

$$\begin{aligned}\left\|x_n - T_{n+j}x_n\right\| &\leq \left\|x_n - x_{n+j}\right\| + \left\|x_{n+j} - T_{n+j}x_{n+j}\right\| + L\left\|x_{n+j} - x_n\right\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.17}$$

Thus, $\|x_n - T_j x_n\| \to 0$, as $n \to \infty$, for all $j \in J$.

Step 4. Since $\{x_n\}$ is bounded, then $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_j}\}$. Suppose $\{x_{n_j}\}$ converges weakly to p. Since C is weakly closed and $\{x_{n_j}\} \subset C$, we have $p \in C$. By Lemma 2.3, $I - T_j$ is demiclosed at 0 for all $j \in J$, and we get $p - T_{jp} = 0 (j \in J)$, that is $p \in F$. Suppose $\{x_n\}$ does not converge weakly to p, then there exists another subsequence $\{x_{n_k}\}$of $\{x_n\}$ which converges weakly to some p_1. Similarly we can prove that $p_1 \in F$. It follows from the proof of above that we know that $\lim_{n\to\infty}\|x_n - p\|$ and $\lim_{n\to\infty}\|x_n - p_1\|$ exist. Since every Hilbert space satisfies Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - p\| &= \lim_{j\to\infty}\left\|x_{n_j} - p\right\| < \lim_{j\to\infty}\left\|x_{n_j} - p_1\right\| \\
&= \lim_{n\to\infty}\|x_n - p_1\| = \lim_{k\to\infty}\|x_{n_k} - p_1\| \\
&< \lim_{k\to\infty}\|x_{n_k} - p\| < \lim_{n\to\infty}\|x_n - p\|.
\end{aligned} \tag{3.18}$$

This is a contradiction. Hence, $\omega_w(x_n) \subseteq F$. Then by virtue of (3.10) and Lemma 2.4, we conclude that $x_n \to q$ as $n \to \infty$, where $q = P_F x_0$. □

Recall that a mapping T is said to be asymptotically strictly pseudocontractive [29], if there exist $\lambda \in [0,1)$ and a sequence $\{u_n\} \subseteq [0,\infty)$ with $\lim_{n\to\infty} u_n = 0$ such that

$$\left\|T^n x - T^n y\right\|^2 \le (1+u_n)^2\left\|x-y\right\|^2 + \lambda\left\|(I-T^n)x - (I-T^n)y\right\|^2, \tag{3.19}$$

for all n and $x, y \in C$.

Theorem 3.2. *Let C be a bounded closed convex subset of a Hilbert space H, and let $\{T_j\}_{j=0}^{N-1} : C \to C$ be a finite family of asymptotically nonexpansive mappings with $F := \bigcap_{j=0}^{N-1} F(T_j) \neq \emptyset$. Assume that $\{\alpha_n\} \subseteq (0,a)$, for some $0 < a < 1$. Suppose the sequence $\{x_n\}$ generated by*

$$\begin{aligned}
&x_0 \in C \text{ is chosen arbitrary}, \\
&y_n = \alpha_n x_n + (1-\alpha_n) T_{i(n)}^{k(n)} x_n, \\
&C_n = \left\{ z \in C : \left\|y_n - z\right\|^2 \le \|x_n - z\|^2 + \theta_n \right\}, \\
&Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\
&x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{aligned} \tag{3.20}$$

where $\theta_n = (2+u_n)u_n(\operatorname{diam} C)^2$. Then $\{x_n\}$ converges strongly to $P_F x_0$.

Proof. Since T_j is asymptotically nonexpansive if and only if T_j is asymptotically strictly pseudocontractive mapping with $\lambda = 0$. Then, the rest of proof follows from Theorem 3.2 of Osilike and Shehu [14] and Theorem 2.2 of Qin et al. [17] directly by letting $\lambda = 0$. □

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