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Research Article

Quasilinearization of the Initial Value Problem for Difference Equations with "Maxima"

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The object of investigation of the paper is a special type of difference equations containing the maximum value of the unknown function over a past time interval. These equations are adequate models of real processes which present state depends significantly on their maximal value over a past time interval. An algorithm based on the quasilinearization method is suggested to solve approximately the initial value problem for the given difference equation. Every successive approximation of the unknown solution is the unique solution of an appropriately constructed initial value problem for a linear difference equation with "maxima," and a formula for its explicit form is given. Also, each approximation is a lower/upper solution of the given mixed problem. It is proved the quadratic convergence of the successive approximations. The suggested algorithm is realized as a computer program, and it is applied to an example, illustrating the advantages of the suggested scheme.

1. Introduction

In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in the control theory correspond to the maximal deviation of the regulated quantity. In the case when the dynamic of these problems is modeled discretely, the corresponding equations are called difference equations with "maxima". The presence of the maximum function in the equation requires not only more complicated calculations but also a development of new methods for qualitative investigations of the behavior of their solutions(see, e.g., the monograph [1]). The character of the maximum function leads to variety of the types of difference equations. Some special types of difference equations are studied in [2–18]. At the same time, when the unknown function at any point is presented in both sides of the equation nonlinearly as well as it is involved in the maximum function, the given equation is not possible to be solved in an explicit form. It requires development of some approximate methods for their solving.

In the present paper, a nonlinear difference equation of delayed type is considered. The equation contains the maximum of the unknown function over a discrete past time interval. The main goal of the paper is suggesting an algorithm for an approximate solving of an initial value problem for the given difference equation.

2. Preliminary Notes

Let $\mathbb{R}_+ = [0,\infty)$, $\mathbb{Z}$ be the set of all integers, let $h \geq 0$ be a given fixed integer and $a, b \in \mathbb{Z}$ be such that a $< b$. Denote by $\mathbb{Z}[a,b] = \{z \in \mathbb{Z} : a \leq z \leq b\}$.

Note that for any function $Q : \mathbb{Z}[m,n] \to \mathbb{R}$, $m < n$, the equalities $\sum_{i=n}^{m} Q(i) = 0$ and $\prod_{i=n}^{m} Q(i) = 1$ hold.

Consider the following nonlinear difference equation with "maxima":

$$\Delta u(k-1) = f\left(k, u(k), \max_{s\in\mathbb{Z}[k-h,k]} u(s)\right) \quad \text{for } k \in \mathbb{Z}[a+1,T] \tag{2.1}$$

with an initial condition

$$u(k) = \varphi(k) \quad \text{for } k \in \mathbb{Z}[a-h+1,a], \tag{2.2}$$

where the functions $u : \mathbb{Z}[a-h+1,T] \to \mathbb{R}$, $\Delta u(k-1) = u(k)-u(k-1)$, $f : \mathbb{Z}[a+1,T]\times\mathbb{R}^2 \to \mathbb{R}$, and $\varphi : \mathbb{Z}[a-h+1,a] \to \mathbb{R}$, the points $a, T \in \mathbb{Z}$ are such that $0 \leq a < T$.

Definition 2.1. One will say that the function $\alpha : \mathbb{Z}[a+1-h,T] \to \mathbb{R}$ is a lower (upper) solution of the problem (2.1), (2.2), if

$$\begin{gathered}\Delta\alpha(k-1) \leq (\geq) f\left(k, \alpha(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha(s)\right) \quad \text{for } k \in \mathbb{Z}[a+1,T], \\ \alpha(k) \leq (\geq)\ \varphi(k) \quad \text{for } k \in \mathbb{Z}[a+1-h,a].\end{gathered} \tag{2.3}$$

Let $\alpha, \beta : \mathbb{Z}[a+1-h,T] \to \mathbb{R}$ be given functions such that $\alpha(k) \leq \beta(k)$ for $k \in \mathbb{Z}[a+1-h,T]$. Define the following sets:

$$\begin{gathered}S(\alpha,\beta) = \{u : \mathbb{Z}[a+1-h,T] \to \mathbb{R} : \alpha(k) \leq u(k) \leq \beta(k), k \in \mathbb{Z}[a+1-h,T]\}, \\ \Omega(\alpha,\beta) = \left\{\begin{matrix}(k,x_1,x_2) \in \mathbb{Z}[a+1,T]\times\mathbb{R}^2 : \alpha(k) \leq x_1 \leq \beta(k), \\ \max_{s\in\mathbb{Z}[k-h,k]} \alpha(s) \leq x_2 \leq \max_{s\in\mathbb{Z}[k-h,k]} \beta(s)\end{matrix}\right\}.\end{gathered} \tag{2.4}$$

3. Comparison Results

In our further investigations, we will use the following results for difference inequalities with "maxima".

Lemma 3.1 (existence and uniqueness). *Let the following conditions be fulfilled:*

(1) *The function* $Q : \mathbb{Z}[a+1,T] \to \mathbb{R}$.

(2) *The functions* $M : \mathbb{Z}[a+1,T] \to \mathbb{R}$, $N : \mathbb{Z}[a+1,T] \to \mathbb{R}_+$ *are such that*

$$M(k) + N(k) < 1 \quad \text{for } k \in \mathbb{Z}[a+1,T]. \tag{3.1}$$

Then the initial value problem for linear difference equation with "maxima"

$$\begin{gathered}\Delta u(k-1) = Q(k) + M(k)u(k) + N(k) \max_{s\in\mathbb{Z}[k-h,k]} u(s) \quad \text{for } k \in \mathbb{Z}[a+1,T],\\ u(k) = \varphi(k) \quad \text{for } k \in \mathbb{Z}[a+1-h,a]\end{gathered} \tag{3.2}$$

has an unique solution on the interval $\mathbb{Z}[a+1-h,T]$.

Proof. We will use the step method to solve the initial value problem (3.2). Assume for a fixed $k \in \mathbb{Z}[a+1,T]$ all values $u(j)$, $j \in \mathbb{Z}[\mathrm{a}+1-h,k-1]$ are known. Then from (3.2), we obtain $(1-M(k))u(k) = u(k-1) + Q(k) + N(k)\max_{s\in\mathbb{Z}[k-h,k]}u(s)$.

Consider the following two possible cases.

Case 1. Let $\max_{l\in\mathbb{Z}[1,h]}u(k-l) \le (Q(k)+u(k-1))/(1-M(k)-N(k))$, or $(Q(k)+u(k-1))/(1-M(k)-N(k)) \ge u(k-l)$ for $l \in \mathbb{Z}[1,h]$.

Therefore, $\max_{s\in\mathbb{Z}[k-h,k]}u(s) = u(k)$. Then applying the inequality (3.1), we obtain the unique solution $u(k)$ of the problem (3.2) given by the equality

$$u(k) = \frac{Q(k) + u(k-1)}{1 - M(k) - N(k)}. \tag{3.3}$$

Case 2. Let $(Q(k)+u(k-1))/(1-M(k)-N(k)) < \max_{l\in\mathbb{Z}[1,h]}u(k-l) = u(k-m)$ where $m \in \mathbb{Z}[1,h]$.

If we assume $u(k) \ge u(k-m)$ then from (3.2), we obtain $u(k-m) \le u(k) = (Q(k) + u(k-1))/(1-M(k)-N(k)) < u(k-m)$. The obtained contradiction proves $u(k) < u(k-m)$ and $\max_{s\in\mathbb{Z}[k-h,k]}u(s) = u(k-m)$.

From the inequalities (3.1) and $N(k) \ge 0$, it follows that $M(k) < 1$ and therefore, the unique solution of problem (3.2) is given by the equality

$$u(k) = \frac{u(k-1) + Q(k) + N(k)u(k-m)}{1 - M(k)}. \tag{3.4}$$

Thus, we receive the value $u(k)$, $k \in Z[a+1,T]$. □

Lemma 3.2. *Let the following conditions be fulfilled:*

(1) *The functions* $M, N : \mathbb{Z}[a+1,T] \to \mathbb{R}$ *satisfy the inequality*

$$M(k) + N(k) < 1 \quad \text{for } k \in \mathbb{Z}[a+1,T]. \tag{3.5}$$

(2) *The function* $u : \mathbb{Z}[a-h+1, T] \to \mathbb{R}$ *satisfies the inequalities*

$$\Delta u(k-1) \leq M(k)u(k) + N(k) \max_{s\in\mathbb{Z}[k-h,k]} u(s), \quad k \in \mathbb{Z}[a+1, T],$$
$$u(k) \leq 0, \quad k \in \mathbb{Z}[a-h+1, a]. \tag{3.6}$$

Then $u(k) \leq 0$ *for* $k \in \mathbb{Z}[a-h+1, T]$.

Proof. Assume the claim of Lemma 3.2 is not true. Then there exists $j \in \mathbb{Z}[a+1, T]$ such that $u(j) > 0$ and $u(k) \leq 0$ for $k \in \mathbb{Z}[a-h+1, j-1]$. Therefore, $\max_{s\in\mathbb{Z}[j-h,j]} u(s) = u(j)$ and according to inequality (3.6), we get

$$u(j) \leq u(j) - u(j-1) \leq (M(j) + N(j))u(j). \tag{3.7}$$

The above inequality contradicts (3.5). □

Remark 3.3. Note if both $M(k), N(k) \leq 0$ for $k \in \mathbb{Z}[a+1, T]$, then inequality (3.5) is satisfied.

Now, we will prove a linear difference inequality in which at any point both the unknown function and its maximum over a past time interval are involved also in the right side of the inequality.

In the proof of our preliminary results, we will need the following lemma.

Lemma 3.4 (see [2], Theorem 4.1.1). *Let* $f, q : \mathbb{Z}(a, \infty) \to \mathbb{R}_+$, $p, u : \mathbb{Z}(a, \infty) \to \mathbb{R}$ *and*

$$u(k) \leq p(k) + q(k)\sum_{l=a}^{k-1} f(l)u(l) \quad \text{for } k \in \mathbb{Z}(a, \infty). \tag{3.8}$$

Then for all $k \in \mathbb{Z}(a, \infty)$*, the following inequality is valid:*

$$u(k) \leq p(k) + q(k)\sum_{l=a}^{k-1} p(l)f(l)\prod_{\tau=l+1}^{k-1}(1 + q(\tau)f(\tau)). \tag{3.9}$$

Now, we will solve a generalized linear difference inequality with "maxima."

Lemma 3.5. *Let the following conditions be fulfilled:*

(1) *The functions* $q, Q : \mathbb{Z}[a+1, T] \to \mathbb{R}_+$ *and*

$$q(k) + Q(k) < 1 \quad \text{for } k \in \mathbb{Z}[a+1, T]. \tag{3.10}$$

(2) *The function* $u : \mathbb{Z}[a+1-h, T] \to \mathbb{R}_+$ *and satisfies the inequalities*

$$u(k) \leq C + \sum_{l=a+1}^{k} q(l)u(l) + \sum_{l=a+1}^{k} Q(l) \max_{s\in\mathbb{Z}[l-h,l]} u(s), \quad k \in \mathbb{Z}[a+1, T],$$
$$u(k) \leq C, \quad k \in \mathbb{Z}[a+1-h, a], \tag{3.11}$$

where $C = \text{const} \geq 0$.

Then,

$$u(k) \leq \frac{C}{\prod_{\tau=a+1}^{k}\left(1 - q(\tau) - Q(\tau)\right)} \quad \text{for } k \in \mathbb{Z}[a+1, T]. \tag{3.12}$$

Proof. Define a function $z : \mathbb{Z}[a+1-h, T] \to \mathbb{R}_+$ by the equalities

$$z(k) = \begin{cases} C + \displaystyle\sum_{l=a+1}^{k} q(l)u(l) + \sum_{l=a+1}^{k} Q(l) \max_{s\in\mathbb{Z}[l-h,l]} u(s), & k \in \mathbb{Z}[a+1, T], \\ C, & k \in \mathbb{Z}[a+1-h, a]. \end{cases} \tag{3.13}$$

It is easy to check that for any $k \in \mathbb{Z}[a+1, T]$, the inequality $z(k+1) \geq z(k)$ holds. Also, from the definition of $z(k)$, it follows that $u(k) \leq z(k), k \in \mathbb{Z}[a+1, T]$, and $\max_{s\in\mathbb{Z}[k-h,k]} u(s) \leq \max_{s\in\mathbb{Z}[k-h,k]} z(s) = z(k)$ for $k \in \mathbb{Z}[a+1, T]$. Therefore, for any $k \in \mathbb{Z}[a+1, T]$, we obtain

$$z(k) \leq C + \sum_{l=a+1}^{k} \left(q(l) + Q(l)\right) z(l). \tag{3.14}$$

From inequality (3.14), it follows

$$z(k) \leq \frac{C}{\left(1 - q(k) - Q(k)\right)} + \frac{1}{\left(1 - q(k) - Q(k)\right)} \sum_{l=a+1}^{k-1} \left(q(l) + Q(l)\right) z(l). \tag{3.15}$$

According to Lemma 3.4 from inequality (3.15), we get for $k \in \mathbb{Z}[a+1, T]$

$$\begin{aligned} z(k) &\leq \frac{C}{\left(1 - q(k) - Q(k)\right)} \\ &\quad \times \left\{1 + \sum_{l=a+1}^{k-1} \frac{q(l) + Q(l)}{1 - q(l) - Q(l)} \prod_{\tau=l+1}^{k-1} \left(1 + \frac{q(\tau) + Q(\tau)}{1 - q(\tau) - Q(\tau)}\right)\right\} \\ &= \frac{C}{\left(1 - q(k) - Q(k)\right)} \\ &\quad \times \left\{1 + \sum_{l=a+1}^{k-1} \left[\frac{q(l) + Q(l)}{1 - q(l) - Q(l)} \; \frac{1}{\prod_{\tau=l+1}^{k-1}\left(1 - q(\tau) - Q(\tau)\right)}\right]\right\} \\ &\leq \frac{C}{\prod_{\tau=a+1}^{k}\left(1 - q(\tau) - Q(\tau)\right)}. \end{aligned} \tag{3.16}$$

Inequality (3.16) implies the validity of the required inequality (3.12). □

4. Quasilinearization

We will apply the method of quasilinearization to obtain approximate solution of the IVP for the nonlinear difference equation with "maxima" (2.1), (2.2). We will prove the convergence of the sequence of successive approximations is quadratic.

Theorem 4.1. *Let the following conditions be fulfilled:*

(1) *The functions* $\alpha_0, \beta_0 : \mathbb{Z}[a+1-h, T] \to \mathbb{R}$ *are a lower and an upper solutions of the problem* (2.1), (2.2), *respectively, and such that* $\alpha_0(k) \le \beta_0(k)$ *for* $k \in \mathbb{Z}[a+1-h, T]$.

(2) *The function* $f : \mathbb{Z}[a+1, T] \times \mathbb{R}^2 \to \mathbb{R}$ *satisfies for* $(k, x, y) \in \Omega(\alpha_0, \beta_0)$ *the equality*

$$f(k,x,y) = F(k,x,y) - G(k,x,y), \tag{4.1}$$

where the functions $F, G : \Omega(\alpha_0, \beta_0) \to \mathbb{R}$ *are continuous and twice continuously differentiable with respect to their second and third arguments and the following inequalities are valid for* $k \in \mathbb{Z}[a+1, T]$, $(k, x, y) \in \Omega(\alpha_0, \beta_0)$*:*

$$\begin{aligned} &F_{xx}(k,x,y) \ge 0, \qquad F_{xy}(k,x,y) \ge 0, \qquad F_{yy}(k,x,y) \ge 0, \\ &G_{xx}(k,x,y) \ge 0, \qquad G_{xy}(k,x,y) \ge 0, \qquad G_{yy}(k,x,y) \ge 0, \end{aligned} \tag{4.2}$$

$$G_x\left(k, \beta_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_0(s)\right) \le F_x\left(k, \alpha_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_0(s)\right), \tag{4.3}$$

$$G_y\left(k, \beta_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_0(s)\right) \le F_y\left(k, \alpha_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_0(s)\right), \tag{4.4}$$

$$M(k) + N(k) < 1, \tag{4.5}$$

where

$$\begin{aligned} M(k) &= F_x\left(k, \beta_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_0(s)\right) - G_x\left(k, \alpha_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_0(s)\right), \\ N(k) &= F_y\left(k, \beta_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_0(s)\right) - G_y\left(k, \alpha_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_0(s)\right). \end{aligned} \tag{4.6}$$

(3) *The function* $\varphi : \mathbb{Z}[a-h+1, a] \to \mathbb{R}$.

Then there exist two sequences $\{\alpha_n(k)\}_{n=0}^{\infty}$ *and* $\{\beta_n(k)\}_{n=0}^{\infty}$, $k \in \mathbb{Z}[a+1-h, T]$, *such that:*

(a) *The functions* $\alpha_n : \mathbb{Z}[a+1-h, T] \to \mathbb{R}$, $(n = 1, 2, \ldots)$ *are lower solutions of IVP* (2.1), (2.2).

(b) *The functions* $\beta_n : \mathbb{Z}[a+1-h, T] \to \mathbb{R}$, $(n = 1, 2, \ldots)$ *are upper solutions of IVP* (2.1), (2.2).

(c) *The following inequalities hold for* $k \in [a+1-h, T]$

$$\alpha_0(k) \le \alpha_1(k) \le \cdots \le \alpha_n(k) \le \cdots \le \beta_n(k) \le \cdots \le \beta_1(k) \le \beta_0(k). \tag{4.7}$$

(d) *Both sequences are convergent on* $\mathbb{Z}[a+1-h,T]$ *and their limits* $V(k) = \lim_{n\to\infty}\alpha_n(k)$ *and* $W(k) = \lim_{n\to\infty}\beta_n(k)$ *are the minimal and the maximal solutions of IVP* (2.1), (2.2) *in* $S(\alpha_0,\beta_0)$. *In the case, IVP* (2.1), (2.2) *has an unique solution in* $S(\alpha_0,\beta_0)$ *both limits coincide, that is,* $V(k) = W(k)$.

(e) *The convergence is quadratic, that is, there exist constants* λ_i, $\mu_i > 0$, $i = 1,2$ *such that for the solution* $x(k)$ *of IVP* (2.1), (2.2) *in* $S(\alpha_0,\beta_0)$, *the inequalities*

$$\begin{aligned} \|x-\alpha_{n+1}\| &\le \lambda_1\|x-\alpha_n\|^2 + \lambda_2\|x-\beta_n\|^2, \\ \|x-\beta_{n+1}\| &\le \mu_1\|x-\alpha_n\|^2 + \mu_2\|x-\beta_n\|^2 \end{aligned} \tag{4.8}$$

hold, where $\|u\| = \max_{s\in\mathbb{Z}[a+1-h,T]}|u(s)|$ *for any function* $u : \mathbb{Z}[a+1-h,T] \to \mathbb{R}$.

Proof. From Taylor formula and condition (2) of Theorem 4.1 for $(k,x_1,y_1), (k,x_2,y_2) \in \Omega(\alpha_0,\beta_0)$, the following inequalities are valid:

$$\begin{aligned} f(k,x_2,y_2) - f(k,x_1,y_1) &\le F_x(k,x_2,y_2)(x_2-x_1) + F_y(k,x_2,y_2)(y_2-y_1) \\ &\quad - G(k,x_2,y_2) + G(k,x_1,y_1), \\ -G(k,x_2,y_2) + G(k,x_1,y_1) &\le G_x(k,x_1,y_1)(x_1-x_2) + G_y(k,x_1,y_1)(y_1-y_2). \end{aligned} \tag{4.9}$$

Consider the initial value problem for the linear difference equation with "maxima"

$$\Delta x(k-1) = Q_0(k)x(k) + q_0(k)\max_{s\in\mathbb{Z}[k-h,k]} x(s) + P_0(k), \quad k \in \mathbb{Z}[a+1,T], \tag{4.10}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}[a+1-h,a], \tag{4.11}$$

where the functions $P_0, Q_0, q_0 : \mathbb{Z}[a+1,T] \to \mathbb{R}$ are defined by the equalities

$$\begin{aligned} P_0(k) &= f\left(k,\alpha_0(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_0(s)\right) - Q_0(k)\alpha_0(k) - q_0(k)\max_{s\in\mathbb{Z}[k-h,k]}\alpha_0(s), \\ Q_0(k) &= F_x\left(k,\alpha_0(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_0(s)\right) - G_x\left(k,\beta_0(k),\max_{s\in\mathbb{Z}[k-h,k]}\beta_0(s)\right), \\ q_0(k) &= F_y\left(k,\alpha_0(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_0(s)\right) - G_y\left(k,\beta_0(k),\max_{s\in\mathbb{Z}[k-h,k]}\beta_0(s)\right). \end{aligned} \tag{4.12}$$

From inequality (4.4), it follows that $q_0(k) \ge 0$ and from inequalities (4.2), (4.5), we get $Q_0(k) + q_0(k) < 1$ for $k \in \mathbb{Z}[a+1,T]$. According to Lemma 3.1 the IVP (4.10), (4.11) has an unique solution $\alpha_1(k)$, defined on the interval $\mathbb{Z}[a+1-h,T]$.

Define a function $p_1 : \mathbb{Z}[a+1-h,T] \to \mathbb{R}$ by the equality $p_1(k) = \alpha_0(k) - \alpha_1(k)$. Then we get $p_1(k) = 0$ for $k \in \mathbb{Z}[a+1-h,a]$.

Let $k \in \mathbb{Z}[a+1,T]$. From the choice of the function $\alpha_0(k)$ and (4.10) for the function $\alpha_1(k)$, we get

$$\Delta p_1(k-1) \leq Q_0(k)p_1(k) + q_0(k)\max_{s\in\mathbb{Z}[k-h,k]} p_1(s). \tag{4.13}$$

According to Lemma 3.2 for the function $p_1(k)$, it follows that $p_1(k) \leq 0$ for $k \in \mathbb{Z}[a+1-h,T]$. Therefore, $\alpha_0(k) \leq \alpha_1(k)$ for $k \in \mathbb{Z}[a+1-h,T]$.

Consider the linear difference equation with "maxima"

$$\Delta x(k-1) = Q_0(k)x(k) + q_0(k)\max_{s\in\mathbb{Z}[k-h,k]} x(s) + R_0(k), \quad k \in \mathbb{Z}[a+1,T], \tag{4.14}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}[a+1-h,a], \tag{4.15}$$

where the functions $q_0(k)$ and $Q_0(k)$ are defined by equalities (4.12) and

$$R_0(k) = f\left(k, \beta_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_0(s)\right) - Q_0(k)\beta_0(k) - q_0(k)\max_{s\in\mathbb{Z}[k-h,k]} \beta_0(s). \tag{4.16}$$

According to Lemma 3.1, the linear initial value problem (4.14), (4.15) has a unique solution $\beta_1(k)$, defined on the interval $\mathbb{Z}[a+1-h,T]$.

Define a function $p_2 : \mathbb{Z}[a+1-h,T] \to \mathbb{R}$ by the equality $p_2(k) = \beta_1(k) - \beta_0(k)$. Then $p_2(k) = 0$ for $k \in \mathbb{Z}[a+1-h,a]$.

Now, let $k \in \mathbb{Z}[a+1,T]$. From the choice of the function $\beta_0(k)$ and (4.14) for the function $\beta_1(k)$, we get

$$\begin{aligned}\Delta p_2(k-1) &\leq Q_0(k)\left[\beta_1(k) - \beta_0(k)\right] \\ &\quad + q_0(k)\left[\max_{s\in\mathbb{Z}[k-h,k]} \beta_1(s) - \max_{s\in\mathbb{Z}[k-h,k]} \beta_0(s)\right] \\ &\leq Q_0(k)p_2(k) + q_0(k)\max_{s\in\mathbb{Z}[k-h,k]} p_2(s).\end{aligned} \tag{4.17}$$

Inequality (4.17) proves the function $p_2(k)$ satisfies inequality (3.6). According to Lemma 3.2, it follows that $p_2(k) \leq 0$ for $k \in \mathbb{Z}[a+1-h,T]$. Therefore, $\beta_1(k) \leq \beta_0(k)$ for $k \in \mathbb{Z}[a+1-h,T]$.

Define a function $p_3 : \mathbb{Z}[a+1-h,T] \to \mathbb{R}$ by the equality $p_3(k) = \alpha_1(k) - \beta_1(k)$. Then $p_3(k) \leq 0$ for $k \in \mathbb{Z}[a+1-h,a]$.

Let $k \in \mathbb{Z}[a+1,T]$. Then for the function $p_3(k)$, we get

$$\begin{aligned}\Delta p_3(k-1) &= f\left(k, \alpha_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_0(s)\right) - f\left(k, \beta_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_0(s)\right) \\ &\quad + Q_0(k)\left[\alpha_1(k) - \alpha_0(k)\right] - Q_0(k)\left[\beta_1(k) - \beta_0(k)\right] \\ &\quad + q_0(k)\left[\max_{s\in\mathbb{Z}[k-h,k]} \alpha_1(s) - \max_{s\in\mathbb{Z}[k-h,k]} \alpha_0(s)\right]\end{aligned}$$

$$
\begin{aligned}
&- q_0(k)\left[\max_{s\in\mathbb{Z}[k-h,k]}\beta_1(s) - \max_{s\in\mathbb{Z}[k-h,k]}\beta_0(s)\right] \\
&\le Q_0(k)p_3(k) + q_0(k)\max_{s\in\mathbb{Z}[k-h,k]}p_3(s).
\end{aligned}
\tag{4.18}
$$

Inequality (4.18) proves the function $p_3(k)$ satisfies inequality (3.6). According to Lemma 3.2, it follows that $p_3(k) \le 0$ for $k \in \mathbb{Z}[a+1-h,T]$. Therefore, $\alpha_1(k) \le \beta_1(k)$ for $k \in \mathbb{Z}[a+1-h,T]$.

Furthermore, the functions $\alpha_1(k)$ and $\beta_1(k) \in S(\alpha_0,\beta_0)$.

Now, we will prove that the function $\alpha_1(k)$ is a lower solution of (2.1), (2.2) on the interval $\mathbb{Z}[a+1-h,T]$.

Let $k \in \mathbb{Z}[a+1,T]$. From the inequalities $\alpha_0(k) \le \alpha_1(k) \le \beta_0(k)$ for $k \in \mathbb{Z}[a+1-h,T]$, $\max_{s\in\mathbb{Z}[k-h,k]}\alpha_0(s) \le \max_{s\in\mathbb{Z}[k-h,k]}\alpha_1(s) \le \max_{s\in\mathbb{Z}[k-h,k]}\beta_0(s)$ for $k \in \mathbb{Z}[a+1,T]$, inequalities (4.9), definitions (4.12), and inequalities (4.2) which prove the monotonic property of the first derivatives of the functions $F(k,x,y)$ and $G(k,x,y)$ we get

$$
\begin{aligned}
\Delta\alpha_1(k-1) &= f\left(k,\alpha_0(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_0(s)\right) + Q_0(k)[\alpha_1(k)-\alpha_0(k)] \\
&\quad + q_0(k)\left[\max_{s\in\mathbb{Z}[k-h,k]}\alpha_1(s) - \max_{s\in\mathbb{Z}[k-h,k]}\alpha_0(s)\right] \\
&= f\left(k,\alpha_1(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_1(s)\right) + Q_0(k)[\alpha_1(k)-\alpha_0(k)] \\
&\quad + \left[f\left(k,\alpha_0(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_0(s)\right) - f\left(k,\alpha_1(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_1(s)\right)\right] \\
&\quad + q_0(k)\left[\max_{s\in\mathbb{Z}[k-h,k]}\alpha_1(s) - \max_{s\in\mathbb{Z}[k-h,k]}\alpha_0(s)\right] \\
&\le f\left(k,\alpha_1(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_1(s)\right).
\end{aligned}
\tag{4.19}
$$

Thus, the function $\alpha_1(k)$ is a lower solution of (2.1), (2.2) on $\mathbb{Z}[a+1-h,T]$.

In a similar way, we can prove that the function $\beta_1(k)$ is an upper solution of (2.1), (2.2) on the interval $\mathbb{Z}[a+1-h,T]$.

Analogously, we can construct two sequences of functions $\{\alpha_n(k)\}_{n=1}^{\infty}$ and $\{\beta_n(k)\}_{n=1}^{\infty}$. If the functions $\alpha_j(k)$ and $\beta_j(k)$, $j = 1,2,\dots,n-1,n$, are obtained such that α_j, $\beta_j \in S(\alpha_0,\beta_0)$ and the claims (a), (b), (c) of Theorem 4.1 are satisfied, then we consider the initial value problem for the linear difference equation with "maxima"

$$
\Delta x(k-1) = Q_n(k)x(k) + q_n(k)\max_{s\in\mathbb{Z}[k-h,k]}x(s) + P_n(k), \quad k \in \mathbb{Z}[a+1,T], \tag{4.20}
$$

$$
x(k) = \varphi(k), \quad k \in \mathbb{Z}[a+1-h,a], \tag{4.21}
$$

and the initial value problem for the linear difference equation with "maxima"

$$\begin{aligned} \Delta x(k-1) &= Q_n(k)x(k) + q_n(k)\max_{s\in\mathbb{Z}[k-h,k]} x(s) + R_n(k), \quad k\in\mathbb{Z}[a+1,T], \\ x(k) &= \varphi(k), \quad k\in\mathbb{Z}[a+1-h,a], \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} P_n(k) &= f\left(k,\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s)\right) - Q_n(k)\alpha_n(k) - q_n(k)\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s), \\ R_n(k) &= f\left(k,\beta_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)\right) - Q_n(k)\beta_n(k) - q_n(k)\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s), \\ Q_n(k) &= F_x\left(k,\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s)\right) - G_x\left(k,\beta_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)\right) \geq 0, \\ q_n(k) &= F_y\left(k,\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s)\right) - G_y\left(k,\beta_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)\right) \geq 0. \end{aligned} \tag{4.23}$$

Since $\alpha_j,\beta_j\in S(\alpha_0,\beta_0)$, the first derivatives of the function $F(k,x,y)$ and $G(k,x,y)$ are nondecreasing in $\Omega(\alpha_0,\beta_0)$, and inequalities (4.7) hold, we obtain $Q_n(k)\leq M(k)$, $q_n(k)\geq q_0(k)\geq 0$ and $q_n(k)\leq N(k)$, that is, $Q_n(k)+q_n(k)\leq M(k)+N(k)<1$. Therefore, according to Lemma 3.1, the initial value problems (4.20), (4.21), and (4.22) have unique solutions $\alpha_{n+1}(k)$ and $\beta_{n+1}(k)$, $k\in\mathbb{Z}[a+1-h,T]$, correspondingly.

The proof that the functions $\alpha_{n+1},\beta_{n+1}\in S(\alpha_n,\beta_n)$, $\alpha_{n+1}(k)\leq\beta_{n+1}(k)$, and they are lower/upper solutions of (2.1), (2.2) on the interval $\mathbb{Z}[a+1-h,T]$ is the same as in the case of $n=1$ and we omit it.

For any fixed $k\in\mathbb{Z}[a+1-h,T]$, the sequences $\{\alpha_n(k)\}_{n=0}^{\infty}$ and $\{\beta_n(k)\}_{n=0}^{\infty}$ are monotone nondecreasing and monotone nonincreasing, respectively, and they are bounded by $\alpha_0(k)$ and $\beta_0(k)$. Therefore, they are convergent on $\mathbb{Z}[a+1-h,T]$, that is, there exist functions $V,W:\mathbb{Z}[a+1-h,T]\to\mathbb{R}$ such that

$$\lim_{n\to\infty}\alpha_n(k) = V(k), \qquad \lim_{n\to\infty}\beta_n(k) = W(k). \tag{4.24}$$

From inequalities (4.7), it follows that $V,W\in S(\alpha_0,\beta_0)$.

Now, we will prove that for any $k\in\mathbb{Z}[a+1,T]$ the following equality holds:

$$\lim_{n\to\infty}\left[\max_{\xi\in\mathbb{Z}[k-h,k]}\alpha_n(\xi)\right] = \max_{\xi\in\mathbb{Z}[k-h,k]}\left[\lim_{n\to\infty}\alpha_n(\xi)\right]. \tag{4.25}$$

Let $k\in\mathbb{Z}[a+1,T]$ be fixed. We denote $\max_{\xi\in\mathbb{Z}[k-h,k]}\alpha_n(\xi)=A_n$. From inequalities (4.7) for every $\xi\in\mathbb{Z}[k-h,k]$, the inequalities $\alpha_{n-1}(\xi)\leq\alpha_n(\xi)\leq A_n$ hold and thus, $A_{n-1}\leq A_n$, that is, the sequence $\{A_n\}_{n=0}^{\infty}$ is monotone nondecreasing and bounded from above by $\beta_0(k)$. Therefore, there exists the limit $A=\lim_{n\to\infty}A_n$.

From the monotonicity of the sequence of the lower solutions $\alpha_n(k)$, we get that for $\xi\in\mathbb{Z}[k-h,k]$ it is fulfilled $\alpha_n(\xi)\leq V(\xi)$. Let $\eta\in\mathbb{Z}[k-h,k]$ be such that $\max_{\xi\in\mathbb{Z}[k-h,k]}V(\xi)=V(\eta)$.

From the inequalities $\alpha_n(\eta) \leq A_n \leq A$ for every $n = 0, 1, 2, \ldots$ it follows $V(\eta) \leq A$. Assume that $V(\eta) < A$. Then there exists a natural number N such that the inequalities $V(\eta) < A_N \leq A$ hold. Therefore, there exists $\xi \in \mathbb{Z}[k-h, k]$ such that $\alpha_N(\xi) = \max_{\xi \in \mathbb{Z}[k-h,k]} \alpha_N(\xi) = A_N$ or $V(\eta) < \alpha_N(\xi) \leq V(\xi)$. The obtained contradiction proves the validity of the required inequality (4.25).

Analogously, we can show that the functions $\beta_n(k)$ also satisfy (4.25).

Now, we will prove that the function $V(k)$ is a solution of the IVP (2.1), (2.2) on $\mathbb{Z}[a+1-h, T]$.

Let $k \in \mathbb{Z}[a+1-h, a]$. Take a limit as $n \to \infty$ in (4.21) and get $V(k) = \varphi(k)$.

Therefore, the function $V(k)$ satisfies equality (2.2) for $k \in \mathbb{Z}[a+1-h, a]$.

Let $k \in \mathbb{Z}[a+1, T]$. Taking a limit in (4.20) as $n \to \infty$ and applying (4.25), we obtain the function $V(k)$ satisfies equality (2.1) for $k \in \mathbb{Z}[a+1, T]$.

In a similar way, we can prove that $W(k)$ is a solution of the IVP (2.1), (2.2).

Therefore, we obtain two solutions of (2.1), (2.2) in $S(\alpha_0, \beta_0)$.

In the case of uniqueness of the solution of (2.1), (2.2) in $S(\alpha_0, \beta_0)$, we have $V(k) = W(k)$ for $k \in \mathbb{Z}[a-h+1, T]$. In the case of nonuniqueness, let $u \in S(\alpha_0, \beta_0)$ be another solution of (2.1), (2.2). Then it is easy to prove that $V(k) \leq u(k) \leq W(k)$, that is, V is the minimal solution and W is the maximal solution of (2.1), (2.2) in $S(\alpha_0, \beta_0)$.

We will prove that the convergence of the sequences $\{\alpha_n(k)\}_{n=0}^{\infty}$, $\{\beta_n(k)\}_{n=0}^{\infty}$ is quadratic. Let $x(k)$ be a solution of (2.1), (2.2) in $S(\alpha_0, \beta_0)$.

Define the functions $\widetilde{A}_{n+1}, \widetilde{B}_{n+1} : \mathbb{Z}[a+1-h, T] \to \mathbb{R}_+$, $n = 0, 1, \ldots$ by the equalities

$$\widetilde{A}_{n+1}(k) = x(k) - \alpha_{n+1}(k), \qquad \widetilde{B}_{n+1}(k) = \beta_{n+1}(k) - x(k). \tag{4.26}$$

It is obvious that $\widetilde{A}_{n+1}(k) = 0$ for $k \in \mathbb{Z}[a+1-h, a]$.

Let $k \in [a+1, T]$. According to the definitions of the functions $\widetilde{A}_{n+1}(k)$, $\alpha_{n+1}(k)$ and the condition (2) of Theorem 4.1, we get

$$\begin{aligned}
\Delta \widetilde{A}_{n+1}(k-1) = {} & f\left(k, x(k), \max_{s \in \mathbb{Z}[k-h,k]} x(s)\right) - f\left(k, \alpha_n(k), \max_{s \in \mathbb{Z}[k-h,k]} \alpha_n(s)\right) \\
& + Q_n(k) \widetilde{A}_{n+1}(k) + q_n(k) \left(\max_{s \in \mathbb{Z}[k-h,k]} x(s) - \max_{s \in \mathbb{Z}[k-h,k]} \alpha_{n+1}(s)\right) \\
& - Q_n(k) \widetilde{A}_n(k) - q_n(k) \left(\max_{s \in \mathbb{Z}[k-h,k]} x(s) - \max_{s \in \mathbb{Z}[k-h,k]} \alpha_n(s)\right) \\
\leq {} & Q_n(k) \widetilde{A}_{n+1}(k) + q_n(k) \max_{s \in \mathbb{Z}[k-h,k]} \widetilde{A}_{n+1}(s) \\
& + \left[F_x\left(k, x(k), \max_{s \in \mathbb{Z}[k-h,k]} x(s)\right) - G_x\left(k, \alpha_n(k), \max_{s \in \mathbb{Z}[k-h,k]} \alpha_n(s)\right) - Q_n(k)\right] \widetilde{A}_n(k) \\
& + \left[F_y\left(k, x(k), \max_{s \in \mathbb{Z}[k-h,k]} x(s)\right) - G_y\left(k, \alpha_n(k), \max_{s \in \mathbb{Z}[k-h,k]} \alpha_n(s)\right) - q_n(k)\right] \\
& \times \left(\max_{s \in \mathbb{Z}[k-h,k]} x(s) - \max_{s \in \mathbb{Z}[k-h,k]} \alpha_n(s)\right).
\end{aligned} \tag{4.27}$$

According to the mean value theorem, there exist points ζ_i and ξ_j, $i = 1, 3$, $j = 2, 4$ such that

$$\begin{aligned} &\alpha_n(k) \leq \zeta_i \leq x(k), \qquad \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s) \leq \zeta_j \leq \max_{s\in\mathbb{Z}[k-h,k]} x(s), \\ &\alpha_n(k) \leq \xi_i \leq \beta_n(k), \qquad \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s) \leq \xi_j \leq \max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s), \end{aligned} \tag{4.28}$$

$$\begin{aligned} &F_x\left(k, x(k), \max_{s\in\mathbb{Z}[k-h,k]} x(s)\right) - G_x\left(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)\right) - Q_n(k) \\ &\quad = F_x\left(k, x(k), \max_{s\in\mathbb{Z}[k-h,k]} x(s)\right) - F_x\left(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} x(s)\right) \\ &\qquad + F_x\left(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} x(s)\right) - F_x\left(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)\right) \\ &\qquad + G_x\left(k, \beta_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)\right) - G_x\left(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)\right) \\ &\qquad + G_x\left(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)\right) - G_x\left(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)\right) \\ &\quad = F_{xx}\left(k, \zeta_1, \max_{s\in\mathbb{Z}[k-h,k]} x(s)\right) \widetilde{A}_n(k) \\ &\qquad + F_{xy}(k, \alpha_n(k), \zeta_2)\left(\max_{s\in\mathbb{Z}[k-h,k]} x(s) - \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)\right) \\ &\qquad + G_{xx}\left(k, \zeta_1, \max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)\right)(\beta_n(k) - \alpha_n(k)) \\ &\qquad + G_{xy}(k, \alpha_n(k), \zeta_2)\left(\max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s) - \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)\right), \end{aligned} \tag{4.29}$$

$$\begin{aligned} &F_y\left(k, x(k), \max_{s\in\mathbb{Z}[k-h,k]} x(s)\right) - G_y\left(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)\right) - q_n(k) \\ &\quad = F_{yx}\left(k, \xi_3, \max_{s\in\mathbb{Z}[k-h,k]} x(s)\right) \widetilde{A}_n(k) \\ &\qquad + F_{yy}(k, \alpha_n(k), \xi_4)\left(\max_{s\in\mathbb{Z}[k-h,k]} x(s) - \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)\right) \\ &\qquad + G_{yx}\left(k, \xi_3, \max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)\right)(\beta_n(k) - \alpha_n(k)) \\ &\qquad + G_{yy}(k, \alpha_n(k), \xi_4)\left(\max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s) - \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)\right). \end{aligned} \tag{4.30}$$

Then the following inequalities are valid:

$$\begin{gathered}
\widetilde{A}_n(k)\left[\beta_n(k)-\alpha_n(k)\right] \le \widetilde{A}_n(k)\left[\widetilde{A}_n(k)+\widetilde{B}_n(k)\right] \le \frac{3}{2}\widetilde{A}_n^2(k)+\frac{1}{2}\widetilde{B}_n^2(k), \\
\widetilde{A}_n(k)\left[\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)-\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s)\right] \le \frac{3}{2}\left\|\widetilde{A}_n\right\|^2+\frac{1}{2}\left\|\widetilde{B}_n\right\|^2, \\
\widetilde{A}_n(k)\left[\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)-\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s)\right] \le \frac{3}{2}\left\|\widetilde{A}_n\right\|^2+\frac{1}{2}\left\|\widetilde{B}_n\right\|^2, \\
\max_{s\in\mathbb{Z}[k-h,k]}\widetilde{A}_n(s)\left[\beta_n(k)-\alpha_n(k)\right] \le \frac{3}{2}\left\|\widetilde{A}_n\right\|^2+\frac{1}{2}\left\|\widetilde{B}_n\right\|^2, \\
\max_{s\in\mathbb{Z}[k-h,k]}\widetilde{A}_n(s)\left[\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)-\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s)\right] \le \frac{3}{2}\left\|\widetilde{A}_n\right\|^2+\frac{1}{2}\left\|\widetilde{B}_n\right\|^2.
\end{gathered} \tag{4.31}$$

From inequalities (4.29) and (4.31), we obtain

$$\begin{aligned}
&\left[F_x\left(k,x(k),\max_{s\in\mathbb{Z}[k-h,k]}x(s)\right)-G_x\left(k,\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s)\right)-Q_n(k)\right]\widetilde{A}_n(k) \\
&\quad\le F_{xx}\left(k,\zeta_1,\max_{s\in\mathbb{Z}[k-h,k]}x(s)\right)\widetilde{A}_n^2(k) \\
&\qquad+F_{xy}(k,\alpha_n(k),\zeta_2)\left(\frac{3}{2}\left|\left|\widetilde{A}_n\right|\right|^2+\frac{1}{2}\left|\left|\widetilde{B}_n\right|\right|^2\right) \\
&\qquad+G_{xx}\left(k,\zeta_1,\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)\right)\left(\frac{3}{2}\left\|\widetilde{A}_n\right\|^2+\frac{1}{2}\left\|\widetilde{B}_n\right\|^2\right) \\
&\qquad+G_{xy}(k,\alpha_n(k),\zeta_2)\left(\frac{3}{2}\left\|\widetilde{A}_n\right\|^2+\frac{1}{2}\left\|\widetilde{B}_n\right\|^2\right).
\end{aligned} \tag{4.32}$$

From inequalities (4.30), (4.31), we get

$$\begin{aligned}
&\left[F_y\left(k,x(k),\max_{s\in\mathbb{Z}[k-h,k]}x(s)\right)-G_y\left(k,\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s)\right)-q_n(k)\right] \\
&\quad\times\left(\max_{s\in\mathbb{Z}[k-h,k]}x(s)-\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s)\right) \\
&\quad\le F_{yx}\left(k,\xi_3,\max_{s\in\mathbb{Z}[k-h,k]}x(s)\right)\widetilde{A}_n^2(k) \\
&\qquad+F_{yy}(k,\alpha_n(k),\xi_4)\left(\frac{3}{2}\left\|\widetilde{A}_n\right\|^2+\frac{1}{2}\left\|\widetilde{B}_n\right\|^2\right) \\
&\qquad+G_{yx}\left(k,\xi_3,\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)\right)\left(\frac{3}{2}\left\|\widetilde{A}_n\right\|^2+\frac{1}{2}\left\|\widetilde{B}_n\right\|^2\right) \\
&\qquad+G_{yy}(k,\alpha_n(k),\xi_4)\left(\frac{3}{2}\left\|\widetilde{A}_n\right\|^2+\frac{1}{2}\left\|\widetilde{B}_n\right\|^2\right).
\end{aligned} \tag{4.33}$$

Since the second derivatives of the functions F, G are continuous and bounded in $\Omega(\alpha_0, \beta_0)$, it follows from inequalities in (4.27), (4.32), and (4.33), there exist positive constants L_k, N_k such that

$$\begin{aligned}\Delta \widetilde{A}_{n+1}(k-1) \leq{}& Q_n(k)\widetilde{A}_{n+1}(k) + q_n(k)\max_{s\in\mathbb{Z}[k-h,k]}\widetilde{A}_{n+1}(s)\\ &+ L_k\left\|\widetilde{A}_n\right\|^2 + N_k\left\|\widetilde{B}_n\right\|^2.\end{aligned}\tag{4.34}$$

Therefore,

$$\begin{aligned}\widetilde{A}_{n+1}(k) \leq{}& \widetilde{A}_{n+1}(k-1) + Q_n(k)\widetilde{A}_{n+1}(k) + q_n(k)\max_{s\in\mathbb{Z}[k-h,k]}\widetilde{A}_{n+1}(s)\\ &+ L_k\left\|\widetilde{A}_n\right\|^2 + N_k\left\|\widetilde{B}_n\right\|^2,\\ &\cdots \quad \cdots \quad \cdots\\ \widetilde{A}_{n+1}(k-1) \leq{}& \widetilde{A}_{n+1}(k-2) + Q_n(k-1)\widetilde{A}_{n+1}(k-1)\\ &+ q_n(k-1)\max_{s\in\mathbb{Z}[k-h-1,k-1]}\widetilde{A}_{n+1}(s) + L_{k-1}\left\|\widetilde{A}_n\right\|^2 + N_{k-1}\left\|\widetilde{B}_n\right\|^2,\\ \widetilde{A}_{n+1}(a+1) \leq{}& \widetilde{A}_{n+1}(a) + Q_n(a+1)\widetilde{A}_{n+1}(a+1)\\ &+ q_n(a+1)\max_{s\in\mathbb{Z}[a+1-h,a+1]}\widetilde{A}_{n+1}(s) + L_{a+1}\left\|\widetilde{A}_n\right\|^2 + N_{a+1}\left\|\widetilde{B}_n\right\|^2.\end{aligned}\tag{4.35}$$

From inequalities in (4.35), we obtain

$$\begin{aligned}\widetilde{A}_{n+1}(k) \leq{}& \widetilde{L}_n\left(\left\|\widetilde{A}_n\right\|^2 + \left\|\widetilde{B}_n\right\|^2\right) + \sum_{l=a+1}^{k}\left[Q_n(l)\widetilde{A}_{n+1}(l) + q_n(l)\max_{\eta\in[l-h,l]}\widetilde{A}_{n+1}(\eta)\right],\\ &\qquad k\in\mathbb{Z}[a+1,T],\\ \widetilde{A}_{n+1}(k) ={}& 0 \leq \widetilde{L}_n\left(\left\|\widetilde{A}_n\right\|^2 + \left\|\widetilde{B}_n\right\|^2\right), \quad k\in\mathbb{Z}[a-h+1,a],\end{aligned}\tag{4.36}$$

where $\widetilde{L}_n = \sum_{j=a+1}^{T}(L_j + N_j)$.

According to Lemma 3.5 from inequalities in (4.36), it follows

$$\widetilde{A}_{n+1}(k) \leq \frac{\widetilde{L}_n\left(\left\|\widetilde{A}_n\right\|^2 + \left\|\widetilde{B}_n\right\|^2\right)}{\prod_{l=a+1}^{k}\left(1 - q_n(l) - Q_n(l)\right)}.\tag{4.37}$$

From (4.37) and the condition (2) of Theorem 4.1, it follows that there exist positive constants λ_i, where $i = 1, 2$, such that

$$\widetilde{A}_{n+1}(k) \leq \lambda_1 \left\| \widetilde{A}_n \right\|^2 + \lambda_2 \left\| \widetilde{B}_n \right\|^2, \quad k \in \mathbb{Z}[a+1, T]. \tag{4.38}$$

In a similar way, we can prove that there exist positive constants μ_j, where $j = 1, 2$, such that

$$\widetilde{B}_{n+1}(k) \leq \mu_1 \left\| \widetilde{B}_n \right\|^2 + \mu_2 \left\| \widetilde{A}_n \right\|^2, \quad k \in \mathbb{Z}[a+1, T]. \tag{4.39}$$

Inequalities (4.38), (4.39) and the definitions of the functions $\widetilde{A}_{n+1}(k)$, $\widetilde{B}_{n+1}(k)$ imply the validity of (4.8), that is, the convergence of the monotone sequences $\{\alpha_n(k)\}_{n=0}^{\infty}$ and $\{\beta_n(k)\}_{n=0}^{\infty}$ is quadratic. $\square$

5. Application

Now, we will give an example to illustrate the suggested above scheme for approximate obtaining of a solution.

Consider the following nonlinear difference equation with "maxima":

$$\Delta u(k-1) = \frac{1}{2 - 0.5u(k)} - \frac{1}{2 + 0.5 \max_{s \in \mathbb{Z}[k-2,k]} u(s)} - u(k-1), \quad k \in \mathbb{Z}[1, 3], \tag{5.1}$$

with an initial condition

$$u(k) = 0, \quad k \in \mathbb{Z}[-1, 0]. \tag{5.2}$$

The function $\alpha_0(k) = -1$, $k \in \mathbb{Z}[-1, 3]$, is a lower solution of (5.1), (5.2) because the inequality $0 < 1/(2 - 0.5(-1)) - 1/(2 + 0.5(-1)) + 1 = -4/15 + 1 = 11/15$ holds.

The function $\beta_0(k) = 1$, $k \in \mathbb{Z}[-1, 3]$, is an upper solution of (5.1), (5.2) because the inequality $0 > 1/1.5 - 1/2.5 - 1 = -11/15$ holds.

The conditions of Theorem 4.1 are satisfied since $F_{xx}(k, x, y) = 0.5/(2 - 0.5x)^3 > 0$ and $G_{yy}(k, x, y) = 0.5/(2 + 0.5y)^3$ for $-1 \leq x, y \leq 1$. Also, the inequality (4.5) holds, because in this case $M(k) = 0.5/(1.5)^2 = 2/9$, $N(k) = -(-0.5/(1.5)^2) = 2/9$ and $M(k) + N(k) < 1$.

According to Theorem 4.1, the initial value problem (5.1), (5.2) has a solution which is between $\alpha_0(k) = -1$ and $\beta_0(k) = 1$. It is obviously the problem (5.1), (5.2) has a zero solution. This solution also could be obtained by constructing two sequences of successive approximations.

Table 1: Values of successive approximations $\alpha_n(k)$ and $\beta_n(k)$, $n = 1, 2, 3, 4, 5$.

k	1	2	3
$\beta_1(k)$	0.126984126984127	0.126984126984127	0.126984126984127
$\beta_2(k)$	0.00245312888643079	0.00245312888643079	0.0025469207078195
$\beta_3(k)$	5.93137225955726E-07	5.93137225955726E-07	9.79796674723704E-07
$\beta_4(k)$	2.96900520626677E-14	2.96900520626677E-14	9.99338575141566E-14
$\beta_5(k)$	2.13083324289133E-17	2.13083324289133E-17	1.43434637320051E-16
$\alpha_5(k)$	−1.16621056078124E-17	−1.16621056078124E-17	−6.48937233257787E-18
$\alpha_4(k)$	−4.08173696273438E-17	−4.08173696273438E-17	−2.74428886272319E-15
$\alpha_3(k)$	−7.22563191957548E-09	−7.22563191957548E-09	−2.43398432836776E-07
$\alpha_2(k)$	−0.000449855202856569	−0.000449855202856569	−0.00207365792650451
$\alpha_1(k)$	−0.115942028985507	−0.115942028985507	−0.126023944549464

The successive approximation $\alpha_n(k)$ is a solution of (4.20), (4.21) which is reduced to the following initial value problem:

$$\begin{aligned}\Delta\alpha_n(k-1) = {} & Q_{n-1}(k)\alpha_n(k) + q_{n-1}(k)\max_{s\in\mathbb{Z}[k-2,k]}\alpha_n(s)\\ & + \frac{1}{2-0.5\alpha_{n-1}(k)} - \alpha_n(k-1) - \frac{1}{2+0.5\max_{s\in\mathbb{Z}[k-2,k]}\alpha_{n-1}(s)}\\ & - Q_{n-1}(k)\alpha_{n-1}(k) - q_{n-1}(k)\max_{s\in\mathbb{Z}[k-2,k]}\alpha_{n-1}(s), \quad k\in\mathbb{Z}[1,3],\\ & \alpha_n(k) = 0, \quad k\in\mathbb{Z}[-1,0],\end{aligned} \tag{5.3}$$

and the successive approximation $\beta_n(k)$ is a solution of (4.22) which is reduced to the following initial value problem:

$$\begin{aligned}\Delta\beta_n(k-1) = {} & Q_{n-1}(k)\beta_n(k) + q_{n-1}(k)\max_{s\in\mathbb{Z}[k-2,k]}\beta_n(s)\\ & + \frac{1}{2-0.5\beta_{n-1}(k)} - \beta_n(k-1) - \frac{1}{2+0.5\max_{s\in\mathbb{Z}[k-2,k]}\beta_{n-1}(s)}\\ & - Q_{n-1}(k)\beta_{n-1}(k) - q_{n-1}(k)\max_{s\in\mathbb{Z}[k-2,k]}\beta_{n-1}(s), \quad k\in\mathbb{Z}[1,3],\\ & \beta_n(k) = 0, \quad k\in\mathbb{Z}[-1,0],\end{aligned} \tag{5.4}$$

where

$$Q_{n-1}(k) = \frac{0.5}{(2-0.5\alpha_{n-1}(k))^2}, \qquad q_{n-1}(k) = \frac{0.5}{\left(2+0.5\max_{s\in\mathbb{Z}[k-2,k]}\beta_{n-1}(s)\right)^2}. \tag{5.5}$$

Initial value problems (5.3) and (5.4) are solved by a computer program, using the algorithm given in the proof of Lemma 3.1 and the results are written in Table 1.

Table 1 demonstrates both sequences monotonically approach the exact zero solution. This illustrates the application of the proved above procedure for approximately obtaining of the solution.

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