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Research Article

An Iterative Algorithm for the Generalized Reflexive Solutions of the Generalized Coupled Sylvester Matrix Equations

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An iterative algorithm is constructed to solve the generalized coupled Sylvester matrix equations $(AXB - CYD, EXF - GYH) = (M, N)$, which includes Sylvester and Lyapunov matrix equations as special cases, over generalized reflexive matrices X and Y. When the matrix equations are consistent, for any initial generalized reflexive matrix pair $[X_1, Y_1]$, the generalized reflexive solutions can be obtained by the iterative algorithm within finite iterative steps in the absence of round-off errors, and the least Frobenius norm generalized reflexive solutions can be obtained by choosing a special kind of initial matrix pair. The unique optimal approximation generalized reflexive solution pair $[\widehat{X}, \widehat{Y}]$ to a given matrix pair $[X_0, Y_0]$ in Frobenius norm can be derived by finding the least-norm generalized reflexive solution pair $[\widetilde{X}^*, \widetilde{Y}^*]$ of a new corresponding generalized coupled Sylvester matrix equation pair $(A\widetilde{X}B - C\widetilde{Y}D, E\widetilde{X}F - G\widetilde{Y}H) = (\widetilde{M}, \widetilde{N})$, where $\widetilde{M} = M - AX_0B + CY_0D$, $\widetilde{N} = N - EX_0F + GY_0H$. Several numerical examples are given to show the effectiveness of the presented iterative algorithm.

1. Introduction

In this paper, the following notations are used. Let $\mathcal{R}^{m\times n}$ denote the set of all $m \times n$ real matrices. We denote by the superscript T the transpose of a matrix. In matrix space $\mathcal{R}^{m\times n}$, define inner product as $\langle A, B\rangle = \mathrm{tr}(B^T A)$ for all $A, B \in \mathcal{R}^{m\times n}$, where $\mathrm{tr}(A)$ denotes the trace of a matrix A. $\|A\|$ represents the Frobenius norm of A. $\mathcal{R}(A)$ represents the column space of A. $\mathrm{vec}(\cdot)$ represents the vector operator, that is, $\mathrm{vec}(A) = (\mathbf{a}_1^T, \mathbf{a}_2^T, \ldots, \mathbf{a}_n^T)^T \in \mathcal{R}^{mn}$ for the matrix $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) \in \mathcal{R}^{m\times n}$, $\mathbf{a}_i \in R^m$, $i = 1, 2, \ldots, n$. $A \otimes B$ stands for the Kronecker product of matrices A and B, $\text{diag}(A, B)$ denotes the block diagonal matrix with A and B and being the main diagonal elements orderly. I_n denotes the n-order identity matrix.

Definition 1.1 (see [1, 2]). A matrix $P \in \mathcal{R}^{n\times n}$ is said to be a generalized reflection matrix if P satisfies that $P^T = P$, $P^2 = I$.

Definition 1.2 (see [1, 2]). Let $P \in \mathcal{R}^{n\times n}$ and $Q \in \mathcal{R}^{n\times n}$ be two generalized reflection matrices. A matrix $A \in \mathcal{R}^{n\times n}$ is called generalized reflexive (or generalized antireflexive) with respect to the matrix pair (P, Q) if $PAQ = A$ (or $PAQ = -A$). The set of all n-by-n generalized reflexive matrices with respect to matrix pair (P, Q) is denoted by $\mathcal{R}_r^{n\times n}(P, Q)$.

The generalized reflexive and antireflexive matrices have many special properties and usefulness in engineering and scientific computations [1–6]. In particular, let $P = Q$, then a generalized reflexive matrix is called a reflexive matrix, which plays an important role in many areas and has been studied in [7–11]. Specially, let $X^T = X$, then a reflexive matrix X is called a generalized bisymmetric matrix, which has been studied in [12, 13]. Moreover, let $P = Q = J_n$, then a generalized reflexive matrix is the well-known centrosymmetric matrix, which has been widely and extensively studied in [14–17].

The generalized coupled Sylvester systems play a fundamental role in the various fields of engineering theory, particularly in control systems. The numerical solution of the generalized coupled Sylvester systems has been addressed in a large body of literature. Kågström and Westin [18] developed a generalized Schur method by applying the QZ algorithm to solve $(AXB - CYD, EXF - GYH) = (M, N)$. Ding and Chen [19] presented an iterative least squares solutions of $(AXB - CYD, EXF - GYH) = (M, N)$ based on a hierarchical identification principle [20], in addition, by applying the hierarchical identification principle, Kılıçman and Zhour [21] developed an iterative algorithm for obtaining the weighted least-squares solution. Recently, some finite iterative algorithms have also been developed to solve matrix equations. For more detail, we refer to [11, 13, 22–30]. Wang [31, 32] gave the bi(skew)symmetric and centrosymmetric solutions to the system of quaternion matrix equations $A_1X = C_1$, $A_3XB_3 = C_3$. Wang [33] also solved a system of matrix equations over arbitrary regular rings with identity. Chang and Wang [34] gave the necessary and sufficient conditions for the existence of and the expressions for the symmetric solutions of the matrix equations $AX + YA = C$, $AXA^T + BYB^T = C$, and $(A^TXA, B^TXB) = (C, D)$. Ding and Chen [25] also presented the gradient-based iterative algorithms by applying the gradient search principle and the hierarchical identification principle for the general coupled matrix equations $\sum_{j=1}^{p} A_{ij}X_jB_{ij} = M_i$, $i = 1, 2, \ldots, p$. Zhou et al. [35] proposed gradient-based iterative algorithms for solving the general coupled matrix equations with weighted least squares solutions. Wu et al. [36, 37] gave the finite iterative solutions to coupled Sylvester-conjugate matrix equations. Wu et al. [38] gave the finite iterative solutions to a class of complex matrix equations with conjugate and transpose of the unknowns. Jonsson and Kågström [39] proposed recursive block algorithms for solving the one-sided and coupled Sylvester matrix equations $(AX - YB, DX - YE) = (C, F)$. Jonsson and Kågström [40] also proposed recursive block algorithms for the two-sided and generalized Sylvester and Lyapunov matrix equations. Dehghan and Hajarian [7, 8] gave the reflexive and generalized bisymmetric matrices solutions of the generalized coupled Sylvester matrix equations $(AY - ZB, CY - ZD) = (E, F)$. Very recently, Dehghan and Hajarian [12] constructed an iterative algorithm to solve the generalized coupled Sylvester matrix equations $(AXB + CYD, EXF + GYH) = (M, N)$ over generalized bisymmetric matrices.

Huang et al. [13] present an iterative algorithm for the generalized coupled Sylvester matrix equations $(AY - ZB, CY - ZD) = (E, F)$ and its optimal approximation problem over generalized reflexive matrices solutions. In [30], the similar but different iterative algorithm is constructed to solve the generalized coupled Sylvester matrix equations $(AXB-CYD, EXF-GYH) = (M, N)$ and the optimal approximation problem over reflexive matrices. However, the generalized coupled Sylvester matrix equations $(AXB - CYD, EXF - GYH) = (M, N)$ and the optimal approximation over generalized reflexive matrices have not been solved.

In this paper, we will consider the following two problems.

Problem 1. Let $P \in \mathcal{R}^{m\times m}$, $Q \in \mathcal{R}^{n\times n}$, $R \in \mathcal{R}^{s\times s}$, and $S \in \mathcal{R}^{t\times t}$ be generalized reflection matrices. For given matrices $A \in \mathcal{R}^{p\times m}$, $B \in \mathcal{R}^{n\times q}$, $C \in \mathcal{R}^{p\times s}$, $D \in \mathcal{R}^{t\times q}$, $M \in \mathcal{R}^{p\times q}$, $E \in \mathcal{R}^{k\times m}$, $F \in \mathcal{R}^{n\times l}$, $G \in \mathcal{R}^{k\times s}$, $H \in \mathcal{R}^{t\times l}$, $N \in \mathcal{R}^{k\times l}$, find a pair of matrices $X \in \mathcal{R}_r^{m\times n}(P, Q)$, $Y \in \mathcal{R}_r^{s\times t}(R, S)$ such that

$$\begin{aligned} AXB - CYD &= M, \\ EXF - GYH &= N. \end{aligned} \tag{1.1}$$

Problem 2. When Problem 1 is consistent, let S_E denote the set of the generalized reflexive solutions of Problem 1, that is,

$$S_E = \left\{[X, Y] \mid AXB - CYD = M,\ EXF - GYH = N,\ Y \in \mathcal{R}_r^{m\times n}(P, Q),\ Z \in \mathcal{R}_r^{s\times t}(R, S)\right\}. \tag{1.2}$$

For a given matrix pair $[Y_0, Z_0] \in \mathcal{R}_r^{m\times n}(P, Q) \times \mathcal{R}_r^{s\times t}(R, S)$, find $[\widehat{Y}, \widehat{Z}] \in S_E$ such that

$$\left\|\widehat{Y} - Y_0\right\|^2 + \left\|\widehat{Z} - Z_0\right\|^2 = \min_{[Y,Z]\in S_E}\left\{\|Y - Y_0\|^2 + \|Z - Z_0\|^2\right\}. \tag{1.3}$$

The two-sided and generalized coupled Sylvester matrix equations (1.1) play a fundamental role in wide applications in several areas, such as stability theory, control theory, perturbation analysis, and some other fields of pure and applied mathematics. In addition, as special type of generalized coupled Sylvester matrix equations (1.1), the generalized Sylvester matrix equation $(AX - YB, CX - YD) = (E, F)$ arises in computing the deflating subspace of descriptor linear systems [18]. Wu et al. [36] presented some examples to show a motivation for studying (1.1). Problem 2 occurs frequently in experiment design, see for instance [41].

This paper is organized as follows. In Section 2, we will solve Problem 1 by constructing an iterative algorithm, that is, if Problem 1 is consistent, then for an arbitrary initial matrix pair $[Y_1, Z_1] \in \mathcal{R}_r^{m\times n}(P, Q) \times \mathcal{R}_r^{s\times t}(R, S)$, we can obtain a solution pair $[Y^*, Z^*]$ of Problem 1 within finite iterative steps in the absence of round-off errors. Let $X_1 = A^TKB^T + E^TLF^T + PA^TKB^TQ + PE^TLF^TQ$ and $Y_1 = -C^TKD^T - G^TLH^T - RC^TKD^TS - RG^TLH^TS$, where $K \in \mathcal{R}^{p\times q}$, $L \in \mathcal{R}^{k\times l}$ are arbitrary matrices, or more especially, let $X_1 = 0$ and $Y_1 = 0$, we can obtain the least Frobenius norm solutions of Problem 1. Then, in Section 3, we give the optimal approximate solution pair of Problem 2 by finding the least Frobenius norm generalized reflexive solution pair of the corresponding generalized coupled Sylvester matrix equations. In Section 4, several numerical examples are given to illustrate the application of our method. At last, some conclusions are drawn in Section 5.

2. An Iterative Algorithm for Solving Problem 1

In this section, we will first introduce an iterative algorithm to solve Problem 1, then prove that it is convergent. Then, we will give the least-norm generalized reflexive solutions of Problem 1 when an appropriate initial iterative matrix pair is chosen.

For the purpose of simplification, we introduce the following operators:

$$\Phi(X,Y) = AXB - CYD,$$
$$\Psi(X,Y) = EXF - GYH. \tag{2.1}$$

Algorithm 2.1. We have the following steps.

Step 1. Input matrices $A \in \mathcal{R}^{p\times m}$, $B \in \mathcal{R}^{n\times q}$, $C \in \mathcal{R}^{p\times s}$, $D \in \mathcal{R}^{t\times q}$, $M \in \mathcal{R}^{p\times q}$, $E \in \mathcal{R}^{k\times m}$, $F \in \mathcal{R}^{n\times l}$, $G \in \mathcal{R}^{k\times s}$, $H \in \mathcal{R}^{t\times l}$, $N \in \mathcal{R}^{k\times l}$, and four generalized reflection matrices $P \in \mathcal{R}^{m\times m}$, $Q \in \mathcal{R}^{n\times n}$, $R \in \mathcal{R}^{s\times s}$, $S \in \mathcal{R}^{t\times t}$.

Step 2. Choose two arbitrary matrices $X_1 \in \mathcal{R}_r^{m\times n}(P,Q)$, $Y_1 \in \mathcal{R}_r^{s\times t}(R,S)$. Compute

$$\begin{aligned}
R_1 &= \operatorname{diag}(M - \Phi(X_1,Y_1), N - \Psi(X_1,Y_1)),\\
U_1 &= \frac{1}{2}\Big[A^T(M-\Phi(X_1,Y_1))B^T + E^T(N-\Psi(X_1,Y_1))F^T\\
&\quad + PA^T(M-\Phi(X_1,Y_1))B^TQ + PE^T(N-\Psi(X_1,Y_1))F^TQ\Big],\\
V_1 &= \frac{1}{2}\Big[-C^T(M-\Phi(X_1,Y_1))D^T - G^T(N-\Psi(X_1,Y_1))H^T\\
&\quad - RC^T(M-\Phi(X_1,Y_1))D^TS - RG^T(N-\Psi(X_1,Y_1))H^TS\Big],\\
k &:= 1.
\end{aligned} \tag{2.2}$$

Step 3. If $R_k = 0$, then stop and $[X_k, Y_k]$ is the solution of the generalized coupled Sylvester matrix equation (1.1); else if $R_k \neq 0$, but $U_k = 0$ and $V_k = 0$, then stop and the generalized coupled Sylvester matrix equations (1.1) are not consistent over generalized reflexive matrices; else $k := k+1$.

Step 4. Compute

$$\begin{aligned}
X_k &= X_{k-1} + \frac{\|R_{k-1}\|^2}{\|U_{k-1}\|^2 + \|V_{k-1}\|^2}U_{k-1},\\
Y_k &= Y_{k-1} + \frac{\|R_{k-1}\|^2}{\|U_{k-1}\|^2 + \|V_{k-1}\|^2}V_{k-1},\\
R_k &= \operatorname{diag}(M - \Phi(X_k,Y_k), N - \Psi(X_k,Y_k))\\
&= R_{k-1} - \frac{\|R_{k-1}\|^2}{\|U_{k-1}\|^2 + \|V_{k-1}\|^2}\operatorname{diag}(\Phi(U_{k-1},V_{k-1}), \Psi(U_{k-1},V_{k-1})),
\end{aligned}$$

$$U_k = \frac{1}{2}\Big[A^T(M-\Phi(X_k,Y_k))B^T + E^T(N-\Psi(X_k,Y_k))F^T$$
$$+PA^T(M-\Phi(X_k,Y_k))B^TQ + PE^T(N-\Psi(X_k,Y_k))F^TQ\Big] + \frac{\|R_k\|^2}{\|R_{k-1}\|^2}U_{k-1},$$
$$V_k = \frac{1}{2}\Big[-C^T(M-\Phi(X_{k-1},Y_{k-1}))D^T - G^T(N-\Psi(X_{k-1},Y_{k-1}))H^T$$
$$-RC^T(M-\Phi(X_{k-1},Y_{k-1}))D^TS - RG^T(N-\Psi(X_{k-1},Y_{k-1}))H^TS\Big] + \frac{\|R_k\|^2}{\|R_{k-1}\|^2}V_{K-1}. \tag{2.3}$$

Step 5. Go to Step 3.

Obviously, it can be seen that $X_k, U_k \in R_r^{m\times n}(P,Q)$, $Y_k, V_k \in R_r^{s\times t}(R,S)$, where $k = 1,2,\ldots$.

Lemma 2.2. *For the sequences $\{R_i\}$, $\{U_i\}$, and $\{V_i\}$ generated by Algorithm 2.1, and $s \geq 2$, we have*

$$\operatorname{tr}\left(R_i^T R_j\right) = 0, \quad \operatorname{tr}\left(U_i^T U_j + V_i^T V_j\right) = 0, \quad i,j = 1,2,\ldots,s, \ i \neq j. \tag{2.4}$$

The proof of Lemma 2.2 is presented in Appendix A.

Lemma 2.3. *Suppose $[X^*, Y^*]$ is an arbitrary solution pair of Problem 1, then for any initial generalized reflexive matrix pair $[X_1, Y_1]$, we have*

$$\operatorname{tr}\left((X^* - X_i)^T U_i + (Y^* - Y_i)^T V_i\right) = \|R_i\|^2, \quad k = 1,2,\ldots, \tag{2.5}$$

where the sequences $\{X_i\}$,$\{Y_i\}$, $\{U_i\}$, $\{V_i\}$, and $\{R_i\}$ are generated by Algorithm 2.1.

The proof of Lemma 2.3 is presented in Appendix B.

Remark 2.4. If there exist, a positive number k such that $U_k = 0$ and $V_k = 0$ but $R_k \neq 0$, then by Lemma 2.3, we have that the generalized coupled Sylvester matrix equations (1.1) are not consistent over generalized reflexive matrices.

Theorem 2.5. *Suppose that Problem 1 is consistent, then for an arbitrary initial matrix pair $[X_1, Y_1] \in \mathcal{R}_r^{m\times n}(P,Q) \times \mathcal{R}_r^{s\times t}(R,S)$, a generalized reflexive solution pair of Problem 1 can be obtained with finite iteration steps in the absence of round-off errors.*

Proof. If $R_i \neq \mathbf{0}$, $i = 1,2,\ldots,pq+st$, by Lemma 2.3, we have $U_i \neq \mathbf{0}, V_i \neq \mathbf{0}$, $i = 1,2,\ldots,pq+st$, then we can compute $[X_{pq+st+1}, Y_{pq+st+1}]$ by Algorithm 2.1.

By Lemma 2.2, we have

$$\operatorname{tr}\left(R_{pq+st+1}^T R_i\right) = 0, \quad i = 1,2,\ldots,pq+st,$$
$$\operatorname{tr}\left(R_i^T R_j\right) = 0, \quad i,j = 1,2,\ldots,pq+st, \ i \neq j. \tag{2.6}$$

It can be seen that the set of $R_1, R_2, \ldots, R_{pq+st}$ is an orthogonal basis of the matrix subspace

$$S = \left\{ L \mid L = \operatorname{diag}(L_1, L_2),\ L_1 \in \mathcal{R}^{p\times q},\ L_2 \in \mathcal{R}^{s\times t} \right\}, \tag{2.7}$$

which implies that $R_{pq+st+1} = 0$, that is, $[X_{pq+st+1}, Y_{pq+st+1}] \in \mathcal{R}_r^{m\times n}(P,Q) \times \mathcal{R}_r^{s\times t}(R,S)$ is a solution pair of Problem 1. This completes the proof. □

To show the least Frobenius norm generalized reflexive solutions of Problem 1, we first introduce the following result.

Lemma 2.6 (see [42, Lemma 2.4]). *Suppose that the consistent system of linear equation $Ax = b$ has a solution $x^* \in R(A^T)$, then x^* is a unique least Frobenius norm solution of the system of linear equation.*

By Lemma 2.6, the following result can be obtained.

Theorem 2.7. *Suppose that Problem 1 is consistent. If we choose the initial iterative matrices $X_1 = A^TKB^T + E^TLF^T + PA^TKB^TQ + PE^TLF^TQ$ and $Y_1 = -C^TKD^T - G^TLH^T - RC^TKD^TS - RG^TLH^TS$, where $K \in \mathcal{R}^{p\times q}$, $L \in \mathcal{R}^{k\times l}$ are arbitrary matrices, especially, $X_1 = 0 \in \mathcal{R}^{m\times n}(P,Q)$ and $Y_1 = 0 \in \mathcal{R}^{s\times t}(R,S)$, then the solution pair $[Y^*, Z^*]$ generated by Algorithm 2.1 is the unique least Frobenius norm generalized reflexive solutions of Problem 1.*

Proof. We know the solvability of the generalized coupled Sylvester matrix equations (1.1) over generalized reflexive matrices is equivalent to the following matrix equations:

$$\begin{gathered} AXB - CYD = M, \\ EXF - GYH = N, \\ APXQB - CRYSD = M, \\ EPXQF - GRYSH = N. \end{gathered} \tag{2.8}$$

Then, the system of matrix equations (2.8) is equivalent to

$$\begin{pmatrix} B^T \otimes A & -D^T \otimes C \\ F^T \otimes E & -H^T \otimes G \\ B^TQ \otimes AP & -D^TS \otimes CR \\ F^TQ \otimes EP & -H^TS \otimes GR \end{pmatrix} \begin{pmatrix} \operatorname{vec}(X) \\ \operatorname{vec}(Y) \end{pmatrix} = \begin{pmatrix} \operatorname{vec}(M) \\ \operatorname{vec}(N) \\ \operatorname{vec}(M) \\ \operatorname{vec}(N) \end{pmatrix}. \tag{2.9}$$

Let $X_1 = A^TKB^T + E^TLF^T + PA^TKB^TQ + PE^TLF^TQ$ and $Y_1 = -C^TKD^T - G^TLH^T - RC^TKD^TS - RG^TLH^TS$, where $K \in \mathcal{R}^{p\times q}$, $L \in \mathcal{R}^{k\times l}$ are arbitrary matrices, then

$$\begin{aligned}
\begin{pmatrix} \operatorname{vec}(X_1) \\ \operatorname{vec}(Y_1) \end{pmatrix} &= \begin{pmatrix} \operatorname{vec}\left(A^TKB^T + E^TLF^T + PA^TKB^TQ + PE^TLF^TQ\right) \\ \operatorname{vec}\left(-C^TKD^T - G^TLH^T - RC^TKD^TS - RG^TLH^TS\right) \end{pmatrix} \\
&= \begin{pmatrix} B \otimes A^T & F \otimes E^T & QB \otimes PA^T & QF \otimes PE^T \\ -D \otimes C^T & -H \otimes G^T & -SD \otimes RC^T & -SH \otimes RG^T \end{pmatrix} \begin{pmatrix} \operatorname{vec}(K) \\ \operatorname{vec}(L) \\ \operatorname{vec}(K) \\ \operatorname{vec}(L) \end{pmatrix} \\
&= \begin{pmatrix} B^T \otimes A & -D^T \otimes C \\ F^T \otimes E & -H^T \otimes G \\ B^TQ \otimes AP & -D^TS \otimes CR \\ F^TQ \otimes EP & -H^TS \otimes GR \end{pmatrix}^T \begin{pmatrix} \operatorname{vec}(K) \\ \operatorname{vec}(G) \\ \operatorname{vec}(K) \\ \operatorname{vec}(G) \end{pmatrix} \\
&\in \mathcal{R}\left(\begin{pmatrix} B^T \otimes A & -D^T \otimes C \\ F^T \otimes E & -H^T \otimes G \\ B^TQ \otimes AP & -D^TS \otimes CR \\ F^TQ \otimes EP & -H^TS \otimes GR \end{pmatrix}^T \right).
\end{aligned} \tag{2.10}$$

Furthermore, we can see that all X_k, Y_k generated by Algorithm 2.1 satisfy

$$\begin{pmatrix} \operatorname{vec}(X_k) \\ \operatorname{vec}(Y_k) \end{pmatrix} \in \mathcal{R}\left(\begin{pmatrix} B^T \otimes A & -D^T \otimes C \\ F^T \otimes E & -H^T \otimes G \\ B^TQ \otimes AP & -D^TS \otimes CR \\ F^TQ \otimes EP & -H^TS \otimes GR \end{pmatrix}^T \right), \tag{2.11}$$

by Lemma 2.6, we know that $[X^*, Y^*]$ is the least Frobenius norm generalized reflexive solution pair of the system of linear equations (2.9). Since vector operator is isomorphic, $[X^*, Y^*]$ is the unique least Frobenius norm generalized reflexive solution pair of the system of matrix equations (2.8), then $[X^*, Y^*]$ is the unique least Frobenius norm generalized reflexive solution pair of Problem 1. □

3. The Solution of Problem 2

In this section, we will show that the optimal approximate solutions of Problem 2 for a given generalized reflexive matrix pair can be derived by finding the least Frobenius norm generalized reflexive solutions of the corresponding generalized coupled Sylvester matrix equations.

When Problem 1 is consistent, the set of generalized reflexive solutions of Problem 1 denoted by S_E is not empty. For a given matrix pair $[X_0, Y_0] \in \mathcal{R}_r^{m\times n}(P,Q) \times \mathcal{R}_r^{s\times t}(R,S)$, we have

$$\begin{cases} AXB - CYD = M \\ EXF - GYH = N \end{cases} \Longleftrightarrow \begin{cases} A(X - X_0)B - C(Y - Y_0)D = M - AX_0B + CY_0D \\ E(X - X_0)F - G(Y - Y_0)H = N - EX_0F + GY_0H \end{cases} \tag{3.1}$$

Set $\widetilde{X} = X - X_0$, $\widetilde{Y} = Y - Y_0$, $\widetilde{M} = M - AX_0B + CY_0D$, $\widetilde{N} = N - EX_0F + GY_0H$, then Problem 2 is equivalent to that of finding the least Frobenius norm generalized reflexive solutions pair $[\widetilde{X}^*, \widetilde{Y}^*]$ of the corresponding generalized coupled Sylvester matrix equations

$$\begin{aligned} A\widetilde{X}B - C\widetilde{Y}D &= \widetilde{M}, \\ E\widetilde{X}F - G\widetilde{Y}H &= \widetilde{N}. \end{aligned} \tag{3.2}$$

By using Algorithm 2.1, let initial iteration matrix $\widetilde{X}_1 = A^TKB^T + E^TLF^T + PA^TKB^TQ + PE^TLF^TQ$ and $\widetilde{Y}_1 = -C^TKD^T - G^TLH^T - RC^TKD^TS - RG^TLH^TS$, or more especially, let $\widetilde{X}_1 = 0 \in \mathcal{R}_r^{m\times n}(P,Q)$ and $\widetilde{Y}_1 = 0 \in \mathcal{R}_r^{s\times t}(R,S)$, then we can get the least Frobenius norm generalized reflexive solution pair $[\widetilde{X}^*, \widetilde{Y}^*]$ of (3.2). Thus, the generalized reflexive solution pair of the problem 2 can be represented as $[\widehat{X}, \widehat{Y}] = [\widetilde{X}^* + X_0, \widetilde{Y}^* + Y_0]$.

4. Numerical Experiments

In this section, we will show several numerical examples to illustrate our results. All the tests are performed by MATLAB 7.8.

Example 4.1. Consider the generalized reflexive solutions of the generalized coupled Sylvester matrix equations $AXB - CYD = M, EXY - GYH = N$, where

$$A = \begin{pmatrix} 1 & 3 & -5 & 7 & -9 \\ 2 & 0 & 4 & 6 & -1 \\ 0 & -2 & 9 & 6 & -8 \\ 3 & 6 & 2 & 2 & -3 \\ -5 & 5 & -22 & -1 & -11 \\ 8 & 4 & -6 & -9 & -9 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 8 & -5 & 4 \\ -1 & 5 & -2 & 3 \\ 3 & 9 & 2 & -6 \\ -2 & 7 & -8 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 6 & -5 & 7 & -9 \\ 2 & 4 & 6 & -11 \\ 9 & -12 & 3 & -8 \\ 13 & 6 & 4 & -15 \\ -5 & 15 & -13 & -11 \\ 2 & 9 & -6 & -9 \end{pmatrix}, \quad D = \begin{pmatrix} 7 & 1 & 8 & -6 \\ -4 & 5 & -2 & 3 \\ 3 & -12 & 0 & 8 \\ 1 & 6 & 9 & 4 \\ -5 & 8 & -2 & 9 \end{pmatrix},$$

$$E=\begin{pmatrix}14&5&-1&7&1\\-2&3&-2&5&4\\13&4&2&-3&6\\-8&1&-5&4&8\end{pmatrix},\qquad F=\begin{pmatrix}1&3&-5&8&2\\-11&5&-6&2&5\\13&2&7&-9&7\\-9&6&-5&12&1\end{pmatrix},$$

$$G=\begin{pmatrix}1&2&-5&8\\-5&5&-7&3\\2&4&9&-6\\-3&7&-12&11\end{pmatrix},\qquad H=\begin{pmatrix}2&4&8&-5&4\\7&-1&5&-2&3\\6&3&9&2&-6\\5&-2&7&-8&1\\1&4&-3&-2&6\end{pmatrix},$$

$$M=\begin{pmatrix}519&1177&1701&1632\\-103&1583&-100&2382\\82&1800&1029&3308\\-514&839&-493&2458\\-753&1132&2683&-762\\-1164&258&858&408\end{pmatrix},\qquad N=\begin{pmatrix}-2426&964&-2653&2092&603\\-65&247&-919&291&788\\-1331&1547&-17&992&712\\-1684&-659&-2730&1756&-765\end{pmatrix}. \tag{4.1}$$

Let

$$P=\begin{pmatrix}0&0&0&0&-1\\0&0&0&1&0\\0&0&-1&0&0\\0&1&0&0&0\\-1&0&0&0&0\end{pmatrix},\qquad Q=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix},$$
$$R=\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&-1&0\\1&0&0&0\end{pmatrix},\qquad S=\begin{pmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&-1&0&0\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix} \tag{4.2}$$

be generalized reflection matrices.

We will find the generalized reflexive solutions of the matrix equations $AXB - CYD = M, EXY - GYH = N$ by using Algorithm 2.1. It can be verified that the matrix equations are consistent over generalized reflexive matrices and the solutions are

$$X^* = \begin{pmatrix} -2 & 9 & 2 & 5 \\ 3 & 1 & 11 & -1 \\ 7 & 3 & -7 & 3 \\ 11 & 1 & 3 & -1 \\ -2 & 5 & 2 & 9 \end{pmatrix}, \quad Y^* = \begin{pmatrix} 14 & 16 & -1 & 3 & 4 \\ 9 & 7 & 0 & 9 & 7 \\ -3 & -8 & -8 & 3 & 8 \\ 3 & 4 & 1 & 14 & 16 \end{pmatrix}. \tag{4.3}$$

Because of the influence of the error of calculation, the residual R_i is usually unequal to zero in the process of the iteration, where $i = 1, 2, \ldots$. For any chosen positive number ε; however, small enough, for example, $\varepsilon = 1.0000e - 010$, whenever $\|R_k\| < \varepsilon$, stop the iteration, X_k and Y_k are regarded to be generalized reflexive solutions of the matrix equations $AXB - CYD = M$, $EXY - GYH = N$. Choose an initially iterative matrix pair $[X_1, Y_1] \in \mathcal{R}_r^{5\times 4}(P, Q) \times \mathcal{R}_r^{4\times 5}(R, S)$, such as

$$X_1 = \begin{pmatrix} -1 & 2 & 2 & 4 \\ 6 & -1 & 3 & 2 \\ 7 & 8 & -7 & 8 \\ 3 & -2 & 6 & 1 \\ -2 & 4 & 1 & 2 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 3 & 4 & -1 & 3 & 7 \\ 9 & 1 & 0 & 9 & 1 \\ -3 & 1 & -8 & 3 & -1 \\ 3 & 7 & 1 & 3 & 4 \end{pmatrix}. \tag{4.4}$$

By Algorithm 2.1, we have

$$\begin{gathered} X_{30} = \begin{pmatrix} -2.0000 & 9.0000 & 2.0000 & 5.0000 \\ 3.0000 & 1.0000 & 11.0000 & -1.0000 \\ 7.0000 & 3.0000 & -7.0000 & 3.0000 \\ 11.0000 & 1.0000 & 3.0000 & -1.0000 \\ -2.0000 & 5.0000 & 2.0000 & 9.0000 \end{pmatrix}, \\ Y_{30} = \begin{pmatrix} 14.0000 & 16.0000 & -1.0000 & 3.0000 & 4.0000 \\ 9.0000 & 7.0000 & 0 & 9.0000 & 7.0000 \\ -3.0000 & -8.0000 & -8.0000 & 3.0000 & 8.0000 \\ 3.0000 & 4.0000 & 1.0000 & 14.0000 & 16.0000 \end{pmatrix}, \\ \|R_{30}\| = 2.9703e - 012 < \varepsilon. \end{gathered} \tag{4.5}$$

So we obtain the generalized reflexive solutions of the matrix equations $AXB - CYD = M$, $EXY - GYH = N$. The relative error of the solutions and the residual are shown in

Figure 1, where the relative error $REk = (\|X_k - X^*\| + \|Y_k - Y^*\|)/(\|X^*\| + \|Y^*\|)$ and the residual $Rk = \|R_k\|$.

Let

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.6}$$

by Algorithm 2.1, we have

$$\begin{gathered} X^* = X_{30} = \begin{pmatrix} -2.0000 & 9.0000 & 2.0000 & 5.0000 \\ 3.0000 & 1.0000 & 11.0000 & -1.0000 \\ 7.0000 & 3.0000 & -7.0000 & 3.0000 \\ 11.0000 & 1.0000 & 3.0000 & -1.0000 \\ -2.0000 & 5.0000 & 2.0000 & 9.0000 \end{pmatrix}, \\ Y^* = Y_{30} = \begin{pmatrix} 14.0000 & 16.0000 & -1.0000 & 3.0000 & 4.0000 \\ 9.0000 & 7.0000 & 0 & 9.0000 & 7.0000 \\ -3.0000 & -8.0000 & -8.0000 & 3.0000 & 8.0000 \\ 3.0000 & 4.0000 & 1.0000 & 14.0000 & 16.0000 \end{pmatrix}, \\ \|R_{30}\| = 8.2565e - 012 < \varepsilon. \end{gathered} \tag{4.7}$$

The relative error of the solutions and the residual are shown in Figure 2.

Example 4.2. Consider the unique least-norm generalized reflexive solutions of the matrix equations in Example 4.1. Let

$$\begin{gathered} K = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & 1 & -3 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & -2 & -1 \end{pmatrix}, \qquad L = \begin{pmatrix} -1 & 1 & -1 & 0 & 5 \\ 0 & 1 & -1 & 3 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & 0 & 1 & -3 & 6 \end{pmatrix}, \\ X_1 = A^T K B^T + C^T L D^T + P A^T K B^T Q + P C^T L D^T Q, \\ Y_1 = -E^T K F^T - G^T L H^T - R E^T K F^T S - R G^T L H^T S. \end{gathered} \tag{4.8}$$

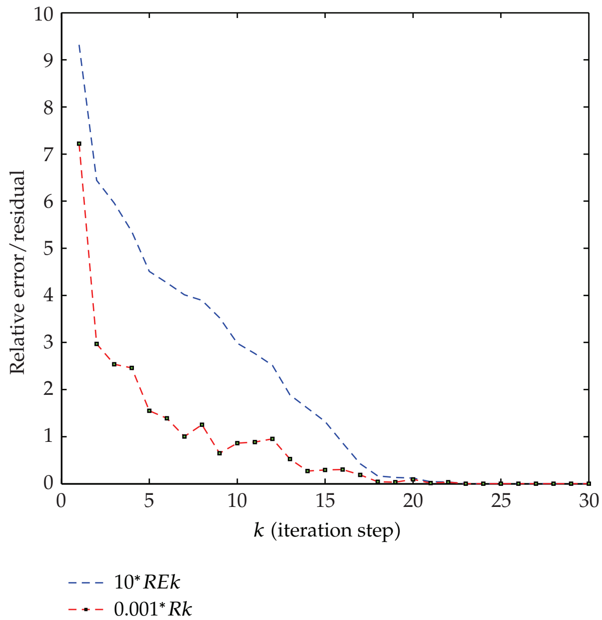

Figure 1: The relative error of the solutions and the residual for Example 4.1 with $X_1 \neq 0$, $Y_1 \neq 0$.

By using Algorithm 2.1, we have the least-norm generalized reflexive solutions of the matrix equations $AXB - CYD = M$, $EXY - GYH = N$ as follows:

$$X^* = X_{30} = \begin{pmatrix} -2.0000 & 9.0000 & 2.0000 & 5.0000 \\ 3.0000 & 1.0000 & 11.0000 & -1.0000 \\ 7.0000 & 3.0000 & -7.0000 & 3.0000 \\ 11.0000 & 1.0000 & 3.0000 & -1.0000 \\ -2.0000 & 5.0000 & 2.0000 & 9.0000 \end{pmatrix},$$

$$Y^* = Y_{30} = \begin{pmatrix} 14.0000 & 16.0000 & -1.0000 & 3.0000 & 4.0000 \\ 9.0000 & 7.0000 & 0 & 9.0000 & 7.0000 \\ -3.0000 & -8.0000 & -8.0000 & 3.0000 & 8.0000 \\ 3.0000 & 4.0000 & 1.0000 & 14.0000 & 16.0000 \end{pmatrix},$$

$$\|R_{30}\| = 2.3986e - 011e - 012 < \varepsilon. \tag{4.9}$$

The relative error of the solutions and the residual are shown in Figure 3.

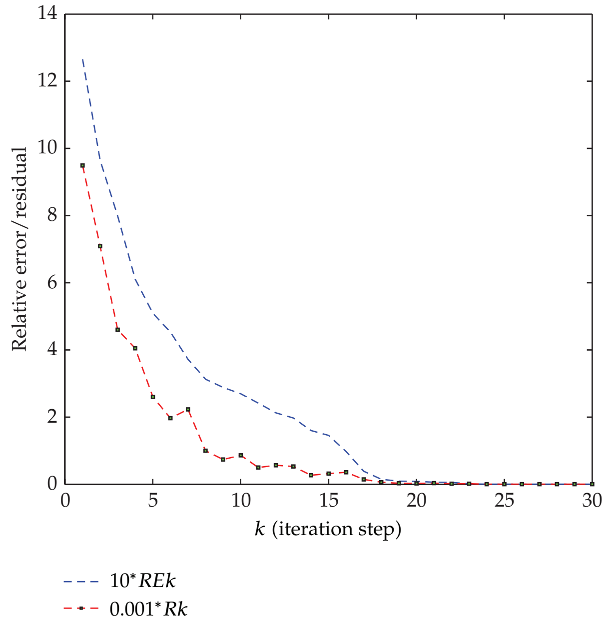

Figure 2: The relative error of the solutions and the residual for Example 4.1 with $X_1 = 0$, $Y_1 = 0$.

Example 4.3. Let S_E denote the set of all generalized reflexive solutions of the matrix equations in Example 4.1. For a given generalized reflexive matrices

$$X_0 = \begin{pmatrix} 3 & -1 & 2 & 2 \\ 3 & -2 & 0 & 0 \\ 1 & -3 & -1 & -3 \\ 0 & 0 & 3 & 2 \\ -2 & 2 & -3 & -1 \end{pmatrix}, \qquad Y_0 = \begin{pmatrix} 2 & 4 & -2 & 2 & 0 \\ 1 & 3 & 0 & 1 & 3 \\ 5 & -2 & 2 & -5 & 2 \\ 2 & 0 & 2 & 2 & 4 \end{pmatrix}, \tag{4.10}$$

we will find $[\widehat{X}, \widehat{Y}] \in S_E$, such that

$$\left\| \widehat{X} - X_0 \right\| + \left\| \widehat{Y} - Y_0 \right\| = \min_{[X,Y] \in S_E} \|X - X_0\| + \|Y - Y_0\|, \tag{4.11}$$

that is, find the optimal approximate generalized reflexive solution pair to the matrix pair $[X_0, Y_0]$ in S_E in Frobenius norm.

Let $\widetilde{X} = X - X_0$, $\widetilde{Y} = Y - Y_0$, $\widetilde{M} = M - AX_0B + CY_0D$, $\widetilde{N} = N - EX_0F + GY_0H$, by the method mentioned in Section 3, we can obtain the least-norm generalized reflexive solution pair $[\widetilde{X}^*, \widetilde{Y}^*]$ of the matrix equations $A\widetilde{X}B + C\widetilde{Y}D = \widetilde{M}, E\widetilde{X}F + G\widetilde{Y}H = \widetilde{N}$ by choosing the initial iteration matrices $\widetilde{X}_1 = 0$ and $\widetilde{Y}_1 = 0$, then by Algorithm 2.1, we have that

$$\widetilde{X}^* = \widetilde{X}^*_{30} = \begin{pmatrix} -5.0000 & 10.0000 & 0.0000 & 3.0000 \\ -0.0000 & 3.0000 & 11.0000 & -1.0000 \\ 6.0000 & 6.0000 & -6.0000 & 6.0000 \\ 11.0000 & 1.0000 & -0.0000 & -3.0000 \\ -0.0000 & 3.0000 & 5.0000 & 10.0000 \end{pmatrix},$$

$$\widetilde{Y}^* = \widetilde{Y}^*_{30} = \begin{pmatrix} 12.0000 & 12.0000 & 1.0000 & 1.0000 & 4.0000 \\ 8.0000 & 4.0000 & 0 & 8.0000 & 4.0000 \\ -8.0000 & -6.0000 & -10.0000 & 8.0000 & 6.0000 \\ 1.0000 & 4.0000 & -1.0000 & 12.0000 & 12.0000 \end{pmatrix}, \tag{4.12}$$

$$\|R_{30}\| = 6.3482e - 010 < \varepsilon = 1.0000e - 010$$

and the optimal approximate generalized reflexive solutions to the matrix pair $[X_0, Y_0]$ in Frobenius norm are

$$\widehat{X} = \widetilde{X}^*_{30} + X_0 = \begin{pmatrix} -2.0000 & 9.0000 & 2.0000 & 5.0000 \\ 3.0000 & 1.0000 & 11.0000 & -1.0000 \\ 7.0000 & 3.0000 & -7.0000 & 3.0000 \\ 11.0000 & 1.0000 & 3.0000 & -1.0000 \\ -2.0000 & 5.0000 & 2.0000 & 9.0000 \end{pmatrix},$$

$$\widehat{Y} = \widetilde{Y}^*_{30} + Y_0 = \begin{pmatrix} 14.0000 & 16.0000 & -1.0000 & 3.0000 & 4.0000 \\ 9.0000 & 7.0000 & 0 & 9.0000 & 7.0000 \\ -3.0000 & -8.0000 & -8.0000 & 3.0000 & 8.0000 \\ 3.0000 & 4.0000 & 1.0000 & 14.0000 & 16.0000 \end{pmatrix}. \tag{4.13}$$

The relative error of the solutions and the residual are shown in Figure 4, where the relative error $REk = (\|\widetilde{X}_k + X_0 - X^*\| + \|\widetilde{Y}_k + Y_0 - Y^*\|)/(\|X^*\| + \|Y^*\|)$ and the residual $Rk = \|R_k\|$.

5. Conclusions

In this paper, an efficient iterative algorithm is presented to solve the generalized coupled Sylvester matrix equations $AXB - CYD = M$, $EXY - GYH = N$ over generalized reflexive matrix pair $[X, Y] \in \mathcal{R}_r^{m\times n}(P, Q) \times \mathcal{R}_r^{s\times t}(R, S)$. When the matrix equations $AXB - CYD = M$, $EXY - GYH = N$ are consistent over generalized reflexive matrices X and Y, for any generalized reflexive initial iterative matrix pair $[X_1, Y_1] \in \mathcal{R}_r^{m\times n}(P, Q) \times \mathcal{R}_r^{s\times t}(R, S)$, the generalized reflexive solutions can be obtained by the iterative algorithm within finite

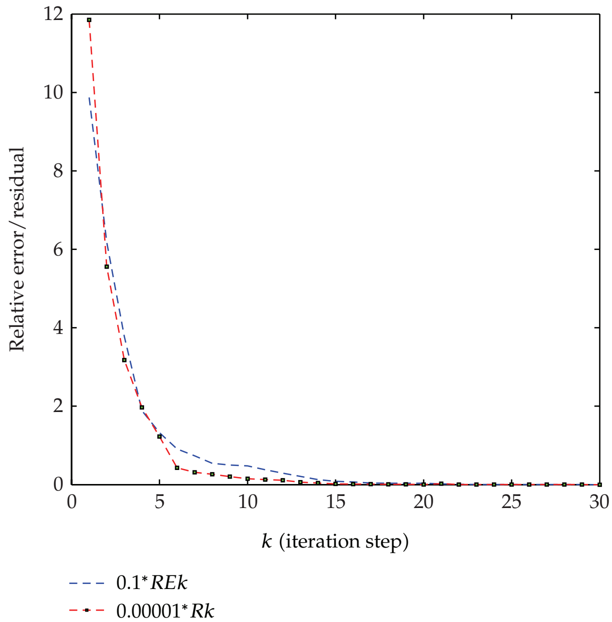

Figure 3: The relative error of the solutions and the residual for Example 4.2.

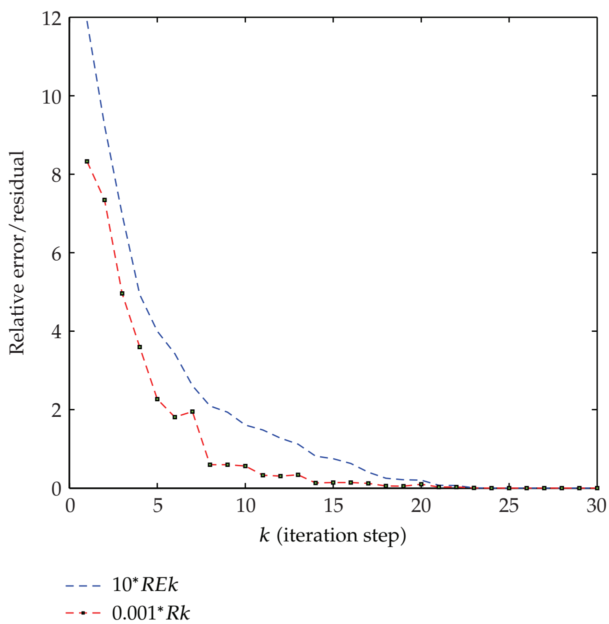

Figure 4: The relative error of the solutions and the residual for Example 4.3.

iterative steps in the absence of round-off errors. Let initial matrices $X_1 = A^TKB^T + E^TLF^T + PA^TKB^TQ + PE^TLF^TQ$ and $Y_1 = -C^TKD^T - G^TLH^T - RC^TKD^TS - RG^TLH^TS$, where $K \in \mathcal{R}^{p\times q}$, $L \in \mathcal{R}^{k\times l}$ are arbitrary matrices, especially, let $X_1 = 0 \in \mathcal{R}_r^{m\times n}(P,Q)$ and $Y_1 = 0 \in \mathcal{R}_r^{s\times t}(R,S)$, the unique least-norm generalized reflexive solutions of the matrix equations can be derived. Furthermore, the optimal approximate solutions of $AXB - CYD = M, EXY - GYH = N$ for a given generalized reflexive matrix pair $[X_0, Y_0] \in \mathcal{R}_r^{m\times n}(P,Q) \times \mathcal{R}_r^{s\times t}(R,S)$ can be derived by finding the least-norm generalized reflexive solutions of two new corresponding generalized coupled Sylvester matrix equations. Finally, several numerical examples are given to illustrate that our iterative algorithm is quite effective.

The results presented in this paper generalize some previous results [7, 12, 13, 30]. When $B = I$, $C = I$, $F = I$, $G = I$, $P = Q$, and $R = S$, then our results reduce to those in [7]. When $P = Q$, $R = S$, $X^T = X$, and $Y^T = Y$, the results in this paper reduce to those in [12]. When $B = I$, $C = I$, $F = I$, and $G = I$, then the results in this paper reduce to those in [13]. When $P = Q$ and $R = S$, then the results in this paper reduce to those in [30].

Appendices

A. The Proof of Lemma 2.2

Since $\mathrm{tr}(R_i^T R_j) = \mathrm{tr}(R_j^T R_i)$, $\mathrm{tr}(U_i^T U_j) = \mathrm{tr}(U_j^T U_i)$, and $\mathrm{tr}(V_i^T V_j) = \mathrm{tr}(V_j^T V_i)$ for all $i, j = 1, 2, \ldots, s$, we only need to prove that

$$\mathrm{tr}\left(R_i^T R_j\right) = 0, \quad \mathrm{tr}\left(U_i^T U_j + V_i^T V_j\right) = 0, \quad 1 \le j < i \le s. \tag{A.1}$$

We prove the conclusion by induction, and two steps are required.

Step 1. We will show that

$$\mathrm{tr}\left(R_{i+1}^T R_i\right) = 0, \quad \mathrm{tr}\left(U_{i+1}^T U_i + V_{i+1}^T V_i\right) = 0, \quad i = 1, 2, \ldots, s-1. \tag{A.2}$$

To prove this conclusion, we also use induction.

For $i = 1$, by Algorithm 2.1, we have that

$$\begin{aligned}
&\mathrm{tr}\left(R_2^T R_1\right) \\
&= \mathrm{tr}\left(\left[R_1 - \frac{\|R_1\|^2}{\|U_1\|^2 + \|V_1\|^2} \mathrm{diag}(\Phi(U_1, V_1), \Psi(U_1, V_1))\right]^T R_1\right) \\
&= \|R_1\|^2 - \frac{\|R_1\|^2}{\|U_1\|^2 + \|V_1\|^2} \mathrm{tr}\Big((\mathrm{diag}(\Phi(U_1, V_1), \Psi(U_1, V_1)))^T \\
&\qquad \times \mathrm{diag}(M - \Phi(X_1, Y_1), N - \Psi(X_1, Y_1))\Big) \\
&= \|R_1\|^2 - \frac{\|R_1\|^2}{\|U_1\|^2 + \|V_1\|^2} \\
&\quad \times \mathrm{tr}\left((\Phi(U_1, V_1))^T (M - \Phi(X_1, Y_1)) + (\Psi(U_1, V_1))^T (N - \Psi(X_1, Y_1))\right) \\
&= \|R_1\|^2 - \frac{\|R_1\|^2}{\|U_1\|^2 + \|V_1\|^2} \mathrm{tr}\Big(U_1^T A^T (M - \Phi(X_1, Y_1)) B^T + U_1^T E^T (N - \Psi(X_1, Y_1)) F^T \\
&\qquad - V_1^T C^T (M - \Phi(X_1, Y_1)) D^T - V_1^T G^T (N - \Psi(X_1, Y_1)) H^T\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \|R_1\|^2 - \frac{\|R_1\|^2}{\|U_1\|^2 + \|V_1\|^2} \\
&\quad \times \operatorname{tr}\Bigg(U_1^T \Bigg[\frac{A^T(M-\Phi(X_1,Y_1))B^T + E^T(N-\Psi(X_1,Y_1))F^T}{2} \\
&\qquad + \frac{A^T(M-\Phi(X_1,Y_1))B^T + E^T(N-\Psi(X_1,Y_1))F^T}{2} \\
&\qquad + \frac{PA^T(M-\Phi(X_1,Y_1))B^TQ + PE^T(N-\Psi(X_1,Y_1))F^TQ}{2} \\
&\qquad - \frac{PA^T(M-\Phi(X_1,Y_1))B^TQ + PE^T(N-\Psi(X_1,Y_1))F^TQ}{2} \Bigg] \\
&\quad + V_1^T \Bigg[\frac{-C^T(M-\Phi(X_1,Y_1))D^T - G^T(N-\Psi(X_1,Y_1))H^T}{2} \\
&\qquad + \frac{-C^T(M-\Phi(X_1,Y_1))D^T - G^T(N-\Psi(X_1,Y_1))H^T}{2} \\
&\qquad + \frac{-RC^T(M-\Phi(X_1,Y_1))D^TS - RG^T(N-\Psi(X_1,Y_1))H^TS}{2} \\
&\qquad - \frac{-RC^T(M-\Phi(X_1,Y_1))D^TS - RG^T(N-\Psi(X_1,Y_1))H^TS}{2} \Bigg] \Bigg) \\
&= \|R_1\|^2 - \frac{\|R_1\|^2}{\|U_1\|^2 + \|V_1\|^2} \\
&\quad \times \operatorname{tr}\Bigg(U_1^T \Bigg[\frac{A^T(M-\Phi(X_1,Y_1))B^T + E^T(N-\Psi(X_1,Y_1))F^T}{2} \\
&\qquad + \frac{PA^T(M-\Phi(X_1,Y_1))B^TQ + PE^T(N-\Psi(X_1,Y_1))F^TQ}{2} \Bigg] \\
&\quad + V_1^T \Bigg[\frac{-C^T(M-\Phi(X_1,Y_1))D^T - G^T(N-\Psi(X_1,Y_1))H^T}{2} \\
&\qquad + \frac{-RC^T(M-\Phi(X_1,Y_1))D^TS - RG^T(N-\Psi(X_1,Y_1))H^TS}{2} \Bigg] \Bigg) \\
&= \|R_1\|^2 - \frac{\|R_1\|^2}{\|U_1\|^2 + \|V_1\|^2} \operatorname{tr}\left(U_1^T U_1 + V_1^T V_1 \right) = 0, \\
&\operatorname{tr}\left(U_2^T U_1\right) + \operatorname{tr}\left(V_2^T V_1\right) \\
&= \operatorname{tr}\Bigg(\Bigg[\frac{A^T(M-\Phi(X_2,Y_2))B^T + E^T(N-\Psi(X_2,Y_2))F^T}{2} \\
&\qquad + \frac{PA^T(M-\Phi(X_2,Y_2))B^TQ + PE^T(N-\Psi(X_2,Y_2))F^TQ}{2} + \frac{\|R_2\|^2}{\|R_1\|^2} U_1 \Bigg]^T U_1 \Bigg)
\end{aligned}
$$

$$+\operatorname{tr}\left(\left[\frac{-C^T(M-\Phi(X_2,Y_2))D^T-G^T(N-\Psi(X_2,Y_2))H^T}{2}\right.\right.$$
$$\left.\left.+\frac{-RC^T(M-\Phi(X_2,Y_2))D^TS-RG^T(N-\Psi(X_2,Y_2))H^TS}{2}+\frac{\|R_2\|^2}{\|R_1\|^2}V_1\right]^T V_1\right)$$
$$=\operatorname{tr}\left(\left[A^T(M-\Phi(X_2,Y_2))B^T+E^T(N-\Psi(X_2,Y_2))F^T+\frac{\|R_2\|^2}{\|R_1\|^2}U_1\right]^T U_1\right)$$
$$+\operatorname{tr}\left(\left[-C^T(M-\Phi(X_2,Y_2))D^T-G^T(N-\Psi(X_2,Y_2))H^T+\frac{\|R_2\|^2}{\|R_1\|^2}V_1\right]^T V_1\right)$$
$$=\operatorname{tr}\Big(U_1^T\left[A^T(M-\Phi(X_2,Y_2))B^T+E^T(N-\Psi(X_2,Y_2))F^T\right]$$
$$+V_1^T\left[-C^T(M-\Phi(X_2,Y_2))D^T-G^T(N-\Psi(X_2,Y_2))H^T\right]\Big)+\frac{\|R_2\|^2}{\|R_1\|^2}\left(\|U_1\|^2+\|V_1\|^2\right)$$
$$=\operatorname{tr}\Big((M-\Phi(X_2,Y_2))^T AU_1B+(N-\Psi(X_2,Y_2))^T EU_1F-(M-\Phi(X_2,Y_2))^T CV_1D$$
$$-(N-\Psi(X_2,Y_2))^T GV_1H\Big)+\frac{\|R_2\|^2}{\|R_1\|^2}\left(\|U_1\|^2+\|V_1\|^2\right)$$
$$=\operatorname{tr}\left(\operatorname{diag}\left((M-\Phi(X_2,Y_2))^T,(N-\Psi(X_2,Y_2))^T\right)\operatorname{diag}(\Phi(U_1,V_1),\Psi(U_1,V_1))\right)$$
$$+\frac{\|R_2\|^2}{\|R_1\|^2}\left(\|U_1\|^2+\|V_1\|^2\right)$$
$$=\frac{\|U_1\|^2+\|V_1\|^2}{\|R_1\|^2}\operatorname{tr}\left(R_2^T(R_1-R_2)\right)+\frac{\|R_2\|^2}{\|R_1\|^2}\left(\|U_1\|^2+\|V_1\|^2\right)=0. \tag{A.3}$$

Assume that (A.2) holds for $i=k-1$, that is, $\operatorname{tr}(R_k^T R_{k-1})=0$, $\operatorname{tr}(U_k^T U_{k-1}+V_k^T V_{k-1})=0$. When $i=k$, we have that

$$\operatorname{tr}\left(R_{k+1}^T R_k\right)$$
$$=\operatorname{tr}\left(\left[R_k-\frac{\|R_k\|^2}{\|U_k\|^2+\|V_k\|^2}\operatorname{diag}(\Phi(U_k,V_k),\Psi(U_k,V_k))\right]^T R_k\right)$$
$$=\|R_k\|^2-\frac{\|R_k\|^2}{\|U_k\|^2+\|V_k\|^2}\operatorname{tr}\left(\operatorname{diag}(\Phi(U_k,V_k),\Psi(U_k,V_k))\right)^T$$
$$\times\operatorname{diag}(M-\Phi(X_k,Y_k),N-\Psi(X_k,Y_k))$$

$$
\begin{aligned}
&= \|R_k\|^2 - \frac{\|R_k\|^2}{\|U_k\|^2 + \|V_k\|^2} \\
&\quad \times \operatorname{tr}\Big((\Phi(U_k, V_k))^T (M - \Phi(X_k, Y_k)) + (\Psi(U_k, V_k))^T (N - \Psi(X_k, Y_k)) \Big) \\
&= \|R_k\|^2 - \frac{\|R_k\|^2}{\|U_k\|^2 + \|V_k\|^2} \operatorname{tr}\Big(U_k^T A^T (M - \Phi(X_k, Y_k)) B^T + U_k^T E^T (N - \Psi(X_k, Y_k)) F^T \\
&\qquad - V_k^T C^T (M - \Phi(X_k, Y_k)) D^T - V_k^T G^T (N - \Psi(X_k, Y_k)) H^T \Big) \\
&= \|R_k\|^2 - \frac{\|R_k\|^2}{\|U_k\|^2 + \|V_k\|^2} \operatorname{tr}\Bigg(U_k^T \Bigg[\frac{A^T (M - \Phi(X_k, Y_k)) B^T + E^T (N - \Psi(X_k, Y_k)) F^T}{2} \\
&\qquad + \frac{A^T (M - \Phi(X_k, Y_k)) B^T + E^T (N - \Psi(X_k, Y_k)) F^T}{2} \\
&\qquad + \frac{P A^T (M - \Phi(X_k, Y_k)) B^T Q + P E^T (N - \Psi(X_k, Y_k)) F^T Q}{2} \\
&\qquad - \frac{P A^T (M - \Phi(X_k, Y_k)) B^T Q + P E^T (N - \Psi(X_k, Y_k)) F^T Q}{2} \Bigg] \\
&\qquad + V_k^T \Bigg[\frac{-C^T (M - \Phi(X_k, Y_k)) D^T - G^T (N - \Psi(X_k, Y_k)) H^T}{2} \\
&\qquad + \frac{-C^T (M - \Phi(X_k, Y_k)) D^T - G^T (N - \Psi(X_k, Y_k)) H^T}{2} \\
&\qquad + \frac{-R C^T (M - \Phi(X_k, Y_k)) D^T S - R G^T (N - \Psi(X_k, Y_k)) H^T S}{2} \\
&\qquad - \frac{-R C^T (M - \Phi(X_k, Y_k)) D^T S - R G^T (N - \Psi(X_k, Y_k)) H^T S}{2} \Bigg] \Bigg) \\
&= \|R_k\|^2 - \frac{\|R_k\|^2}{\|U_k\|^2 + \|V_k\|^2} \operatorname{tr}\Bigg(U_k^T \Bigg[\frac{A^T (M - \Phi(X_k, Y_k)) B^T + E^T (N - \Psi(X_k, Y_k)) F^T}{2} \\
&\qquad + \frac{P A^T (M - \Phi(X_k, Y_k)) B^T Q + P E^T (N - \Psi(X_k, Y_k)) F^T Q}{2} \Bigg] \\
&\qquad + V_k^T \Bigg[\frac{-C^T (M - \Phi(X_k, Y_k)) D^T - G^T (N - \Psi(X_k, Y_k)) H^T}{2} \\
&\qquad + \frac{-R C^T (M - \Phi(X_k, Y_k)) D^T S - R G^T (N - \Psi(X_k, Y_k)) H^T S}{2} \Bigg] \Bigg) \\
&= \|R_k\|^2 - \frac{\|R_k\|^2}{\|U_k\|^2 + \|V_k\|^2} \operatorname{tr}\Big(U_k^T U_k + V_k^T V_k \Big) = 0,
\end{aligned}
$$

$$
\operatorname{tr}\Big(U_{k+1}^T U_k \Big) + \operatorname{tr}\Big(V_{k+1}^T V_k \Big)
$$

$$
\begin{aligned}
&= \operatorname{tr}\Bigg(\Bigg[\frac{A^T(M-\Phi(X_{k+1},Y_{k+1}))B^T + E^T(N-\Psi(X_{k+1},Y_{k+1}))F^T}{2} \\
&\qquad + \frac{PA^T(M-\Phi(X_{k+1},Y_{k+1}))B^TQ + PE^T(N-\Psi(X_{k+1},Y_{k+1}))F^TQ}{2} \\
&\qquad + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}U_k\Bigg]^T U_k\Bigg) \\
&\quad + \operatorname{tr}\Bigg(\Bigg[\frac{-C^T(M-\Phi(X_{k+1},Y_{k+1}))D^T - G^T(N-\Psi(X_{k+1},Y_{k+1}))H^T}{2} \\
&\qquad + \frac{-RC^T(M-\Phi(X_{k+1},Y_{k+1}))D^TS - RG^T(N-\Psi(X_{k+1},Y_{k+1}))H^TS}{2} \\
&\qquad + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}V_k\Bigg]^T V_k\Bigg) \\
&= \operatorname{tr}\Bigg(\Bigg[A^T(M-\Phi(X_{k+1},Y_{k+1}))B^T + E^T(N-\Psi(X_{k+1},Y_{k+1}))F^T + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}U_k\Bigg]^T U_k\Bigg) \\
&\quad + \operatorname{tr}\Bigg(\Bigg[-C^T(M-\Phi(X_{k+1},Y_{k+1}))D^T - G^T(N-\Psi(X_{k+1},Y_{k+1}))H^T + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}V_k\Bigg]^T V_k\Bigg) \\
&= \operatorname{tr}\Big(U_k^T\Big[A^T(M-\Phi(X_{k+1},Y_{k+1}))B^T + E^T(N-\Psi(X_{k+1},Y_{k+1}))F^T\Big] \\
&\qquad + V_k^T\Big[-C^T(M-\Phi(X_{k+1},Y_{k+1}))D^T - G^T(N-\Psi(X_{k+1},Y_{k+1}))H^T\Big]\Big) \\
&\qquad + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}\Big(\|U_k\|^2 + \|V_k\|^2\Big) \\
&= \operatorname{tr}\Big((M-\Phi(X_{k+1},Y_{k+1}))^T AU_kB + (N-\Psi(X_{k+1},Y_{k+1}))^T EU_kF \\
&\quad - (M-\Phi(X_{k+1},Y_{k+1}))^T CV_kD \\
&\quad - (N-\Psi(X_{k+1},Y_{k+1}))^T GV_kH\Big) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}\Big(\|U_k\|^2 + \|V_k\|^2\Big) \\
&= \operatorname{tr}\Big(\operatorname{diag}\Big((M-\Phi(X_{k+1},Y_{k+1}))^T, (N-\Psi(X_{k+1},Y_{k+1}))^T\Big)\operatorname{diag}(\Phi(U_k,V_k), \Psi(U_k,V_k))\Big) \\
&\quad + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}\Big(\|U_k\|^2 + \|V_k\|^2\Big) \\
&= \frac{\|U_k\|^2 + \|V_k\|^2}{\|R_k\|^2}\operatorname{tr}\Big(R_{k+1}^T(R_k - R_{k+1})\Big) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}\Big(\|U_k\|^2 + \|V_k\|^2\Big) = 0.
\end{aligned} \tag{A.4}
$$

Hence, (A.2) holds for $i = k$. Therefore, (A.2) holds by the principle of induction.

Step 2. We show that

$$\operatorname{tr}\left(R_{i+1}^T R_j\right) = 0, \quad \operatorname{tr}\left(U_{i+1}^T U_j + V_{i+1}^T V_j\right) = 0, \quad j = 1, 2, \ldots, i, \ \forall i \geq 1. \tag{A.5}$$

When $i = 1$, (A.5) holds.
Assume that

$$\operatorname{tr}\left(R_i^T R_j\right) = 0, \quad \operatorname{tr}\left(U_i^T U_j + V_i^T V_j\right) = 0, \quad j = 1, 2, \ldots, s-1, \ \forall s \geq 2, \tag{A.6}$$

then we show that

$$\operatorname{tr}\left(R_{i+1}^T R_j\right) = 0, \quad \operatorname{tr}\left(U_{i+1}^T U_j + V_{i+1}^T V_j\right) = 0, \quad j = 1, 2, \ldots, s. \tag{A.7}$$

In fact, we have that

$$\begin{aligned}
&\operatorname{tr}\left(R_{i+1}^T R_j\right) \\
&= \operatorname{tr}\left(\left[R_i - \frac{\|R_i\|^2}{\|U_i\|^2 + \|V_i\|^2} \operatorname{diag}(\Phi(U_i, V_i), \Psi(U_i, V_i))\right]^T R_j\right) \\
&= \operatorname{tr}\left(R_i^T R_j\right) - \frac{\|R_i\|^2}{\|U_i\|^2 + \|V_i\|^2} \operatorname{tr}\left(\operatorname{diag}(\Phi(U_i, V_i), \Psi(U_i, V_i))\right)^T \\
&\quad \times \operatorname{diag}(M - \Phi(X_j, Y_j), N - \Psi(X_j, Y_j)) \\
&= -\frac{\|R_i\|^2}{\|U_i\|^2 + \|V_i\|^2} \operatorname{tr}\left((\Phi(U_i, V_i))^T (M - \Phi(X_j, Y_j)) + (\Psi(U_i, V_i))^T (N - \Psi(X_j, Y_j))\right) \\
&= -\frac{\|R_i\|^2}{\|U_i\|^2 + \|V_i\|^2} \operatorname{tr}\Big(U_i^T A^T (M - \Phi(X_j, Y_j)) B^T + U_i^T E^T (N - \Psi(X_j, Y_j)) F^T \\
&\qquad\qquad - V_i^T C^T (M - \Phi(X_j, Y_j)) D^T - V_i^T G^T (N - \Psi(X_j, Y_j)) H^T\Big) \\
&= -\frac{\|R_i\|^2}{\|U_i\|^2 + \|V_i\|^2} \operatorname{tr}\Bigg(U_i^T \Bigg[\frac{A^T (M - \Phi(X_j, Y_j)) B^T + E^T (N - \Psi(X_j, Y_j)) F^T}{2} \\
&\qquad + \frac{A^T (M - \Phi(X_j, Y_j)) B^T + E^T (N - \Psi(X_j, Y_j)) F^T}{2} \\
&\qquad + \frac{P A^T (M - \Phi(X_j, Y_j)) B^T Q + P E^T (N - \Psi(X_j, Y_j)) F^T Q}{2} \\
&\qquad - \frac{P A^T (M - \Phi(X_j, Y_j)) B^T Q + P E^T (N - \Psi(X_j, Y_j)) F^T Q}{2}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+ V_i^T\left[\frac{-C^T(M-\Phi(X_j,Y_j))D^T - G^T(N-\Psi(X_j,Y_j))H^T}{2}\right.\\
&\quad + \frac{-C^T(M-\Phi(X_j,Y_j))D^T - G^T(N-\Psi(X_j,Y_j))H^T}{2}\\
&\quad + \frac{-RC^T(M-\Phi(X_j,Y_j))D^TS - RG^T(N-\Psi(X_j,Y_j))H^TS}{2}\\
&\quad \left.\left. - \frac{-RC^T(M-\Phi(X_j,Y_j))D^TS - RG^T(N-\Psi(X_j,Y_j))H^TS}{2}\right]\right)\\
&= -\frac{\|R_i\|^2}{\|U_i\|^2+\|V_i\|^2}\operatorname{tr}\left(U_i^T\left[\frac{A^T(M-\Phi(X_j,Y_j))B^T + E^T(N-\Psi(X_j,Y_j))F^T}{2}\right.\right.\\
&\quad \left. + \frac{PA^T(M-\Phi(X_j,Y_j))B^TQ + PE^T(N-\Psi(X_j,Y_j))F^TQ}{2}\right]\\
&\quad + V_i^T\left[\frac{-C^T(M-\Phi(X_j,Y_j))D^T - G^T(N-\Psi(X_j,Y_j))H^T}{2}\right.\\
&\quad \left.\left. + \frac{-RC^T(M-\Phi(X_j,Y_j))D^TS - RG^T(N-\Psi(X_j,Y_j))H^TS}{2}\right]\right)\\
&= -\frac{\|R_i\|^2}{\|U_i\|^2+\|V_i\|^2}\operatorname{tr}\left(U_i^T\left(U_j - \frac{\|R_j\|^2}{\|R_{j-1}\|^2}U_{j-1}\right) + V_i^T\left(V_j - \frac{\|R_j\|^2}{\|R_{j-1}\|^2}V_{j-1}\right)\right)\\
&= -\frac{\|R_i\|^2}{\|U_i\|^2+\|V_i\|^2}\operatorname{tr}\left(U_i^TU_j + V_i^TV_j\right) + \frac{\|R_i\|^2\|R_j\|^2}{\left(\|U_i\|^2+\|V_i\|^2\right)\|R_{j-1}\|^4}\\
&\quad \times\left(\operatorname{tr}\left(U_i^TU_{j-1}\right) + \operatorname{tr}\left(V_i^TV_{j-1}\right)\right) = 0.
\end{aligned}
\tag{A.8}
$$

From the above results, we have $\operatorname{tr}(R_{i+1}^T R_{j+1}) = 0,\ j = 1, 2, \ldots, s-1$, and

$$
\begin{aligned}
&\operatorname{tr}\left(U_{i+1}^TU_j\right) + \operatorname{tr}\left(V_{i+1}^TV_j\right)\\
&= \operatorname{tr}\left(\left[\frac{A^T(M-\Phi(X_{i+1},Y_{i+1}))B^T + E^T(N-\Psi(X_{i+1},Y_{i+1}))F^T}{2}\right.\right.\\
&\quad + \frac{PA^T(M-\Phi(X_{i+1},Y_{i+1}))B^TQ + PE^T(N-\Psi(X_{i+1},Y_{i+1}))F^TQ}{2}\\
&\quad \left.\left. + \frac{\|R_{i+1}\|^2}{\|R_i\|^2}U_i\right]^T U_j\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\operatorname{tr}\left(\left[\frac{-C^T(M-\Phi(X_{i+1},Y_{i+1}))D^T-G^T(N-\Psi(X_{i+1},Y_{i+1}))H^T}{2}\right.\right.\\
&\qquad+\frac{-RC^T(M-\Phi(X_{i+1},Y_{i+1}))D^TS-RG^T(N-\Psi(X_{i+1},Y_{i+1}))H^TS}{2}\\
&\qquad\left.\left.+\frac{\|R_{i+1}\|^2}{\|R_i\|^2}V_i\right]^TV_j\right)\\
&=\operatorname{tr}\left(\left[A^T(M-\Phi(X_{i+1},Y_{i+1}))B^T+E^T(N-\Psi(X_{i+1},Y_{i+1}))F^T+\frac{\|R_{i+1}\|^2}{\|R_i\|^2}U_i\right]^TU_j\right)\\
&\quad+\operatorname{tr}\left(\left[-C^T(M-\Phi(X_{i+1},Y_{i+1}))D^T-G^T(N-\Psi(X_{i+1},Y_{i+1}))H^T+\frac{\|R_{i+1}\|^2}{\|R_i\|^2}V_i\right]^TV_j\right)\\
&=\operatorname{tr}\left(U_j^T\left[A^T(M-\Phi(X_{i+1},Y_{i+1}))B^T+E^T(N-\Psi(X_{i+1},Y_{i+1}))F^T\right]\right.\\
&\qquad\left.+V_j^T\left[-C^T(M-\Phi(X_{i+1},Y_{i+1}))D^T-G^T(N-\Psi(X_{i+1},Y_{i+1}))H^T\right]\right)\\
&\quad+\frac{\|R_{i+1}\|^2}{\|R_i\|^2}\left[\operatorname{tr}\left(U_i^TU_j\right)+\operatorname{tr}\left(V_i^TV_j\right)\right]\\
&=\operatorname{tr}\Big((M-\Phi(X_{i+1},Y_{i+1}))^TAU_jB+(N-\Psi(X_{i+1},Y_{i+1}))^TEU_jF\\
&\qquad-(M-\Phi(X_{i+1},Y_{i+1}))^TCV_jD\\
&\qquad-(N-\Psi(X_{i+1},Y_{i+1}))^TGV_jH\Big)+\frac{\|R_{i+1}\|^2}{\|R_i\|^2}\left[\operatorname{tr}\left(U_i^TU_j\right)+\operatorname{tr}\left(V_i^TV_j\right)\right]\\
&=\operatorname{tr}\left(\operatorname{diag}\left((M-\Phi(X_{i+1},Y_{i+1}))^T,(N-\Psi(X_{i+1},Y_{i+1}))^T\right)\operatorname{diag}(\Phi(U_j,V_j),\Psi(U_j,V_j))\right)\\
&\quad+\frac{\|R_{i+1}\|^2}{\|R_i\|^2}\left[\operatorname{tr}\left(U_i^TU_j\right)+\operatorname{tr}\left(V_i^TV_j\right)\right]\\
&=\frac{\|U_j\|^2+\|V_j\|^2}{\|R_j\|^2}\operatorname{tr}\left(R_{i+1}^T(R_j-R_{j+1})\right)+\frac{\|R_{i+1}\|^2}{\|R_i\|^2}\left[\operatorname{tr}\left(U_i^TU_j\right)+\operatorname{tr}\left(V_i^TV_j\right)\right]=0.
\end{aligned}
\tag{A.9}
$$

By the principle of induction, (A.5) holds.

Noting that (A.1) is implied in Steps 1 and 2 by the principle of induction. This completes the proof.

B. The Proof of Lemma 2.3

We proof the conclusion by induction.

For $i = 1$, we have that

$$\begin{aligned}
&\operatorname{tr}\left((X^* - X_1)^T U_1 + (Y^* - Y_1)^T V_1\right)\\
&= \operatorname{tr}\left((X^* - X_1)^T \left[\frac{A^T(M - \Phi(X_1, Y_1))B^T + E^T(N - \Psi(X_1, Y_1))F^T}{2} \right.\right.\\
&\qquad\qquad \left. + \frac{PA^T(M - \Phi(X_1, Y_1))B^T Q + PE^T(N - \Psi(X_1, Y_1))F^T Q}{2} \right]\\
&\qquad + (Y^* - Y_1)^T \left[\frac{-C^T(M - \Phi(X_1, Y_1))D^T - G^T(N - \Psi(X_1, Y_1))H^T}{2} \right.\\
&\qquad\qquad \left.\left. + \frac{-RC^T(M - \Phi(X_1, Y_1))D^T S - RG^T(N - \Psi(X_1, Y_1))H^T S}{2} \right] \right)\\
&= \operatorname{tr}\left((X^* - X_1)^T \left[A^T(M - \Phi(X_1, Y_1))B^T + E^T(N - \Psi(X_1, Y_1))F^T\right]\right.\\
&\qquad \left. + (Y^* - Y_1)^T \left[-C^T(M - \Phi(X_1, Y_1))D^T - G^T(N - \Psi(X_1, Y_1))H^T\right]\right) \qquad \text{(B.1)}\\
&= \operatorname{tr}\left((M - \Phi(X_1, Y_1))^T A(X^* - X_1)B + (N - \Psi(X_1, Y_1))^T E(X^* - X_1)F\right.\\
&\qquad \left. - (M - \Phi(X_1, Y_1))^T C(Z^* - Z_1)D - (N - \Psi(X_1, Y_1))^T G(Y^* - Y_1)H\right)\\
&= \operatorname{tr}\left(\begin{pmatrix} (M - \Phi(X_1, Y_1))^T & 0 \\ 0 & (N - \Psi(X_1, Y_1))^T \end{pmatrix}\right.\\
&\qquad \left.\begin{pmatrix} A(X^* - X_1)B - C(Y^* - Y_1)D & 0 \\ 0 & E(X^* - X_1)F - G(Y^* - Y_1)H \end{pmatrix}\right)\\
&= \operatorname{tr}\left(\begin{pmatrix} M - \Phi(X_1, Y_1) & 0 \\ 0 & N - \Psi(X_1, Y_1) \end{pmatrix}^T \begin{pmatrix} M - \Phi(X_1, Y_1) & 0 \\ 0 & N - \Psi(X_1, Y_1) \end{pmatrix}\right)\\
&= \|R_1\|^2.
\end{aligned}$$

Assume that (2.5) holds for $i = k$. When $i = k + 1$, by Algorithm 2.1, we have that

$$\begin{aligned}
&\operatorname{tr}\left((X^* - X_{k+1})^T U_{k+1} + (Y^* - Y_{k+1})^T V_{k+1}\right)\\
&= \operatorname{tr}\left((X^* - X_{k+1})^T \left[\frac{A^T(M - \Phi(X_{k+1}, Y_{k+1}))B^T + E^T(N - \Psi(X_{k+1}, Y_{k+1}))F^T}{2} \right.\right.\\
&\qquad\qquad + \frac{PA^T(M - \Phi(X_{k+1}, Y_{k+1}))B^T Q + PE^T(N - \Psi(X_{k+1}, Y_{k+1}))F^T Q}{2}\\
&\qquad\qquad \left. + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} U_k \right]
\end{aligned}$$

$$
\begin{aligned}
&+ (Y^* - Y_{k+1})^T \left[\frac{-C^T(M - \Phi(X_{k+1}, Y_{k+1}))D^T - G^T(N - \Psi(X_{k+1}, Y_{k+1}))H^T}{2} \right. \\
&\qquad + \frac{-RC^T(M - \Phi(X_{k+1}, Y_{k+1}))D^T S - RG^T(N - \Psi(X_{k+1}, Y_{k+1}))H^T S}{2} \\
&\qquad \left. + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} V_k \right] \Bigg) \\
&= \operatorname{tr}\Big((X^* - X_{k+1})^T \left[A^T(M - \Phi(X_{k+1}, Y_{k+1}))B^T + E^T(N - \Psi(X_{k+1}, Y_{k+1}))F^T \right] \\
&\quad + (Y^* - Y_{k+1})^T \left[-C^T(M - \Phi(X_{k+1}, Y_{k+1}))D^T - G^T(N - \Psi(X_{k+1}, Y_{k+1}))H^T \right] \Big) \\
&\quad + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} \operatorname{tr}\Big((X^* - X_{k+1})^T U_k + (Y^* - Y_{k+1})^T V_k \Big) \\
&= \operatorname{tr}\Big((M - \Phi(X_{k+1}, Y_{k+1}))^T A(X^* - X_{k+1})B + (N - \Psi(X_{k+1}, Y_{k+1}))^T E(X^* - X_{k+1})F \\
&\qquad - (M - \Phi(X_{k+1}, Y_{k+1}))^T C(Z^* - Z_{k+1})D - (N - \Psi(X_{k+1}, Y_{k+1}))^T G(Y^* - Y_{k+1})H \Big) \\
&\quad + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} \operatorname{tr}\Big((X^* - X_{k+1})^T U_k + (Y^* - Y_{k+1})^T V_k \Big) \\
&= \operatorname{tr}\left(\begin{pmatrix} (M - \Phi(X_{k+1}, Y_{k+1}))^T & 0 \\ 0 & (N - \Psi(X_{k+1}, Y_{k+1}))^T \end{pmatrix} \right. \\
&\qquad \left. \begin{pmatrix} A(X^* - X_{k+1})B - C(Y^* - Y_{k+1})D & 0 \\ 0 & E(X^* - X_{k+1})F - G(Y^* - Y_{k+1})H \end{pmatrix} \right) \\
&\quad + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} \operatorname{tr}\Big((X^* - X_{k+1})^T U_k + (Y^* - Y_{k+1})^T V_k \Big) \\
&= \operatorname{tr}\left(\begin{pmatrix} (M - \Phi(X_{k+1}, Y_{k+1}))^T & 0 \\ 0 & (N - \Psi(X_{k+1}, Y_{k+1}))^T \end{pmatrix} \right. \\
&\qquad \left. \begin{pmatrix} M - \Phi(X_{k+1}, Y_{k+1}) & 0 \\ 0 & N - \Psi(X_{k+1}, Y_{k+1}) \end{pmatrix} \right) \\
&\quad + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} \operatorname{tr}\Big((X^* - X_{k+1})^T U_k + (Y^* - Y_{k+1})^T V_k \Big) \\
&= \|R_{k+1}\|^2 + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} \operatorname{tr}\Big((X^* - X_k)^T U_k + (Y^* - Y_k)^T V_k \Big) \\
&\quad - \frac{\|R_{k+1}\|^2}{\|U_k\|^2 + \|V_k\|^2} \operatorname{tr}\Big(U_k^T U_k + V_k^T V_k \Big) = \|R_{k+1}\|^2.
\end{aligned}
\tag{B.2}
$$

Therefore, (2.5) holds for $i = k + 1$. Thus, (2.5) holds by the principal of induction. This completes the proof.

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